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Characterization of generalized surfaces of revolution

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Abstract : We study the so-called generalized surfaces of revolution in a Euclidean space by considering normal sections.

1. Introduction

We define a generalized surface of revolution in (n+1)-dimensional Euclidean space E^{n+1} : Let C be a plane curve in E^{n+1}. A manifold of dimension n generated by revolving C around an axis is said to be a generalized surface of revolution in E^{n+1}. In the present paper, we characterize a generalized surface of revolution in E^{n+1}.

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2. Preliminaries

Let M = (M, x) be an n-dimensional submanifold in m-dimensional Euclidean space E^m, where x is an isometric immersion from M into E^m. Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections of M and E^m respectively. For any two vector fields X and Y tangent to M, the second fundamental form σ is given by $\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$. For a vector field ξ normal to M and X a vector field tangent to M, we may decompose $\tilde{\nabla}_X\xi$ as $\tilde{\nabla}_X\xi = -A_\xi X + \overset{\perp}{\nabla}_X\xi$, where $-A_\xi X$ and $\overset{\perp}{\nabla}_X\xi$ denote the tangential and normal components of $\tilde{\nabla}_X\xi$, respectively, and $\nabla^\perp$ is called the normal connection of the normal bundle $T^\perp M$. Let < , > be the scalar product of E^m. Then the Weingarten map A_ξ and the second fundamental form σ have the following relationship : $< A_\xi X, Y > = < \sigma(X, Y), \xi >$ for all vector fields X and Y tangent to M and every normal vector field ξ.

For the second fundamental form σ, we define a covariant derivative $\bar{\nabla}\sigma$ by

$$(2.1)\qquad (\bar{\nabla}_X\sigma)(Y, Z) = \overset{\perp}{\nabla}_X\sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$$

for vector fields X, Y and Z tangent to M. Let R be the curvature tensor of M. Then the structure equations of Gauss and Codazzi are given by

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(2.2) $$R(X, Y)Z = A_{\sigma(Y, Z)}X - A_{\sigma(X, Z)}Y,$$

(2.3) $$(\bar{\nabla}_X\sigma)(Y, Z) - (\bar{\nabla}_Y\sigma)(X, Z) = 0$$

for all vector fields X, Y and Z tangent to M.

For a point p in M and a unit vector t tangent to M at p, the vector t and the normal space $T_p^{\perp}M$ to M at p form an (m-n+1)-dimensional affine space E(p,t) in E^m through p. The intersection of M with E(p,t) gives rise to a curve in a neighborhood of p which is called the normal section of M at p in the direction of t (See [2], [3], [4] and [5] etc. for detail).

We shall recall the definition of isotropy in the sense of O'Neill [8]. A submanifold M of E^m is said to be isotropic at p if the length of second fundamental form $\| \sigma(v, v) \|$ is independent of the choice of the unit vector v in tangent space T_pM to M at p. If M is isotropic at every point, then M is said to be isotropic.

3. Submanifolds with geodesic normal sections through a point

Let M be a smooth hypersurface of an (n+1)-dimensional Euclidean space E^{n+1}. A smooth manifold means that every geometric object is assumed to be smooth. We now define a property (*).

(*) There is a point o in M such that every geodesic through o is a normal section.

We suppose that M satisfies (*). Without loss of generality we may assume that o is the origin of E^{n+1}. Let γ be a geodesic through o parametrized by arc length s and let $\gamma(0) = 0$. Then we have

$$\gamma'(s) = X, \ \gamma''(s) = \sigma (X, X), \ \gamma'''(s) = -A_{\sigma(X, X)}X + (\bar{\nabla}_X\sigma)(X, X) .$$

Since every geodesic through o is a normal section, $A_{\sigma(v, v)}v \wedge v = 0$, where $v = X(0)$, i.e.,

(3.1) $$< \sigma(v, v), \sigma(v, u) > = 0,$$

where u is a unit vector tangent to M at o orthogonal to v.

By O'Neill [8], M is isotropic at o. Thus we have proved the following.

Proposition 3.1. *Let* M *be a hypersurface in* E^{n+1} *satisfying* (*). *Then* M *is isotropic at* o.

Since γ is a plane curve, $\gamma(s) \wedge \gamma'(s) \wedge \gamma''(s) = 0$ along γ. Thus we get

$$X \wedge A_{\sigma(X,X)}X \wedge \sigma(X,X) = 0$$

along γ. This gives

$$(3.2) \qquad < \sigma(X,X), \sigma(X,Y) > = 0$$

along γ, where Y is a unit vector field tangent to M orthogonal to X along γ.

Since every geodesic through o is planar, we can write the immersion $x : M \to E^{n+1}$ locally on a neighborhood U of o with geodesic polar coordinate system $(s, y_1, y_2, ..., y_{n-1})$ as

$$(3.3) \qquad x(s, y_1, y_2, ..., y_{n-1}) = h(s, y_1, y_2, ..., y_{n-1})\, e(y_1, y_2, ..., y_{n-1}) + k(s, y_1, y_2, ..., y_{n-1})\, N,$$

where $e(y_1, y_2, ..., y_{n-1})$ is a vector tangent to M at o, h and k functions satisfying $h(0, y_1, y_2, ..., y_{n-1}) = k(0, y_1, y_2, ..., y_{n-1}) = 0$ and N a unique (up to sign) unit vector normal to M at o. We then have mutually orthogonal local tangent vector fields from (3.3)

$$(3.4) \quad x_*(\partial/\partial s) = \frac{\partial h}{\partial s}(s, y_1, y_2, ..., y_{n-1}) e(y_1, y_2, ..., y_{n-1}) + \frac{\partial k}{\partial s}(s, y_1, y_2, ..., y_{n-1})\, N,$$

$$(3.5) \quad x_*(\partial/\partial y_i) = \frac{\partial h}{\partial y_i}(s, y_1, y_2, ..., y_{n-1})\, e(y_1, y_2, ..., y_{n-1}) + h(s, y_1, y_2, ..., y_{n-1}) \frac{\partial e}{\partial y_i}(y_1, y_2, ..., y_{n-1}) + \frac{\partial k}{\partial y_i}(s, y_1, y_2, ..., y_{n-1})\, N$$

for $i = 1, 2, ..., n-1$, where $x_*(\partial/\partial s)(0, y_1, y_2, ..., y_{n-1}) = e(y_1, y_2, ..., y_{n-1})$. Since $< x_*(\partial/\partial s), x_*(\partial/\partial s) > = 1$, we see that

(3.6) $$\left(\frac{\partial h}{\partial s}\right)^2 + \left(\frac{\partial k}{\partial s}\right)^2 = 1,$$

from which, we put

(3.7) $$\frac{\partial h}{\partial s} = \cos f(s, y_1, y_2, \ldots, y_{n-1}), \quad \frac{\partial k}{\partial s} = \sin f(s, y_1, y_2, \ldots, y_{n-1}),$$

where f is a smooth function satisfying $f(0, y_1, y_2, \ldots, y_{n-1}) = 0$ for all $y_1, y_2, \ldots, y_{n-1}$.

Lemma 3.2. *Let* M *be a hypersurface in* E^{n+1} *satisfying* (*). *Then the curvature of geodesics through* o *is independent of the choice of geodesics .*

Proof. Let γ be a geodesic through o. Then $\gamma(s) = x(s, y_1, y_2, \ldots, y_{n-1})$ for some $y_1, y_2, \ldots, y_{n-1}$. The curvature κ is given by

(3.8) $$(\kappa(s, y_1, y_2, \ldots, y_{n-1}))^2 = < \sigma(X, X), \sigma(X, X) >,$$

where $\gamma'(s) = X$. Now we compute

$$\frac{1}{2} x_*\left(\frac{\partial}{\partial y_i}\right) (\kappa(s, y_1, y_2, \ldots, y_{n-1}))^2 = < \nabla^{\perp}_{\partial/\partial y_i} \sigma(\partial/\partial s, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s) >$$

$$= < (\bar{\nabla}_{\partial/\partial y_i}\sigma)(\partial/\partial s, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s) > + 2 < \sigma(\nabla_{\partial/\partial y_i}\partial/\partial s, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s) >$$

$$= < (\bar{\nabla}_{\partial/\partial s}\sigma)(\partial/\partial y_i, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s) > + 2 < \sigma(\nabla_{\partial/\partial y_i}\partial/\partial s, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s) >$$

(By Codazzi equation)

$$= < (\bar{\nabla}_{\partial/\partial s}\sigma)(\partial/\partial y_i, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s) > \quad \text{(By (3.2))}$$

$$= x_*\left(\frac{\partial}{\partial s}\right) < \sigma(\partial/\partial y_i, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s) > - < \sigma(\nabla_{\partial/\partial s}\partial/\partial y_i, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s) >$$

$$- < (\bar{\nabla}_{\partial/\partial s}\sigma)(\partial/\partial s, \partial/\partial s), \sigma(\partial/\partial y_i, \partial/\partial s) >$$

$$= - < (\bar{\nabla}_{\partial/\partial s}\sigma)(\partial/\partial s, \partial/\partial s), \sigma(\partial/\partial y_i, \partial/\partial s) > \text{(By (3.2))}$$

for all i = 1, 2, ..., n-1. Suppose that $<(\bar{\nabla}_{\partial/\partial s}\sigma)(\partial/\partial s, \partial/\partial s), \sigma(\partial/\partial y_i, \partial/\partial s)> \neq 0$ for some s_o in Dom γ, then $(\bar{\nabla}_{\partial/\partial s}\sigma)(\partial/\partial s, \partial/\partial s) \neq 0$ and $\sigma(\partial/\partial y_i, \partial/\partial s) \neq 0$ at $(s_o, y_1, y_2, ..., y_{n-1})$. Thus there exists an open interval J containing s_o such that $(\bar{\nabla}_{\partial/\partial s}\sigma)(\partial/\partial s, \partial/\partial s) \neq 0$ and $\sigma(\partial/\partial y_i, \partial/\partial s) \neq 0$ at $(s, y_1, y_2, ..., y_{n-1})$ for all $s \in J$. Hence $\sigma(\partial/\partial s, \partial/\partial s) = 0$ at $(s, y_1, y_2, ..., y_{n-1})$ for all $s \in J$ because of (3.2), which is a contradiction. Therefore, we get $<(\bar{\nabla}_{\partial/\partial s}\sigma)(\partial/\partial s, \partial/\partial s), \sigma(\partial/\partial y_i, \partial/\partial s)> = 0$ for any s in Dom γ. Thus we obtain

$$x_*(\frac{\partial}{\partial y_i})\,(\kappa(s, y_1, y_2, ..., y_{n-1}))^2 = 0$$

on a neighborhood of o. Hence the curvature of geodesics through o is independent of the choice of geodesics. (Q.E.D.).

Remark. Lemma 3.2 is different from Theorem 1 in [5] because (*) is satisfied at every point of the submanifold in Theorem 1 in [5] but we only assume that (*) is satisfied at a point in this paper. So the curvature of the geodesic through o is not constant in general.

Lemma 3.3. *The functions* h *and* k *are functions of* s *only on a neighborhood of o.*

Proof. Taking the covariant derivative of (3.4) along the geodesic itself and using (3.7), we obtain

$$(3.8) \qquad (\kappa(s, y_1, y_2, ..., y_{n-1}))^2 = (\frac{\partial f}{\partial s})^2.$$

(3.2) implies

$$(3.9) \qquad <\tilde{\nabla}_{x_*(\partial/\partial s)}\, x_*(\partial/\partial s),\ \tilde{\nabla}_{x_*(\partial/\partial y_i)}\, x_*(\partial/\partial s)> = 0,$$

which gives

$$(3.10) \qquad \frac{\partial f}{\partial s}\,\frac{\partial f}{\partial y_i} = 0 \quad \text{for all } i = 1, 2, ..., n-1.$$

From (3.8), we get

$$\frac{\partial f}{\partial s}(s, y_1, y_2, \dots, y_{n-1}) = \varepsilon\, \kappa(s, y_1, y_2, \dots, y_{n-1}),\ \varepsilon = \pm 1.$$

Thus, by Lemma 3.2, we see that $\frac{\partial f}{\partial s}(s, y_1, y_2, \dots, y_{n-1})$ is a function of s only, say, $\phi(s)$, for any $(s, y_1, y_2, \dots, y_{n-1})$ in the domain of x :

$$\frac{\partial f}{\partial s}(s, y_1, y_2, \dots, y_{n-1}) = \phi(s). \tag{3.11}$$

Integrating (3.11) with respect to s, we get

$$f(s, y_1, y_2, \dots, y_{n-1}) - f(0, y_1, y_2, \dots, y_{n-1}) = \int_0^s \phi(t)\,dt + q(y_1, y_2, \dots, y_{n-1})$$

for some function q of $y_1, y_2, \dots, y_{n-1}$ only. Since $f(0, y_1, y_2, \dots, y_{n-1}) = 0$, we see that $q(y_1, y_2, \dots, y_{n-1}) = 0$ and hence

$$f(s, y_1, y_2, \dots, y_{n-1}) = \int_0^s \phi(t)\,dt\,. \tag{3.12}$$

We denote by $\psi(s)$ the function given by the right hand side of the above equality (3.12). Integrating (3.7) and taking account of (3.12), we get

$$h(s, y_1, y_2, \dots, y_{n-1}) - h(0, y_1, y_2, \dots, y_{n-1}) = \int_0^s \cos\psi(t)\,dt + h_1(y_1, y_2, \dots, y_{n-1}),$$

$$k(s, y_1, y_2, \dots, y_{n-1}) - k(0, y_1, y_2, \dots, y_{n-1}) = \int_0^s \sin\psi(t)\,dt + k_1(y_1, y_2, \dots, y_{n-1})$$

for some functions h_1 and k_1 of $y_1, y_2, \dots, y_{n-1}$ only. Since $h(0, y_1, y_2, \dots, y_{n-1}) = 0$ and $k(0, y_1, y_2, \dots, y_{n-1}) = 0$, we see that $h_1(y_1, y_2, \dots, y_{n-1}) = 0$ and $k_1(y_1, y_2, \dots, y_{n-1}) = 0$ and hence h and k are functions of s only given by

$$h(s, y_1, y_2, \dots, y_{n-1}) = \int_0^s \cos\psi(t)\,dt\,,$$

$$k(s, y_1, y_2, \dots, y_{n-1}) = \int_0^s \sin\psi(t)\,dt. \qquad \text{(Q. E. D.).}$$

If a smooth hypersurface M of E^{n+1} satisfies (*), then we can conclude M is locally a generalized surface of revolution with vertex o by observing the equation (3.3), Lemma 3.2 and Lemma 3.3. The converse is trivial. Thus we have

Theorem 3.4. (Characterization). *Let* M *be a smooth hypersurface of* E^{n+1}. *Then* M *is locally a generalized surface of revolution with vertex* o *if and only if there is a point* o *such that every geodesic through* o *is a normal section* .

Corollary 3.5. *Let* M *be a complete connected smooth hypersurface of* E^{n+1}. *Then* M *is a generalized surface of revolution if and only if there is a point* o *such that every geodesic through* o *is a normal section* .

Corollary 3.6. *Let* M *be a smooth surface of* E^3. *Then* M *is a locally surface of revolution with vertex* o *if and only if there is a point* o *such that every geodesic through* o *is a normal section.*

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