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Note on a tensor product of two holonomic systems with support on plane curves

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Kashiwara [5] proved that for any holonomic $\mathcal{D}_X$-Modules $\mathcal{M}_1$ and $\mathcal{M}_2$, the tensor product $\mathcal{M}_1 \otimes \mathcal{M}_2$ and $\mathcal{T}or_k^{\mathcal{O}_X}(\mathcal{M}_1, \mathcal{M}_2)$ become holonomic $\mathcal{D}_X$-Modules. We are interested here in calculating the tensor product of holonomic $\mathcal{D}_X$-Modules supported on plane curves.

In §1 we fix our notation and recall some properties of the tensor products of holonomic $\mathcal{D}_X$-Modules. In §2 we recall the projection formula for $\mathcal{D}_X$-Modules. In §3 as application of the projection formula associated to blow-up, we calculate the tensor product for a typical case.

§1. Tensor products of holonomic systems

Let $(X, \mathcal{O}_X)$ be a complex manifold. Let X_1 and X_2 be two copies of the complex manifold X. Let us denote by p_1 and p_2 the projection from

$X_1 \times X_2$ to X_1 and from $X_1 \times X_2$ to X_2 respectively.

For any two $\mathcal{D}_X$-Modules $\mathcal{M}_1$ and $\mathcal{M}_2$, we set

$$\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2 = \mathcal{D}_{X_1 \times X_2} \otimes_{p_1^{-1}\mathcal{D}_{X_1} \otimes p_2^{-1}\mathcal{D}_{X_2}} (p_1^{-1}\mathcal{M}_1 \otimes p_2^{-1}\mathcal{M}_2).$$

Here we regard $\mathcal{M}_1$ as $\mathcal{D}_{X_1}$-Module and $\mathcal{M}_2$ as $\mathcal{D}_{X_2}$-Module.

The $\mathcal{D}_{X_1 \times X_2}$-Module above is called the exterior tensor product of $\mathcal{M}_1$ and $\mathcal{M}_2$. We have the following quasi-isomorphism :

$$\mathcal{M}_1 \overset{L}{\otimes}_{\mathcal{O}_X} \mathcal{M}_2 = \mathcal{D}_{X \to X_1 \times X_2} \overset{L}{\otimes}_{\mathcal{D}_{X_1 \times X_2}} (\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2).$$

Kashiwara [5] proved the following theorem.

Theorem 1 (Kashiwara [5]).

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two (regular) holonomic $\mathcal{D}_X$-Modules. Then $\mathcal{M}_1 \otimes_{\mathcal{O}_X} \mathcal{M}_2$ and $\mathcal{T}or_k^{\mathcal{O}_X}(\mathcal{M}_1, \mathcal{M}_2)$ are (regular) holonomic systems.

Let X be a domain in C^2. Let F be a curve in X defined by a holomorphic function f : $F = \{(x, y) \in X \mid f(x, y) = 0\}$. Let us denote by $\mathcal{H}^1_{[F]}(\mathcal{O}_X)$ the sheaf of algebraic local cohomology with support in F. Refer to Grothendieck [2] for the notion of the sheaf of algebraic local cohomology. The sheaf $\mathcal{H}^1_{[F]}(\mathcal{O}_X)$ regarded as a left $\mathcal{D}_X$-Module is a regular holonomic system (see Kashiwara [5] and Mebkhout [6]).

It is known that if F is a analytically irreducible curve defined on a domain X in $\mathbf{C}^2$, then $\mathcal{H}^1_{[F]}(\mathcal{O}_X)$ is a simple $\mathcal{D}_X$-Module. (Refer to van Doorn and van den Essen [1] for the proof of this fact.) Hence if we set $\delta(f) = \dfrac{1}{f} \mod \mathcal{O}_X$, then $\delta(f)$ generates $\mathcal{H}^1_{[F]}(\mathcal{O}_X)$ as $\mathcal{D}_X$-Module.

We have the following result.

Proposition 2

Let F and G be analytically irreducible plane curves on X passing through a point P. Assume that F and G meet properly at P. Then we have

(i) $\mathcal{T}or_j^{\mathcal{O}_X}(\mathcal{H}^1_{[F]}(\mathcal{O}_X), \mathcal{H}^1_{[G]}(\mathcal{O}_X)) = 0 \qquad$ for $j \geqq 1$,

(ii) $\mathcal{H}^1_{[F]}(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{H}^1_{[G]}(\mathcal{O}_X)$ is isomorphic to the simple regular holonomic $\mathcal{D}_X$-Module $\mathcal{H}^2_{[P]}(\mathcal{O}_X)$.

Proof. Let us recall the definition of the sheaf of algebraic local cohomology :

$$\mathcal{H}^1_{[F]}(\mathcal{O}_X) = \varinjlim \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X/(f)^k, \mathcal{O}_X).$$

Since

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/(f)^k, \mathcal{O}_X) = \mathrm{Ker}\,(f^k : \mathcal{O}_X \longrightarrow \mathcal{O}_X) = 0,$$

$\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X/(f)^k, \mathcal{O}_X)$ is quasi-isomorphic to the complex

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\quad f^k \quad} \mathcal{O}_X \longrightarrow 0.$$

Thus the tensor product $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X/(f)^k, \mathcal{O}_X) \otimes \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X/(g)^{k'}, \mathcal{O}_X)$ is quasi-isomorphic to the tensor product of the following two complexes :

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\quad f^k \quad} \mathcal{O}_X \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \mathcal{O}_X \xrightarrow{\quad g^{k'} \quad} \mathcal{O}_X \longrightarrow 0.$$

Since f and g are coprime, we have

$$\mathcal{T}or^{\mathcal{O}_X}_j(\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X/(f)^k, \mathcal{O}_X), \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X/(g)^{k'}, \mathcal{O}_X)) = 0,$$

which implies the first assertion. The argument above also implies the second assertion.

Q.E.D.

Under the assumption of Proposition 2, we have the following result.

Corollary 3.

(i) $m=\delta(f)\otimes\delta(g)$ is a generator of the tensor product $\mathcal{H}^1_{[F]}(\mathcal{O}_X)\otimes\mathcal{H}^1_{[G]}(\mathcal{O}_X)$

(ii) the generator m is a linear combination of derivatives of Dirac's delta function.

Example 4 ([10]).

Set $F = \{(x, y) \in X \mid y = 0\}$, $G = \{(x, y) \in X \mid y - x^2 = 0\}$,

$$\delta(y) = \frac{1}{y} \mod \mathcal{O}_X, \quad \delta(y-x^2) = \frac{1}{y-x^2} \mod \mathcal{O}_X,$$

and

$$m = \delta(y) \otimes \delta(y-x^2) = 1_{X \to X_1 \times X_2} \otimes (\delta(y_1) \otimes \delta(y_2 - x_2^2)).$$

(Here $1_{X \longrightarrow X_1 \times X_2}$ is the canonical section of $\mathcal{D}_{X \longrightarrow X_1 \times X_2}$ associated with the diagonal embedding $X \longrightarrow X_1 \times X_2$.) Then we have

$$\mathcal{D}_X m = \mathcal{D}_X \delta(y) \otimes \mathcal{D}_X \delta(y-x^2)$$

$$= \mathcal{D}_X / \mathcal{D}_X x^2 + \mathcal{D}_X (x \frac{\partial}{\partial x} + 2) + \mathcal{D}_X y.$$

$$= \mathcal{D}_X (\frac{\partial}{\partial x} \delta(x, y)),$$

where $\delta(x, y)$ denotes Dirac's delta-function.

We set :

$X_1 = \{(x_1, y_1) \mid x_1, y_1 \in X\}$ and $X_2 = \{(x_2, y_2) \mid x_2, y_2 \in X\}$.

Since

$$\mathcal{D}_X \delta(y) = \mathcal{D}_X / \mathcal{D}_X \frac{\partial}{\partial x} + \mathcal{D}_X y \cong \mathcal{H}^1_{[F]}(\mathcal{O}_X),$$

$$\mathcal{D}_X \delta(y - x^2) = \mathcal{D}_X / \mathcal{D}_X (\frac{\partial}{\partial x} + 2x \frac{\partial}{\partial y}) + \mathcal{D}_X (y - x^2) \cong \mathcal{H}^1_{[G]}(\mathcal{O}_X),$$

we have

$$\mathcal{D}_{X_1\times X_2}(\delta(y_1)\widehat{\otimes}\delta(y_2-x_2^2))$$

$$= \mathcal{D}/\mathcal{D}\frac{\partial}{\partial x_1} + \mathcal{D}y_1 + \mathcal{D}(\frac{\partial}{\partial x_2} + 2x_2\frac{\partial}{\partial y_2}) + \mathcal{D}(y_2-x_2^2)$$

$$\cong \mathcal{H}^1_{[F]}(\mathcal{O}_X)\widehat{\otimes}\mathcal{H}^1_{[G]}(\mathcal{O}_X),$$

where we set $\mathcal{D} = \mathcal{D}_{X_1\times X_2}$. We put $\mathcal{L} = \mathcal{D}_{X_1\times X_2}(\delta(y_1)\widehat{\otimes}\delta(y_2-x_2^2))$.

We have the following equality :

$$\mathcal{D}_X m = \mathcal{L}/(x_1-x_2)\mathcal{L} + (y_1-y_2)\mathcal{L}.$$

Now we set : $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X x^2 + \mathcal{D}_X(x\frac{\partial}{\partial x} + 2) + \mathcal{D}_X y$.

Recall here the following identities :

$$x\,1_{X\to X_1\times X_2} = 1_{X\to X_1\times X_2}(x_1+x_2)/2, \quad y\,1_{X\to X_1\times X_2} = 1_{X\to X_1\times X_2}(y_1+y_2)/2,$$

$$\frac{\partial}{\partial x}1_{X\to X_1\times X_2} = 1_{X\to X_1\times X_2}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}), \text{ etc.}$$

We define a left $\mathcal{D}_X$ linear map

$$\Psi : \mathcal{M} \longrightarrow \mathcal{L}/(x_1-x_2)\mathcal{L} + (y_1-y_2)\mathcal{L}$$

by

$$\Psi(x) = (x_1+x_2)/2, \quad \Psi(\frac{\partial}{\partial x}) = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \text{ etc.}$$

Since

$$\Psi(x^2) = (x_1+x_2)^2/4 = x_2^2 + (x_1-x_2)(x_1+3x_2)/4,$$

$$\Psi(x\frac{\partial}{\partial x}+2) = \frac{1}{2}(x_1+x_2)(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}) + 2$$

$$= 2\frac{\partial}{\partial y_2} y_1 + \frac{1}{2}(x_1+x_2)(\frac{\partial}{\partial x_2} + 2x_2\frac{\partial}{\partial y_2}) + 2\frac{\partial}{\partial y_2}(y_2-x_2^2)$$

$$- (x_1-x_2)x_2\frac{\partial}{\partial y_2} + 2(y_1-y_2)\frac{\partial}{\partial y_2}$$

and

$$\Psi(y) = (y_1+y_2)/2 = y_1 - (y_1-y_2)/2,$$

Ψ is a well-defined map.

We thus get the following equalities :

$$\mathscr{D}_x m = \mathscr{L}/ (x_1-x_2)\mathscr{L} + (y_1-y_2)\mathscr{L}$$

$$= \mathscr{D}_x / \mathscr{D}_x x^2 + \mathscr{D}_x(x\frac{\partial}{\partial x}+2) + \mathscr{D}_x y.$$

Remark (cf. Passare [8]).

$$\frac{1}{y^2(y-x^2)} = (\frac{1}{6}\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y})m.$$

§2. Projection formula

In this section we recall the projection formula for the tensor products of $\mathcal{D}_X$-Modules.

Let X and Z be two complex manifolds. Let Φ be a proper holomorphic map from Z to X.

Theorem 5 (see [3]).

For any $\mathcal{D}_Z$-Module $\mathcal{M}$ and for any $\mathcal{D}_X$-Module $\mathcal{N}$, we have the following projection formula :

$$\left(\int_\Phi \mathcal{M}\right) \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{N} = \int_\Phi \left(\mathcal{M} \overset{L}{\underset{\mathcal{O}_Z}{\otimes}} L\Phi^* \mathcal{N}\right)$$

Proof.

Set : $\mathcal{L} = \mathcal{D}_{Z\times X} \otimes_{p^{-1}\mathcal{D}_Z \otimes q^{-1}\mathcal{D}_X} (p^{-1}\mathcal{M} \hat{\otimes} q^{-1}\mathcal{N})$

where $p : Z\times X \longrightarrow Z$ and $q : Z\times X \longrightarrow X$ are natural projections.
We have

$$\mathcal{M} \overset{L}{\otimes} L\Phi^* \mathcal{N} = \mathcal{D}_{Z \longrightarrow Z\times Z} \overset{L}{\otimes} (\mathcal{D}_{Z\times Z \longrightarrow Z\times X} \overset{L}{\otimes} \mathcal{L})$$

$$= \mathcal{D}_{Z \longrightarrow Z\times X} \overset{L}{\otimes} \mathcal{L},$$

and

$$\left(\int_\Phi \mathcal{M}\right) \otimes \mathcal{N} = R\Phi_* (\mathcal{D}_{X\times X \longleftarrow Z\times X} \overset{L}{\otimes} \mathcal{L}).$$

By the base change formula we have

$$\int_\Phi (\mathcal{M} \overset{L}{\otimes} L\Phi^* \mathcal{N})$$

$$= R\Phi_*((\mathcal{D}_{X \leftarrow Z} \overset{L}{\otimes} \mathcal{D}_{Z \rightarrow Z \times X}) \overset{L}{\otimes} \mathcal{L})$$

$$= \mathcal{D}_{X \rightarrow X \times X} \overset{L}{\otimes} (R\Phi_*(\mathcal{D}_{X \times X \leftarrow Z \times X}) \overset{L}{\otimes} \mathcal{L})$$

$$= \mathcal{D}_{X \rightarrow X \times X} \overset{L}{\otimes} ((\int_\Phi \mathcal{M}) \overset{L}{\otimes} \mathcal{N})$$

Example 6.

We calculate the tensor product $\mathcal{D}_X \delta(y) \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{D}_X \delta(y - x^2)$ by means of the projection formula.

Let $X = \mathbf{C}^2$, $Z = \{(x, y) \in X \mid y = 0\}$ and $i : Z \longrightarrow X$ be the natural embedding map. We put :

$$\mathcal{M} = \mathcal{D}_Z / \mathcal{D}_Z \frac{\partial}{\partial x} \cong \mathcal{O}_Z$$

$$\mathcal{N} = \mathcal{D}_X \delta(y - x^2) = \mathcal{D}_X / \mathcal{D}_X (\frac{\partial}{\partial x} + 2x \frac{\partial}{\partial y}) + \mathcal{D}_X (y - x^2)$$

Then we get

$$\int_i \mathcal{M} = \mathcal{D}_X / \mathcal{D}_X \frac{\partial}{\partial x} + \mathcal{D}_X y,$$

$$Li^* \mathcal{N} = i^* \mathcal{N} = \mathcal{D}_Z / \mathcal{D}_Z x^2 + \mathcal{D}_Z (x \frac{\partial}{\partial x} + 2).$$

Hence we have

$$\mathcal{D}_X\delta(y)\overset{L}{\underset{\mathcal{O}_X}{\otimes}}\mathcal{D}_X\delta(y-x^2) = (\int_i \mathcal{M})\overset{L}{\underset{\mathcal{O}_X}{\otimes}}\mathcal{N} = \int_i (\mathcal{M}\overset{L}{\underset{\mathcal{O}_Z}{\otimes}} Li^*\mathcal{N})$$

$$= \int_i i^*\mathcal{N} = \int_i \mathcal{D}_Z / \mathcal{D}_Z x^2 + \mathcal{D}_Z(x\frac{\partial}{\partial x}+2)$$

$$= \mathcal{D}_X / \mathcal{D}_X x^2 + \mathcal{D}_X(x\frac{\partial}{\partial x}+2) + \mathcal{D}_X y.$$

§3. An example

Let X be a domain in C^2 containing the origin P. Let $\pi : \tilde{X} \longrightarrow X$ be the blou-up of X with center at the origin P.

For any $\mathcal{D}_X$-Module $\mathcal{N}$, we define the total transform of $\mathcal{N}$ by

$$L\pi^*\mathcal{N} = \mathcal{D}_{\tilde{X}\to X}\overset{L}{\underset{\mathcal{D}_X}{\otimes}}\mathcal{N}.$$

We have the following result.

Proposition 7 ([7]).

Let F be a plane curve defined in X. Let $\tilde{F}$ be the total transform of F i.e. $\tilde{F} = \pi^{-1}(F)$. We have the following results.

(i) $\mathcal{D}_{\tilde{X}\to X} \otimes \mathcal{H}^1_{[F]}(\mathcal{O}_X) = \mathcal{H}^1_{[\tilde{F}]}(\mathcal{O}_{\tilde{X}})$,

(ii) $\mathcal{T}or_k^{\mathcal{D}_X}(\mathcal{D}_{\tilde{X}\to X}, \mathcal{H}^1_{[F]}(\mathcal{O}_X)) = 0$ for $k \geqq 1$.

$$= \mathcal{D}_{\tilde{X}} / (\mathcal{D}_{\tilde{X}}(v^2(u^2 - v)) + \mathcal{D}_{\tilde{X}}(u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v} + 6) + \mathcal{D}_{\tilde{X}}(u^2\frac{\partial}{\partial u} - v\frac{\partial}{\partial u} + 2u)).$$

If we set $\mathcal{M} = \mathcal{D}_{\tilde{X}}\delta(u) = \mathcal{D}_{\tilde{X}} / (\mathcal{D}_{\tilde{X}}u + \mathcal{D}_{\tilde{X}}\frac{\partial}{\partial v})$, then we have

$$\int_{\pi} \mathcal{M} = \mathcal{D}_X / (\mathcal{D}_X y + \mathcal{D}_X \frac{\partial}{\partial x}).$$

It is easy to verify that

$$\mathcal{M} \otimes_{\mathcal{O}_{\tilde{X}}} \pi^* \mathcal{N} = \mathcal{D}_{\tilde{X}} / (\mathcal{D}_{\tilde{X}}u + \mathcal{D}_{\tilde{X}}v^3 + \mathcal{D}_{\tilde{X}}(v\frac{\partial}{\partial v} + 3)),$$

and

$$\int_{\pi} \mathcal{D}_{\tilde{X}} / (\mathcal{D}_{\tilde{X}}u + \mathcal{D}_{\tilde{X}}v^3 + \mathcal{D}_{\tilde{X}}(v\frac{\partial}{\partial v} + 3)),$$

$$= \mathcal{D}_X / (\mathcal{D}_X x^3 + \mathcal{D}_X(x\frac{\partial}{\partial x} + 3) + \mathcal{D}_X y).$$

Therefore we have

$$\mathcal{D}_X \mathbf{m} = \mathcal{D}_X / (\mathcal{D}_X x^3 + \mathcal{D}_X (x\frac{\partial}{\partial x} + 3) + \mathcal{D}_X y)$$

$$= \mathcal{D}_X (\frac{\partial^2}{\partial x^2} \delta(x, y)),$$

where $\mathbf{m} = 1_{X \to X_1 \times X_2} \otimes (\delta(y_1) \widehat{\otimes} \delta(y_2^2 - x_2^3))$.

The blow-up $\tilde{X}$ has two coordinate patches $U = \{(u, v) \mid u, v \in C\}$ and $U' = \{(u', v') \mid u', v' \in C\}$ with

$$u' = \frac{1}{u} \quad \text{and} \quad v' = uv \qquad \text{on} \quad U \cap U' = \{(u, v) \mid u \neq 0\}.$$

The map $\pi : \tilde{X} \longrightarrow X$ is given in U by $\pi(u, v) = (v, uv)$ and the exceptional divisor E is given by $v = 0$. The map π is given in U' by $\pi(u', v') = (u'v', v')$ and the exceptional divisor E is given by $v' = 0$.

For example if we set $f(x, y) = y - x^2$ and $\tilde{f} = f \circ \pi$, then we have

$$\mathcal{D}_{\tilde{X} \to X} \otimes \mathcal{D}_X \delta(f) = \mathcal{D}_{\tilde{X}} \delta(\tilde{f})$$

$$= \mathcal{D}_{\tilde{X}} / \mathcal{D}_{\tilde{X}}(u-v)v + \mathcal{D}_{\tilde{X}}((u-v)\frac{\partial}{\partial u} + 1) + \mathcal{D}_{\tilde{X}}(v(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}) + 1)$$

on U.

Now let us calculate the anihilating ideal of $\delta(y) \otimes \delta(y^2 - x^3)$.

Set $F = \{(x, y) \in \mathbf{C}^2 \mid f(x, y) = y = 0\}$ and $G = \{(x, y) \in \mathbf{C}^2 \mid g(x, y) = y^2 - x^3 = 0\}$. Let $\mathcal{N}$ be the sheaf of algebraic local cohomology with support in the cusp G:

$$\mathcal{N} = \mathcal{D}_X \delta(g) \cong \mathcal{H}^1_{[G]}(\mathcal{O}_X)$$

Let (u, v) be local coordinates on $\tilde{X}$ which satisfy $(x, y) = \pi(u, v) = (v, uv)$. Since $\tilde{g} = g \circ \pi = v^2(u^2 - v)$, we have

$$\pi^* \mathcal{N} = \mathcal{D}_{\tilde{X}} \delta(\tilde{g})$$

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