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On Composite Game and Weak Balanced Set

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Abstract

In this paper, the definition of weak balanced set of a game is given and, then, based on the definition, the relationship between composite games and their cores is studied. Moreover, some necessary and sufficient conditions for the existence of core of a composite game are given.

1. Introduction

It has been well concerned whether or not we can get solutions of games with an n-person-cooperations. In [1], C.N.Bondreva and L.S.Shapley introduced the new concept of balanced set as one of such solutions in game theory. By using the concept, we are able to describe the core of the game. So, necessary and sufficient conditions for the existence of the core of the game were obtained; see [2] and [3] for details. Thus, in game theory, it is one of very important subjects to study the properties of balanced sets. Much works on this subject have been made; for example, G.Owen, in [3], investigated some important problems on balanced sets in terms of linear programming.

In this paper, the concept of weak balanced set is defined and, based on the definition, the core of a composite game is studied. Moreover, some necessary and sufficient conditions for the existence of core of the game are given.

2. Main results and their proofs

Let N be a finite player set, and $S_1, S_2, \ldots, S_p$ be p nonempty subsets of N. We introduce the following notations:

$$S = \{S_1, S_2, \ldots, S_p\}$$

and

$$E_i = \{j \in N; i \in S_j\}.$$

DEFINITION 1. If there are non-negative numbers $z_1, \ldots, z_p$ such that, for all $i \in N$,

$$\sum_{j \in E_i} z_j = 1,$$

we call S a weak balanced set, $z = (z_1, z_2, \ldots, z_p)$ its corresponding weak balanced vector, and z_j $(j = 1, 2, \ldots, p)$ its weak balanced coefficients.

DEFINITION 2. A weak balanced set is called the smallest weak balanced set, if it includes no other weak balanced set.

THEOREM 1. *The union of weak balanced sets is a weak balanced set.*

PROOF. Let $C = \{S_1, S_2, \ldots, S_m\}$ and $D = \{T_1, T_2, \ldots, T_k\}$ be two weak balanced sets, $(y_1, y_2, \ldots, y_m)$ and $(z_1, z_2, \ldots, z_k)$ be their weak balanced vectors, respectively. Also, let $C \cup D := \{R_1, R_2, \ldots, R_q\}$, where $q \leq m + k$ and we define, for $j = 1, 2, \ldots, q$

$$w_j(t) = \begin{cases} ty_l & \text{if } R_j = S_l \in C \setminus D, \\ (1-t)z_p & \text{if } R_j = T_p \in D \setminus C, \\ ty_l + (1-t)z_p & \text{if } R_j = S_l = T_p \in C \cap D, \end{cases}$$

where $0 \leq t \leq 1$.

It is easy to show that $W = (W_1(t), W_2(t), \ldots, W_q(t))$ is a weak balanced vector of $C \cup D$ for $0 \leq t \leq 1$. Thus, we can conclude that the union of two weak balanced sets is a weak balanced set. In view of the law of association of union of sets, the proof will be completed. □

A relationship between weak balanced sets and game cores, which is similar to one between balanced sets and game cores, is given by the following theorem.

THEOREM 2. *There exists a nonempty core of an n-person's game with the characteristic function V if and only if, for each smallest weak balanced set $S = \{S_1, S_2, \ldots, S_m\}$ and its weak balanced vector $y = (y_1, y_2, \ldots, y_m)$ of the game, the following inequality holds :*

$$\sum_{j=1}^{m} y_j V(S_j) \leq V(N),$$

where N is the player set of the game.

The theorem is proved by a similar argument to that of Bondreva's theorem in [1].

Next, in order to simplify the proof of Theorem 3, we give a lemma and a definition.

LEMMA 1. *Let M and N be two disjoint player sets, and let $S = \{S_1, S_2, \ldots, S_p\}$ be a weak balanced set of $M \cup N$, $r = (r_1, r_2, \ldots, r_p)$ its weak balanced vector. Then,*

$$S \cap M := \{S_1 \cap M, S_2 \cap M, \ldots, S_p \cap M\} = \{T_1, T_2, \ldots, T_m\}$$

and

$$S \cap N := \{S_1 \cap N, S_2 \cap N, \ldots, S_p \cap N\} = \{R_1, R_2, \ldots, R_q\}$$

are weak balanced sets of M *and* N*, respectively, and their corresponding weak balanced vectors are* $y = (y_1, y_2, \ldots, y_m)$ *and* $z = (z_1, z_2, \ldots, z_q)$*, respectively, where, for* $j = 1, \ldots, m$ *and* $j' = 1, \ldots, q$

$$y_j = \begin{cases} tr_l & \text{if } T_j = S_l \in M, \\ tr_i & \text{if } T_j \subset S_i \in M \cup N; \end{cases}$$

$$z_{j'} = \begin{cases} tr_l & \text{if } R_{j'} = S_l \in N, \\ tr_i & \text{if } R_{j'} \subset S_i \in M \cup N; \end{cases}$$

and $t \geq 0$*,* $m \leq p$*,* $q \leq p$*.*

PROOF. According to the definition of weak balanced set, we can choose a non-negative real number t, for which y and z are weak balanced vectors of $S \cap M$ and $S \cap N$, respectively. This completes the proof of the lemma. □

DEFINITION 3. Let U, V be characteristic functions of two games G_1, G_2, and M, N be their disjoint player sets, respectively. Then, a game G is called the composite game of games G_1 and G_2 if the player set of G is $M \cup N$ and its characteristic function is given as follows:

$$U \oplus V(S) := U(M \cap S) + V(N \cap S) \quad \text{for all } S \subset M \cup N,$$

see [2] and [3] for details.

THEOREM 3. *Let* M*,* N *be two disjoint player sets of the games* G_1*,* G_2*, and* U*,* V *be their characteristic functions. Then, there exists a nonempty core of the composite game* G *of* G_1 *and* G_2 *if and only if, for the smallest weak balanced set* $S = \{S_1, S_2, \ldots, S_p\}$ *of* G *and its weak balanced vector* $r = (r_1, r_2, \ldots, r_p)$*, the following inequality holds:*

$$\sum_{j=1}^{p} r_j U \oplus V(S_j) \leq U \oplus V(M \cup N).$$

PROOF. As used in Lemma 1, let sets T_i, R_i and scalars y_i, z_i be elements of weak balanced sets $S \cap M$, $S \cap N$ and elements of their weak balanced vectors y and z, respectively. From Lemma 1 and Theorem 1, it follows that

$$\begin{aligned} \sum_{j=1}^{p} r_j U \oplus V(S_j) &= \sum_{j=1}^{p} r_j \left[U(S_j \cap M) + V(S_j \cap N)\right] \\ &= \sum_{i=1}^{m} y_i U(T_i) + \sum_{i=1}^{q} z_i V(R_i) \\ &\leq U(M) + V(N) \\ &= U \oplus V(M \cup N). \end{aligned}$$

Thus, the proof is completed. □

DEFINITION 4. We call the smallest weak balanced set a proper set, if there are no common elements between any two sets in the balanced set.

DEFINITION 5. A game is called a proper game, if its characteristic function V has the following superadditivity: for all $S, T \subset N$ and $S \cap T = \emptyset$,

$$V(S \cup T) \geq V(S) + V(T).$$

LEMMA 2. *If two games G_1 and G_2 are proper games, then their composite game G is also proper.*

PROOF. Let M, N be disjoint player sets and U, V characteristic functions of the games G_1, G_2, respectively. For any $S, T \subset M \cup N$ with $S \cap T = \emptyset$, we have

$$\begin{aligned} U \oplus V(S \cup T) &= U[M \cap (S \cup T)] + V[N \cap (S \cup T)] \\ &= U[(M \cap S) \cup (M \cap T)] + V[(N \cap S) \cup (N \cap T)] \\ &\geq U(M \cap S) + U(M \cap T) + V(N \cap S) + V(N \cap T) \\ &= U \oplus V(S) + V \oplus V(T). \end{aligned}$$

This inequality shows that the characteristic function $U \oplus V$ of the composite game G satisfies superadditivity. This completes the proof of the lemma. □

Now, for a weak balanced set $S = \{S_1, S_2, \ldots, S_p\}$ and its weak balanced vector $r = (r_1, r_2, \ldots, r_p)$ of the game G with the characteristic function V, we introduce the following notation:

$$L(S, V) := \sum_{i=1}^{p} r_i V(S_i).$$

LEMMA 3. *Let N be a player set and V a characteristic function of a game G. If $S = \{S_1, S_2, \ldots, S_p\}$ is an improper smallest weak balanced set of G, then $L(S, V) \leq V(N)$ follows from $L(J, V) \leq V(N)$ and the superadditivity of V, where J is the smallest weak balanced set of G, and the disjoint pairs of elements of which are less than that of S.*

PROOF. The proof is given in [1]. □

From Lemma 3 and Theorem 3, we can easily prove the following theorem.

THEOREM 4. *Let M, N be two disjoint player sets of proper games G_1, G_2 and U, V be their characteristic functions, respectively. Then, there exists a nonempty core of the proper composite game G of G_1 and G_2 if and only if, for the proper smallest weak balanced set $S = \{S_1, S_2, \ldots, S_p\}$ and its weak balanced vector $r = (r_1, r_2, \ldots, r_p)$, the following inequality holds :*

$$\sum_{j=1}^{p} r_j U \oplus V(S_j) \leq U \oplus V(M \cup N).$$

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