Nihonkai Math. J.
Vol.2(1991),139-153

A generalization of Clairaut's theorem and umbilic foliations

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1. Introduction

In differential geometry, behavior of geodesics in a Riemannian manifold is an interesting theme. One of famous and classical results in this direction is Clairaut's theorem on surfaces of revolution. R. L. Bishop[1] defined a Clairaut submersion and obtained a generalization of Clairaut's theorem. The total space of a submersion with connected fibers is considered as a foliated manifold. In this note, we consider Riemannian manifolds with umbilic foliations([2]) and discuss the behavior of geodesics in such manifolds. Our result is a generalization of Clairaut's theorem. We also give some examples of umbilic foliations. We shall be in C^{∞}-category.

1980 Mathematics Subject Classification (1985 Revision). Primary 57R30.

2. Statement of the theorem

Let $(M,g,\mathcal{F})$ be an orientable, connected, p+q dimensional manifold with a Riemannian metric g and with a transversally orientable foliation $\mathcal{F}$ of codimension q ([5], [6]). Then, in differential geometry of foliations, the following fact (#) is a fundamental result which was obtained by B. L. Reinhart[5]:

(#) Let $(M,g,\mathcal{F})$ be as above. If g is bundle-like with respect to $\mathcal{F}$ in the sense of Reinhart([5]), then a geodesic γ in M orthogonal to the leaf at one point of γ is to be orthogonal to leaves at all the points of γ.

Let H be the mean curvature field of $\mathcal{F}$, that is, H is a vector field on M such that the restriction of H to a leaf L of $\mathcal{F}$ is the mean curvature vector field along the submanifold L of M ([2], [6], [7]). A leaf of $\mathcal{F}$ is totally geodesic (resp. totally umbilic) if it is a totally geodesic (resp. totally umbilic) submanifold of M ([2], [6]).

Definition([2], [6]). If all the leaves of $\mathcal{F}$ are totally geodesic, then $\mathcal{F}$ is called <u>totally geodesic</u> with respect to g . If some of the leaves of $\mathcal{F}$ are totally geodesic and others are totally umbilic, or if all the leaves of $\mathcal{F}$ are totally umbilic, then $\mathcal{F}$ is called <u>umbilic</u> with respect to g .

Let $\gamma = \gamma(t)$ be a geodesic in M , where t is an affine parameter. The tangent vector field $\dot{\gamma}(t)$ on γ is decomposed into the following form: $\dot{\gamma}(t) = \dot{\gamma}(t)^T + \dot{\gamma}(t)^N$, where $\dot{\gamma}(t)^T$ (resp. $\dot{\gamma}(t)^N$) is tangent (resp. orthogonal) to the leaf at each point $\gamma(t)$. Thus we have

$$g(\, \dot{\gamma}(t),\ \dot{\gamma}(t)\,) = \rho^2 = \text{constant} \quad .$$

We define a function α on γ by

$$\rho^2 \cdot \cos^2 \alpha(\gamma(t)) = g(\, \dot{\gamma}(t)^T,\ \dot{\gamma}(t)^T\,)$$
$$0 \le \alpha \le \pi/2 \quad ,$$

and we call α the angular function of γ with respect to $\mathcal{F}$. For each t , $\alpha(\gamma(t))$ is an angle between the vectors $\dot{\gamma}(t)$ and $\dot{\gamma}(t)^T$ at $\gamma(t)$. We notice that, in general, the function $\cos\alpha$ is not constant on γ . But we have

Theorem. Let $(M, g, \mathcal{F})$ be as above. Suppose that g is bundle-like with respect to $\mathcal{F}$ and that $\mathcal{F}$ is umbilic with respect to g . Let H be the mean curvature field of $\mathcal{F}$. Let $\gamma(t)$ be a geodesic in M and α be the angular function of γ with respect to $\mathcal{F}$. Suppose that $\cos\alpha \neq 0$ on γ . Then a function r ($\neq 0$) on γ satisfies

$$r \cdot \cos \alpha = \text{constant}$$

if and only if r *is given by the form*

$$r(\gamma(t)) = C \cdot \exp(- \int g(\dot{\gamma}(t), H_{\gamma(t)}) dt) ,$$

where C *is a non-zero constant.*

Remark 1. The fact (#) implies that if $\cos \alpha(\gamma(t_o))$ $= 0$ for some point $\gamma(t_o)$ then $\cos \alpha = 0$ on γ . Hence, in this case, we have that $r \cdot \cos \alpha = 0$ on γ .

Remark 2. If $\mathcal{F}$ is totally geodesic with respect to g , then α is a constant function([5, Theorem 4.1]) and $H = 0$. Thus we have that $r = 1$ and $r \cdot \cos \alpha = \text{constant}$.

Remark 3. Let S be a surface of revolution in R^3 defined by $x = f(v) \cdot \cos u$, $y = f(v) \cdot \sin u$, $z = v$, where f is a positive valued function on an interval $I \subset R^1$, $v \in I$, and $0 \leq u < 2\pi$. Then $S = \{ (u, v) \in S^1 \times I \}$ has a metric $g = f^2 \cdot (du)^2 + (1 + (f')^2) \cdot (dv)^2$, where f' $= \frac{df}{dv}$. We consider a foliation $\mathcal{F}$ on S given by $\mathcal{F}$ $= \{ S^1 \times \{v\} \mid v \in I \}$. Then $\mathcal{F}$ is umbilic with respect to g , and g is bundle-like with respect to $\mathcal{F}$. We have that $H = - \frac{1}{1 + (f')^2} \cdot \frac{f'}{f} \cdot \frac{\partial}{\partial v}$. Thus we have that $g(\dot{\gamma}(t), H_{\gamma(t)})$ $= - \frac{d}{dt}(\log f(v(t)))$ and $- \int g(\dot{\gamma}(t), H_{\gamma(t)}) dt$

$= \log f(v(t)) + C_0$ (C_0 is a constant). We set $C_0 = 0$, then we have that $r = f$. Thus we have Clairaut's theorem. Therefore, our result is a generalization of Clairaut's theorem.

Remark 4. In the case of Clairaut foliation, r is given as a function on M and is called the girth of $\mathcal{F}$ ([1], [7]).

Remark 5. In [7], Clairaut's relation is expressed in the form: $r \cdot \sin \omega(\gamma(t)) =$ constant , because $\omega(\gamma(t))$ is an angle between the vectors $\dot{\gamma}(t)$ and $\dot{\gamma}(t)^N$ at $\gamma(t)$.

3. Proof of the theorem

We suppose that g is bundle-like with respect to $\mathcal{F}$ ([5]) and let ∇ be the Levi-Civita connection with respect to g . Let L be a leaf of $\mathcal{F}$. For each $x \in L$ and a flat chart $U(x^i, x^a)$ about x , we can take an orthonormal adapted frame field $\{ X_i, X_a \}$ on U ([5], [7]). Here $1 \leq i, j \leq p$, $p+1 \leq a, b \leq p+q$. Then L is totally geodesic if

$$(\nabla_{X_i} X_j)_x^N = 0$$

for any $x \in L$, where $(\nabla_{X_i} X_j)_x^N$ denotes the orthogonal part of the vector $(\nabla_{X_i} X_j)_x$ at x , that is, $(\nabla_{X_i} X_j)_x^N$

$= \Sigma_a \Gamma^a_{ij}(x)\cdot(X_a)_x$. And L is totally umbilic if

$$H_x \neq 0 \quad \text{and} \quad (\nabla_{X_i} X_j)^N_x = \delta_{ij}\cdot H_x$$

for any $x \in L$, where δ_{ij} denotes the Kronecker's delta, and H is the mean curvature field of $\mathcal{F}$ defined by

$$H_x = \frac{1}{p} \Sigma_a\, g_x((\Sigma_i \nabla_{X_i} X_i)_x, (X_a)_x)\cdot(X_a)_x$$

for each $x \in M$ ([6], [7]). If L is totally geodesic, then $(\nabla_{X_i} X_j)^N_x = 0$ and $H_x = 0$ for any $x \in L$ so that $(\nabla_{X_i} X_j)^N_x = \delta_{ij}\cdot H_x$ for any $x \in L$.

Now, let $\gamma = \gamma(t)$ be a geodesic in M , that is, $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$. Here t is an affine parameter of γ . We suppose that $\mathcal{F}$ is umbilic (see Definition in section 2) and that $\cos\alpha \neq 0$ on γ . Then we have

$$g(\nabla_{\dot\gamma(t)} \dot\gamma(t)^T, \dot\gamma(t)^T)$$

$$= g(\dot\gamma(t)^N, \nabla_{\dot\gamma(t)^T} \dot\gamma(t)^T)$$

(since g is bundle-like and γ is a geodesic ([7, (6.1) and (6.2)]))

$$= g(\dot\gamma(t)^N, g(\dot\gamma(t)^T, \dot\gamma(t)^T)\cdot H_{\gamma(t)}) \quad .$$

(since $\mathcal{F}$ is umbilic)

Thus we have

$$(1) \qquad g(\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t)^T, \dot{\gamma}(t)^T)$$

$$= g(\dot{\gamma}(t)^T, \dot{\gamma}(t)^T)\cdot g(\gamma(t), H_{\gamma(t)}) \quad .$$

Consider a function $r \neq 0$ on γ . Then we can set $r(\gamma(t)) = c\cdot \exp(f(\gamma(t)))$ (c is a non-zero constant), where f is a function defined on the geodesic $\gamma(t)$. Then we have

$$\rho^2\cdot\cos\alpha(\gamma(t))\cdot\frac{d}{dt}(r(\gamma(t))\cdot\cos\alpha(\gamma(t)))$$

$$= \frac{df}{dt}\cdot r\cdot\rho^2\cdot\cos^2\alpha - r\cdot\rho^2\cdot\cos\alpha\cdot\sin\alpha\cdot\frac{d\alpha}{dt}$$

$$= \frac{df}{dt}\cdot r\cdot g(\dot{\gamma}(t)^T, \dot{\gamma}(t)^T) + r\cdot\frac{1}{2}\cdot\frac{d}{dt}(\rho^2\cdot\cos^2\alpha) \quad .$$

Thus we have

$$(2) \qquad \rho^2\cdot\cos\alpha(\gamma(t))\cdot\frac{d}{dt}(r(\gamma(t))\cdot\cos\alpha(\gamma(t)))$$

$$= \frac{df}{dt}\cdot r\cdot g(\dot{\gamma}(t)^T, \dot{\gamma}(t)^T)$$

$$+ r\cdot g(\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t)^T, \dot{\gamma}(t)^T) \quad .$$

By (1) and (2), we have

$$\rho^2\cdot\cos\alpha\cdot\frac{d}{dt}(r\cdot\cos\alpha)$$

$$= r\cdot g(\dot{\gamma}(t)^T, \dot{\gamma}(t)^T)\cdot\{ \frac{df}{dt} + g(\dot{\gamma}(t), H_{\gamma(t)}) \} \quad .$$

We suppose that $r \cdot \cos \alpha$ = constant on γ, then we have

$$\frac{df}{dt} + g(\, \dot{\gamma}(t),\ H_{\gamma(t)}\,) = 0 .$$

Here we notice that $g(\, \dot{\gamma}(t)^T,\ \dot{\gamma}(t)^T\,) \neq 0$ on γ because $\cos \alpha \neq 0$ on γ. The above differential equation has a solution:

$$f(\gamma(t)) = - \int g(\, \dot{\gamma}(t),\ H_{\gamma(t)}\,)\, dt + c_0 ,$$

where c_0 is a constant. Thus we have

$$r(\gamma(t)) = C \cdot \exp(\, - \int g(\, \dot{\gamma}(t),\ H_{\gamma(t)}\,)\, dt\,) ,$$

where C is a non-zero constant. Conversely, if r is given as the above form then it is clear that r satisfies $r \cdot \cos \alpha$ = constant on γ.

4. Examples

We give some examples of umbilic foliations.

Example 1. Let M be a Kenmotsu manifold, that is, M is a $2n+1$ dimensional manifold with the structure tensor fields $(\, \varphi,\ \xi,\ \eta,\ g\,)$ satisfying the following conditions:

$$\varphi^2 = -I + \eta \otimes \xi , \quad \eta(\xi) = 1 ,$$

$$g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\cdot\eta(Y) ,$$

$$(\nabla_X \varphi)(Y) = g(\varphi(X), Y)\cdot\xi - \eta(Y)\cdot\varphi(X) ,$$

for any vector fields X and Y on M ([3], [4]). Then

$$d\eta = 0$$

([3], [4]) and hence $\eta = 0$ defines a foliation $\mathcal{F}$ on M. Pitis[4] proved that g is bundle-like with respect to $\mathcal{F}$, and $\mathcal{F}$ is umbilic with respect to g.

Example 2. Let (F, g_F) and (B, g_B) be orientable, connected Riemannian manifolds, and $\dim F = p$ and $\dim B = q$. We consider a product manifold $M = F \times B$, and let $p_1 : M \longrightarrow F$ and $p_2 : M \longrightarrow B$ be projections. We define a metric g on M by

$$g(X, Y) = h^2 \cdot g_F(p_{1*}X, p_{1*}Y) + g_B(p_{2*}X, p_{2*}Y)$$

for any vector fields X and Y on M. Here h is a function on $M = F \times B$. Then we have a foliation $\mathcal{F}$ on M given by

$$\mathcal{F} = \{ F \times \{b\} \mid b \in B \} .$$

The foliation $\mathcal{F}$ is umbilic with respect to g and the metric g is bundle-like with respect to $\mathcal{F}$ ([2]). Such a Riemannian manifold (M,g) is called an umbilic product manifold([2]). A warped product manifold is a special type of umbilic product manifolds. Let (R^{p+1}, g_o) be the p+1 dimensional Euclidean space, and let $S^p(r)$ be the p dimensional sphere in (R^{p+1}, g_o) centered at the origin and of radius r. Then we have a foliated Riemannian manifold $(M,g,\mathcal{F})$, where $M = R^{p+1}$ - {the origin}, $g = g_o|_M$, and $\mathcal{F} = \{ S^p(r) \mid r > 0 \}$. We easily have that $\mathcal{F}$ is umbilic with respect to g and g is bundle-like with respect to $\mathcal{F}$. This manifold $(M,g,\mathcal{F})$ is a warped product manifold $(S^p(1) \times (0, +\infty), r^2 g_S + (dr)^2)$ (g_S : the induced metric on $S^p(1)$ from g_o).

Example 3. Let $S^3(1)$ be a unit sphere in R^4, that is, $S^3(1) = \{ x = {}^t(x^1,x^2,x^3,x^4) \in R^4 \mid \|x\|^2 = \Sigma_{i=1}^4 (x^i)^2 = 1 \}$. Let $A_2(R^4)$ be the set of all 2-dimensional affine subspaces of R^4, and let $A_2(R^4 ; o)$ be the set of all 2-dimensional affine subspaces of R^4 passing through the origin $o \in R^4$. The set $A_2(R^4 ; o)$ is a subset of $A_2(R^4)$. We have already known that there exists the Hopf fibration $\pi : S^3(1) \longrightarrow S^2$. It is trivial that each fiber of π is a great circle in $S^3(1)$. Thus we can obtain a subset $A_2^H(R^4 ; o)$ of $A_2(R^4 ; o)$ given by

$A_2^H(R^4 ; o)$

$= \{ \alpha \in A_2(R^4 ; o) \mid \alpha \cap S^3(1)$ is a fiber of π $\}$.

We take a unit vector $v \in R^4$ and a sufficiently small $\varepsilon > 0$. We consider a subset $A_2^H(R^4 ; \varepsilon v)$ of $A_2(R^4)$, that is,

$$A_2^H(R^4 ; \varepsilon v) = \{ \varepsilon v + \alpha \mid \alpha \in A_2^H(R^4 ; o) \} .$$

We notice that $\varepsilon v + \alpha$ denotes a 2-dimensional affine subspace passing through the point εv near the origin. Two subspaces α and $\varepsilon v + \alpha$ are parallel in R^4 . For α_1, $\alpha_2 \in A_2^H(R^4 ; o)$ satisfying $\alpha_1 \cap \alpha_2 \cap S^3(1) = \phi$, we have that $(\varepsilon v + \alpha_1) \cap (\varepsilon v + \alpha_2) \cap S^3(1) = \phi$. There exists only one $\alpha_o \in A_2^H(R^4 ; o)$ passing through the point v . Then $(\varepsilon v + \alpha_o) \cap S^3(1)$ is a great circle, because the affine subspace $\varepsilon v + \alpha_o$ in R^4 coincides with α_o . If $\alpha \in A_2^H(R^4 ; o)$ does not pass through the point v then $(\varepsilon v + \alpha) \cap S^3(1)$ is a small circle. Let x be an arbitrary point of $S^3(1)$, and we regard x as a vector in R^4 . Then $y = \frac{1}{\| x - \varepsilon v \|} \cdot (x - \varepsilon v)$ is a point of $S^3(1)$ so that we have only one $\alpha \in A_2^H(R^4 ; o)$ passing through the point y . Thus $\varepsilon v + \alpha \in A_2^H(R^4 ; \varepsilon v)$ passes through the point x .

Let g_o be the canonical metric on $S^3(1)$ induced from the Euclidean metric in R^4 . The metric g_o is of constant curvature 1 . Then we have a foliation $\mathcal{F}$ on $S^3(1)$ given by

$$\mathcal{F} = \{ \tilde{\alpha} \cap S^3(1) \mid \tilde{\alpha} \in A_2^H(R^4 ; \varepsilon v) \} ,$$

and $\mathcal{F}$ is umbilic with respect to g_0 . We notice that $\mathcal{F}$ has one totally geodesic leaf and that g_0 is not bundle-like with respect to $\mathcal{F}$.

Remark 6. The Hopf fibrations $\pi_Q : S^7(1) \longrightarrow S^4$ and $\pi_{Cay} : S^{15}(1) \longrightarrow S^8$ are considered as Riemannian submersions, which are totally geodesic foliations on $S^7(1)$ and $S^{15}(1)$ with respect to the metrics, respectively. Thus, according to Example 3, we have a foliation on $S^7(1)$ (resp. $S^{15}(1)$) via the Hopf fibration π_Q (resp. π_{Cay}). The new foliation on $S^7(1)$ (resp. $S^{15}(1)$) is umbilic with respect to the metric on $S^7(1)$ (resp. $S^{15}(1)$). Moreover the metric on $S^7(1)$ (resp. $S^{15}(1)$) is not bundle-like with respect to the new foliation on $S^7(1)$ (resp. $S^{15}(1)$).

Example 4. Let $(M, g_0, \mathcal{F})$ be a foliated Riemannian manifold with a totally geodesic foliation $\mathcal{F}$ with respect to the Riemannian metric g_0 , and let g_0 be bundle-like with respect to $\mathcal{F}$. We take a positive valued and non-constant foliated function f on M . Here a function f is foliated if f has constant values along the leaves of $\mathcal{F}$. We consider a metric $g = f^2 \cdot g_0$, then $\mathcal{F}$ is umbilic with respect to g , and g is bundle-like with respect to

$\mathcal{F}$. For instance, the Hopf fibration $\pi : (S^3(1), g_o) \longrightarrow (S^2, h)$ is a Riemannian submersion(see Example 3). Let $\mathcal{F}_o = \{ \alpha \cap S^3(1) \mid \alpha \in A^H_2(R^4 ; o) \}$, which is a foliation on $S^3(1)$ (see Example 3). It is trivial that $\mathcal{F}_o$ is totally geodesic with respect to g_o and that g_o is bundle-like with respect to $\mathcal{F}_o$. We take a positive valued and non-constant function $\underline{f}$ on (S^2, h) . Then $f = \underline{f} \circ \pi$ is a positive valued and non-constant foliated function on $(S^3(1), g_o, \mathcal{F}_o)$. By the above discussion, we have a foliated Riemannian manifold $(S^3(1), g, \mathcal{F}_o)$ where $g = f^2 \cdot g_o$. Then $\mathcal{F}_o$ is umbilic with respect to g , and g is bundle-like with respect to $\mathcal{F}_o$.

Example 5. We consider two spaces:

$$(R^2{}_1, g_1) = (\{(x^1, x^2) \mid x^1, x^2 \in R\}, (dx^1)^2 + (dx^2)^2) ,$$
$$(R^2{}_2, g_2) = (\{(y^1, y^2) \mid y^1, y^2 \in R\}, (dy^1)^2 + (dy^2)^2) ,$$

and a positive-valued, non-constant function h on $(R^2{}_2, g_2)$ that is invariant under rotations, for example,

$$h(y^1, y^2) = \exp((y^1)^2 + (y^2)^2) .$$

Let (X, g_X) be a warped product manifold $(R^2{}_1 \times R^2{}_2, h \cdot g_1 + g_2)$ and let $\mathcal{F}' = \{ R^2{}_1 \times \{y\} \mid y \in R^2{}_2 \}$ be a foliation on (X, g_X) . Then $\mathcal{F}'$ is umbilic with respect to

g_X and g_X is bundle-like with respect to $\mathcal{F}'$. Let G be the group consisting of transformations of (X, g_X) :

$$(x^1, x^2, y^1, y^2) \longrightarrow (x^1 + n, x^2, (\cos n\theta)y^1 - (\sin n\theta)y^2, (\sin n\theta)y^1 + (\cos n\theta)y^2),$$

where $\theta = 2\pi/3$ and $n = 0, \pm 1, \pm 2, \cdots$. Each element of G is an isometry of (X, g_X) . Then we have a foliated manifold $(M = X/G, g, \mathcal{F})$, where $\mathcal{F}$ is the foliation on M induced from $\mathcal{F}'$ on X and g is the Riemannian metric on M induced from g_X on X . Hence $\mathcal{F}$ is umbilic with respect to g and g is bundle-like with respect to $\mathcal{F}$. We must notice that $\mathcal{F}$ is a non-regular foliation([5]).

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Received June 21, 1991