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A Note on Actions of Compact Matrix Quantum Groups on von Neumann Algebras

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Abstract. In this paper we consider the object $\widetilde{S_\mu U(2)}$ coming from $S_\mu U(2)$ defined by S. L. Woronowicz, and construct an action of $\widetilde{S_\mu U(2)}$ on the Powers factor R_λ if $\lambda = \mu^2$. Moreover we show that the fixed point algebra under the action is the AFD II_1-factor which is generated by Jones projections.

1. Introduction

In [8] Woronowicz introduced a concept of a compact matrix quantum group (a compact matrix pseudogroup) which is a certain deformation of the dual object of compact groups. Let $G = (A, u)$ be a compact matrix quantum group and $\Phi : A \longrightarrow A \otimes_{min} A$ be a $*$-homomorphism called a comultiplication where A is a unital C^*-algebra as in [8]. The comultiplication Φ is an action of G on itself.

In [3] the author, Nagisa and Watatani constructed an action of G on the Cuntz algebra $\mathcal{O}_n$ or the UHF-algebra M_n^∞ of type n^∞. The forms of the actions ψ and ψ' were represented as follows :

$$\psi : \mathcal{O}_n \longrightarrow \mathcal{O}_n \otimes_{min} A, \qquad \psi' : M_n^\infty \longrightarrow M_n^\infty \otimes_{min} A.$$

Especially in [3] they considered the actions of $S_\mu U(2)$ (Woronowicz, [9]) on $\mathcal{O}_2$ and M_2^∞, and showed the fixed point algebras under the actions were generated by Jones projections. This means a C^*-algebra version of a deformation of the result of the case for the action of $SU(2)$ by Jones in [1] and [2].

In this paper we construct an action of $\widetilde{S_\mu U(2)}$ coming from $S_\mu U(2)$ on the Powers factor R_λ if $\lambda = \mu^2$ using the Kac-Takesaki operator introduced by Nakagami and Takesaki in [4] and [6]. Moreover we show that the fixed point algebra under the action is the AFD II_1-factor which is generated by the Jones projections $\{e_n\}_{n=1}^\infty$ such that

$$e_i e_{i\pm1} e_i = (\lambda + \lambda^{-1} + 2)^{-1} e_i,$$

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$$e_i e_j = e_j e_i, \qquad \text{for } | i - j | > 1.$$

This is a von Neumann algebra version of the above result in [3], and can be regarded as a new method to construct the Temperley-Lieb-Pimsner-Popa representation of the Jones relations by Pimsner and Popa in [7].

2. Jones projections and an action of $\widetilde{S_\mu U(2)}$ on R_λ.

We shall study the Temperley-Lieb-Pimsner-Popa representation of the Jones relations using compact matrix quantum groups. We shall first collect the facts of the properties of the compact matrix quantum group $S_\mu U(2)$.

Let A be the universal C^*-algebra generated by α and γ satisfying

$$\alpha^*\alpha + \gamma^*\gamma = 1, \qquad \alpha\alpha^* + \mu^2\gamma\gamma^* = 1,$$

$$\gamma^*\gamma = \gamma\gamma^*, \qquad \mu\gamma\alpha = \alpha\gamma, \qquad \mu\gamma^*\alpha = \alpha\gamma^*,$$

where $-1 \leq \mu \leq 1$. Let

$$u = \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \in M_2(A).$$

Then $G = (A, u)$ is a compact matrix quantum group which is denoted by $S_\mu U(2)$ as in [9, Theorem 1.4]. The comultiplication Φ associated with $S_\mu U(2)$ is defined by

$$\Phi(\alpha) = \alpha \otimes \alpha - \mu\gamma^* \otimes \gamma, \qquad \Phi(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

Let $\mathcal{A}$ denote the dense $*$-subalgebra of A generated by α and γ with the above relations.
For any $k \in Z$ and $m, n \in N \cup \{0\}$, we set

$$a_{kmn} = \begin{cases} \alpha^k \gamma^{*m} \gamma^n & \text{for } k \geq 0 \\ (\alpha^*)^{-k} \gamma^{*m} \gamma^n & \text{for } k < 1. \end{cases}$$

By [9, Theorem 1.2], the family of all elements a_{kmn} forms a basis of the vector space $\mathcal{A}$. Let h be the Haar measure on $S_\mu U(2)$ in the sense of [8, Theorem 4.2], that is, h is a state on A such that

$$(h \otimes id)\Phi(a) = (id \otimes h)\Phi(a) = h(a)1, \qquad \text{for } a \in A.$$

By [8, Appendices 1], the Haar measure h satisfies

$$h(a_{kmn}) = 0, \qquad \text{for } k \neq 0 \text{ or } m \neq n$$

and

$$h((\gamma^*\gamma)^m) = \frac{1-\mu^2}{1-\mu^{2m+2}}.$$

It is known that the Haar measure h is faithful on A by [5, Corollary 2.3]. Let $\{\pi_h, H_h\}$ be the GNS-representation of A induced by h. Then $\pi_h(A)$ is *-isomorphic to A. So $G' = (\pi_h(A)\ ,\ (id \otimes \pi_h)u)$ is a compact matrix quantum group. Let Φ' be the comultiplication of G'. Let $\{\pi_{h\otimes h}\ ,\ H_{h\otimes h}\ ,\ \Lambda_{h\otimes h}\}$ be the GNS-representation of $A \otimes_{min} A$ induced by $h \otimes h$. It is well known that

$$\{\pi_{h\otimes h}\ ,\ H_{h\otimes h}\} = \{\pi_h \otimes \pi_h\ ,\ H_h \otimes H_h\}.$$

Let W be the Kac-Takesaki operator on the Hilbert space $H_h \otimes H_h$ defined by

$$W\Lambda_{h\otimes h}(a \otimes b) = \Lambda_{h\otimes h}(\Phi(a)(1 \otimes b)), \qquad \text{for } a, b \in A.$$

By the property of the Haar measure h, W is a unitary operator implementing the comultiplication Φ' of G' as in [4, §2] :

$$\Phi'(\pi_h(a)) = W(\pi_h(a) \otimes 1)W^*, \qquad \text{for } a \in A.$$

Therefore we can define an injective normal *-homomorphism $\widetilde{\Phi} : \pi_h(A)'' \longrightarrow \pi_h(A)''\overline{\otimes}\pi_h(A)''$ such that

$$\widetilde{\Phi}(x) = W(x \otimes 1)W^*, \qquad \text{for } x \in \pi_h(A)''.$$

It is easy to see that $\widetilde{\Phi}$ has the property of a coassociativity :

$$(id\overline{\otimes}\widetilde{\Phi}) \circ \widetilde{\Phi} = (\widetilde{\Phi}\overline{\otimes}id) \circ \widetilde{\Phi}.$$

Put $\widetilde{S_\mu U(2)} = (\pi_h(A)''\ ,\ (id \otimes \pi_h)u)$.

DEFINITION 1. *Let M be a von Neumann algebra and $\delta : M \longrightarrow M\overline{\otimes}\pi_h(A)''$ be a normal *-homomorphism. Then δ is called an action of $\widetilde{S_\mu U(2)}$ on M if it satisfies the following condition :*

$$(\delta\overline{\otimes}id) \circ \delta = (id\overline{\otimes}\widetilde{\Phi}) \circ \delta.$$

DEFINITION 2. *Let M be a von Neumann algebra and δ be an action of $\widetilde{S_\mu U(2)}$ on M. We define the fixed point subalgebra M^δ of M by δ as follows :*

$$M^\delta = \{x \in M\ ;\ \delta(x) = x \otimes 1\} \quad (= M^{\widetilde{S_\mu U(2)}}).$$

We suppose that $\mu \in (-1, 1) \setminus \{0\}$. We denote by M_2^K the K-times tensor product of M_2, and M_2^∞ the UHF-algebra of type 2^∞. For each m, $n \geq 1$, let $\oplus$ be a bilinear map of $(M_2^m \otimes A) \times (M_2^n \otimes A)$ to $M_2^{m+n} \otimes A$ defined by

$$(x \otimes a) \oplus (y \otimes b) = x \otimes y \otimes ab$$

for any $x \in M_2^m$, $y \in M_2^n$, and $a, b \in A$ (cf. [8, §2]). Let u^K be

$$u^K = \overbrace{u \oplus \cdots \oplus u}^{K \text{ times}}.$$

As in [3, Remark 4], there exists a *-homomorphism $\Gamma_1 : M_2^\infty \longrightarrow M_2^\infty \otimes_{min} A$ called an (product type) action of $S_\mu U(2) = (A, u)$ on M_2^∞ such that

$$\Gamma_1(x) = u^K(x \otimes 1_A)(u^K)^*, \qquad \text{for } x \in M_2^K.$$

Let η be a state on M_2 such that

$$\eta(x) = \frac{1}{1+\lambda} Tr\left(\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} x\right), \qquad \text{for } x \in M_2,$$

where Tr is the canonical trace on M_2. Let φ be a state on M_2^∞ defined by

$$\varphi(x) = \prod_{i=1}^K \eta(x_i)$$

for $x = \otimes_{i=1}^K x_i \in M_2^K$. Then we have the following :

PROPOSITION 3. *If $\lambda = \mu^2$ then Γ_1 preserves φ, that is,*

$$(\varphi \otimes id)\Gamma_1(x) = \varphi(x)1_A$$

for any $x \in M_2^\infty$.

PROOF. We shall show the proposition by induction on the number of the tensor product of M_2. Let $\{e_{kl}\}_{1 \le k,l \le 2}$ be a system of matrix units of M_2. It is clear that the assertion holds on M_2.

For $x = e_{i_1 j_1} \otimes \cdots \otimes e_{i_K j_K} \in M_2^K$, $\Gamma_1(x)$ is represented as follows ([3, Corollary 3]) :

$$\Gamma_1(x) = \sum_{\substack{a_1, \cdots, a_K \\ b_1, \cdots, b_K}} e_{a_1 b_1} \otimes \cdots \otimes e_{a_K b_K} \otimes u_{a_1 i_1} \cdots u_{a_K i_K} u^*_{b_K j_K} \cdots u^*_{b_1 j_1}.$$

We may assume that the claim holds for K, namely, Γ_1 preserves φ on M_2^K. Then we have

$$\prod_{l=1}^K \eta(e_{i_l j_l})1_A = \sum_{\substack{a_1, \cdots, a_K \\ b_1, \cdots, b_K}} \prod_{l=1}^K \eta(e_{a_l b_l}) u_{a_1 i_1} \cdots u_{a_K i_K} u^*_{b_K j_K} \cdots u^*_{b_1 j_1}.$$

Put $X = \prod_{l=1}^K \eta(e_{i_l j_l})1_A$. For $y = e_{i_1 j_1} \otimes \cdots \otimes e_{i_{K+1} j_{K+1}} \in M_2^{K+1}$, by the induction hypothesis,

$$\varphi(y)1_A = \eta(e_{i_{K+1}, j_{K+1}})X.$$

Now we have

$$
\begin{aligned}
&(\varphi\otimes id)\Gamma_1(y)\\
&= \sum_{\substack{a_1,\cdots,a_{K+1}\\ b_1,\cdots,b_{K+1}}} \Big(\prod_{l=1}^{K}\eta(e_{a_l b_l})\Big)\eta(e_{a_{K+1},b_{K+1}})u_{a_1 i_1}\cdots u_{a_{K+1},i_{K+1}}u^*_{b_{K+1},j_{K+1}}\cdots u^*_{b_1 j_1}\\
&= \frac{1}{1+\lambda}\sum_{\substack{a_1,\cdots,a_K\\ b_1,\cdots,b_K}}\prod_{l=1}^{K}\eta(e_{a_l b_l})u_{a_1 i_1}\cdots u_{a_K i_K}u_{1,i_{K+1}}u^*_{1,j_{K+1}}u^*_{b_K j_K}\cdots u^*_{b_1 j_1}\\
&\quad+\frac{\lambda}{1+\lambda}\sum_{\substack{a_1,\cdots,a_K\\ b_1,\cdots,b_K}}\prod_{l=1}^{K}\eta(e_{a_l b_l})u_{a_1 i_1}\cdots u_{a_K i_K}u_{2,i_{K+1}}u^*_{2,j_{K+1}}u^*_{b_K j_K}\cdots u^*_{b_1 j_1}.
\end{aligned}
$$

If $(i_{K+1}, j_{K+1}) = (1,1)$, the above term is

$$\frac{1}{1+\lambda}X\alpha\alpha^* + \frac{\lambda}{1+\lambda}X\gamma\gamma^* = \frac{1}{1+\lambda}X(\alpha\alpha^* + \mu^2\gamma\gamma^*) = \eta(e_{11})X.$$

Similarly we can calculate for each case (i_{K+1}, j_{K+1}) = (1,2), (2,1) and (2,2). Therefore the assertion holds also for $K+1$. ∎

In the rest of the paper, we suppose that $\lambda = \mu^2$. Let $\{\pi_{\varphi\otimes h}\ ,\ H_{\varphi\otimes h}\ ,\ \Lambda_{\varphi\otimes h}\}$ be the GNS-representation of $M_2^\infty \otimes_{min} A$ induced by $\varphi\otimes h$. It is well known that

$$\{\pi_{\varphi\otimes h}\ ,\ H_{\varphi\otimes h}\} = \{\pi_\varphi\otimes\pi_h\ ,\ H_\varphi\otimes H_h\},$$

where $\{\pi_\varphi, H_\varphi\}$ is the GNS-representation of M_2^∞ induced by φ. Let $\Gamma_2 : \pi_\varphi(M_2^\infty) \longrightarrow \pi_\varphi(M_2^\infty)\otimes_{min}\pi_h(A)$ be the action of $G' = (\pi_h(A)\ ,\ (id\otimes\pi_h)u)$ on $\pi_\varphi(M_2^\infty)$ such that

$$(\pi_\varphi\otimes\pi_h)\circ\Gamma_1 = \Gamma_2\circ\pi_\varphi.$$

Let V be the Kac-Takesaki operator on the Hilbert space $H_\varphi\otimes H_h$ defined by

$$V\Lambda_{\varphi\otimes h}(x\otimes a) = \Lambda_{\varphi\otimes h}(\Gamma_1(x)(1\otimes a)),\quad x\in M_2^\infty,\ a\in A.$$

Then by Proposition 3, we can see that V is a unitary operator on $H_\varphi\otimes H_h$ implementing Γ_2 similarly to [4, §2] (cf. [6, Chapter III. §2]).

Let R_λ be the Powers factor, that is, $R_\lambda = \pi_\varphi(M_2^\infty)''$ in $B(H_\varphi)$. We now have an action of $S_\mu\widetilde{U}(2)$ on R_λ.

THEOREM 4. *The injective normal *-homomorphism* $\Gamma : R_\lambda \longrightarrow R_\lambda\overline{\otimes}\pi_h(A)''$ *as the normal extension of* Γ_2 *is an action of* $S_\mu\widetilde{U}(2)$ *on* R_λ.

We shall determine the fixed point subalgebra of R_λ under the above action Γ. Let e be a projection in $M_2 \otimes M_2$ such that

$$e = \frac{1}{1+\lambda}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -\sqrt{\lambda} & 0 \\ 0 & -\sqrt{\lambda} & \lambda & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\{e_n\}_{n=1}^{\infty}$ be projections such that

$$e_1 = e \otimes 1_{M_2} \otimes \cdots, \quad e_2 = 1_{M_2} \otimes e \otimes 1_{M_2} \cdots, \quad e_3 = 1_{M_2} \otimes 1_{M_2} \otimes e \otimes 1_{M_2} \cdots, \cdots.$$

By [7, 5.5.Notation], each e_i $(i = 1, 2, \cdots)$ is in R_λ and the sequence of the projections satisfies the following Jones relations :

$$e_i e_{i\pm 1} e_i = (\lambda + \lambda^{-1} + 2)^{-1} e_i, \quad e_i e_j = e_j e_i, \quad \text{for } | i - j | > 1.$$

By [3, Proposition 9], it was shown that

$$M_2^{\infty \Gamma_1} = C^*\{1, e_1, e_2, \cdots\}.$$

Let φ_λ be the Powers state on R_λ. Then φ_λ restricting to the *-subalgebra of R_λ generated by the projections e_n is the Markov trace tr of modulus $\lambda + \lambda^{-1} + 2$ by [7, 5.5 Notation]. By [2, Theorem 4.1.1], the von Neumann algebra generated by the projections e_n with tr is the AFD II_1-factor. Therefore by the standard argument (cf. [3]), we can reach the following :

THEOREM 5. *The fixed point algebra $R_\lambda^{\widetilde{S_\mu U(2)}}$ is the AFD II_1-factor which is generated by $\{e_n\}_{n=1}^{\infty}$.*

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