Nihonkai Math. J.
Vol.2(1991), 47-61

On closed regular curves in Riemannian manifolds

Dedicated to Professor Masahisa Adachi on his 60th birthday

Kōjun ABE

§ 0. Introduction.

The purpose of this paper is to classify closed regular curves in Riemannian manifolds by an equivalence relation of some flows on the tubular neighborhoods of the curves. In [1] we considered a class of algebras of smooth sections of smooth vector bundles over Riemannian manifolds with bracket operations. In the case when the manifolds are one dimensional those algebras are closely related to the above flows. The classification is reduced to the problem to solve infinite series of ordinary differential equations which give the Taylor expansions of the equations of the flows. We shall present a method for computing them and calculate first five terms. In the case of symmetric spaces it is more simpler and interesting. Especially the case of Riemannian manifolds of constant curvature or projective spaces we shall see that those calculations are related to the total torsions or the fundamental forms of the projective spaces

([2],[3]).

The paper is organized as follows. In §1 we define an equivalence relation of closed regular curves and indicate a motivation of this paper. In §2 we give a method for computations of the equations of the flows for any Riemannian manifolds. In §3 we calculate the equations in the case of symmetric spaces. In §4 we classify the equivalence classes in the case of three dimensional Riemannian manifolds of constant curvature.

I would like to thank Professors K.Sekigawa, A.Asada and I.Yokota for their useful conversations and advices, and to thank H.Kamiya who helped me using computer.

§ 1. Preliminaries.

Let M be an n-dimensional Riemannian manifold with Riemannian connection ∇. Let $C = \{c(t)\}$ be a closed regular curve in M. In this paper we assume that any closed regular curve is parameterized by arc length. Let U_δ be a δ-tubular neighborhood of C, and $\pi: U \longrightarrow C$ be the natural projection. We define a vector field X on U_δ as follows. For $x \in U_\delta$ with $\pi(x) = c(t)$, X_x is parallel to $c'(t)$ along the geodesic joining x and c(t). Let $\mathfrak{F}_\delta$ be a flow on U_δ given by integral curves of X. Let $\Gamma(\mathfrak{F}_\delta)$ denote the Lie algebra of smooth vector fields on U_δ which are tangent to flows of $\mathfrak{F}_\delta$.

Definition 1.1. We say the above vector field X and the flow $\mathfrak{F}_\delta$ to be the parallel vector field and parallel flow to the curve C on U_δ respectively. Let C' be the other smooth regular closed curve and let $\mathfrak{F}'_\delta$ be the parallel flow to C' on a δ-tubular neighborhood U'_δ. We say that C and C' are equivalent if there exist a positive number δ and a diffeomorphism

$$\sigma: (U_\delta, \mathfrak{F}_\delta) \longrightarrow (U'_\delta, \mathfrak{F}'_\delta)$$

which is a flow map. Let φ_δ and φ'_δ be the Poincaré maps for the flows $\mathfrak{F}_\delta$ and $\mathfrak{F}'_\delta$ respectively (see Irwin [8], Chapter 2).

Proposition 1.2. (1) C and C' are equivalent if and only if the Poincaré maps φ_δ and φ'_δ are differentiably conjugate for some positive number δ.
(2) The Lie algebras $\Gamma(\mathfrak{F}_\delta)$ and $\Gamma(\mathfrak{F}'_\delta)$ are isomorphic for some positive number δ if and only if there exists a flow preserving diffeomorphism $\sigma: (U_\delta,\mathfrak{F}_\delta) \longrightarrow (U'_\delta,\mathfrak{F}'_\delta)$.

Proof. (1) follows from Irwin [8], (5.39) and (5.40). (2) follows from Amemiya [4].

Remark. The starting point of the problem here was Pursell-Shanks type theorem for the algebras $\Gamma(\mathfrak{F}_\delta)$. By Proposition 1.2, it is reduced to classify the equivalence classes of closed regular curves.

§ 2. Closed curves in Riemannian manifolds.

Let M be an n-dimensional Riemannian manifold with Riemannian connection ∇. Let $C = \{c(t)\}$ be a closed regular curve in M and X be the parallel vector field to C on a δ-tubular neighborhood U_δ. Let $\nu(C)$ denote the normal bundle of C in M. Fixed a unit normal vector $v_0 \in \nu_{c(0)}(C)$. Let $\pi\colon U_\delta \longrightarrow C$ be the projection. Let $\psi(t,s)$ $(-\infty<t<\infty,\ -\delta<s<\delta)$ be a family of integral curves of X such that

$$\psi(0,s) = \exp_{c(0)} s v_0 \quad \text{and} \quad \pi(\psi(t,s)) = c(t).$$

There exists a family of unit vector fields

$$Y(t,s) \in \nu_{c(t)}(C) \quad (-\infty<t<\infty,\ -\delta<s<\delta)$$

such that $\psi(t,s) = \exp_{c(t)} sY(t,s)$. Let $\varepsilon\colon (-\infty,\infty) \times (-\delta,\delta) \longrightarrow R$ be a C^∞ function such that

$$\frac{\partial\psi(t,s)}{\partial t} = \varepsilon(t,s)\, X(\psi(t,s)). \tag{2.1}$$

Let

$$Y(t,s) = \sum_{k=0}^{\infty} s^k\, Y_k(t)$$

and

$$\varepsilon(t,s) = \sum_{k=0}^{\infty} s^k\, \varepsilon_k(t)$$

be the the formal Taylor expansions of $Y(t,s)$ and $\varepsilon(t,s)$ respectively. Here $Y_k(t) \in \nu_{c(t)}(C)$ and $\varepsilon_k \in C^\infty(R)$. Put

$$\hat{\psi}(t,s,\lambda) = \exp_{c(t)} \lambda Y(t,s) \quad (-\infty<t<\infty,\ -\delta<s<\delta,\ -\delta<\lambda<\delta).$$

For each $v \in \nu_p(C)$ $(p \in C)$ let

$$(v^*)_q = \tau_{pq} v \qquad \text{for } q \in \pi^{-1}(p),$$

where τ_{pq} is the parallel translation along the geodesic which joins p and q (c.f. Helgason [7], Chapter I, § 6). Let $\hat{\nabla}$ be the induced connection by $\hat{\psi}$ (c.f. Cheeger Ebin [5], Chapt I, § 0).

We shall use the following notations:

$$\hat\psi_\lambda(t,s,\lambda) = \frac{\partial\hat\psi(t,s,\lambda)}{\partial\lambda}, \qquad \hat\psi_s(t,s,\lambda) = \frac{\partial\hat\psi(t,s,\lambda)}{\partial s},$$

$$\hat\nabla_\lambda = \hat\nabla_{\frac{\partial}{\partial\lambda}}, \qquad \hat\nabla_s = \hat\nabla_{\frac{\partial}{\partial s}}, \qquad \nabla_s = \hat\nabla_\lambda + \hat\nabla_s,$$

$$\hat\nabla^{i_1+j_1+i_2+j_2+\cdots}_{\lambda^{i_1}s^{j_1}\lambda^{i_2}s^{j_2}\cdots} = (\hat\nabla_\lambda)^{i_1}(\hat\nabla_s)^{j_1}(\hat\nabla_\lambda)^{i_2}(\hat\nabla_s)^{j_2}\cdots,$$

$$\psi_s(t,s) = \frac{\partial\psi(t,s)}{\partial s}, \qquad \psi_t(t,s) = \frac{\partial\psi(t,s)}{\partial t},$$

$$\nabla_t = \nabla_{\psi_t(t,s)},$$

$$C^j_{i_1\cdots i_k} = \frac{j!}{i_1!\cdots i_k!} \qquad (i_1+\cdots+i_k = j).$$

Lemma 2.1. Let X be a smooth vector field along $\hat\psi$. Then

(1) $$\hat\nabla^{n+1}_{\lambda s^n} X = \hat\nabla^{n+1}_{s^n\lambda} X + \sum_{j=0}^{n-1} \sum_{i_1+i_2+i_3+i_4=j} C^j_{i_1i_2i_3i_4} \cdot (\hat\nabla^{i_1}_{s^{i_1}}R)(\hat\nabla^{i_2}_{s^{i_2}}\hat\psi_\lambda, \hat\nabla^{i_3}_{s^{i_3}}\hat\psi_s)\,\hat\nabla^{n-1-j+i_4}_{s^{n-1-j+i_4}} X.$$

(2) $$\hat\nabla^{m+n}_{\lambda^m s^n} X = \hat\nabla^{m+n}_{s^n\lambda^m} X$$

$$+ \sum_{k=0}^{n-1} \sum_{i_1+i_2+i_3=k} \sum_{\ell=0}^{n-k-1} \sum_{p+q=m-1} \sum_{j_1+j_2+j_3+j_4=p} C^k_{i_1i_2i_3}\, C^{\ell+k}_{\ell\ k} \cdot C^p_{j_1j_2j_3j_4} (\hat\nabla^{i_1+j_1}_{\lambda^{j_1}s^{i_1}} R)(\hat\nabla^{i_2+j_2}_{\lambda^{j_2}s^{i_2}} \hat\psi_\lambda, \hat\nabla^{i_3+j_3}_{\lambda^{j_3}s^{i_3}} \hat\psi_s)\, \hat\nabla^{n-k-1+j_4+q}_{\lambda^{j_4} s^{n-k-1}\lambda^q} X.$$

Proof. (1) follows by easy calculations. Put

$$A_n(X) = \hat\nabla^{n+1}_{\lambda s^n} X - \hat\nabla^{n+1}_{s^n\lambda} X.$$

Then by induction we have

$$\hat\nabla^{m+n}_{\lambda^m s^n} X = \hat\nabla^{m+n}_{s^n\lambda^m} X + \sum_{p+q=m-1} \hat\nabla^p_{\lambda^p} A_n(\hat\nabla^q_{\lambda^q} X).$$

(2) follows from (1).

Let V be a smooth vector field on U_δ. Then V is regarded as a vector field along $\hat{\psi}$.

Lemma 2.2. Let X be a smooth vector field along ψ and let V be a smooth vector field on U_δ. Then

(1) $\left(\hat{\nabla}^{m+n+1}_{s^m \lambda s^n} X \right)_{(t,0,0)} = \left(\hat{\nabla}^{m+n+1}_{s^{m+n}\lambda} X \right)_{(t,0,0)}$.

(2) $\left(\hat{\nabla}^{m+n+2}_{s^m \lambda^2 s^n} V \right)_{(t,0,0)} = \left(\hat{\nabla}^{m+n+2}_{s^{m+n}\lambda^2} V \right)_{(t,0,0)}$

$+ \sum_{k=0}^{n-1} \sum_{i_1+i_2=k} C^k_{i_1 i_2} \left(\sum_{\ell=0}^{n-k-1} C^{k+\ell}_{k\ \ell} \right) \sum_{j_1+j_2+j_3=m} C^m_{j_1 j_2 j_3} (i_1+j_1)!$

$\cdot (i_2+j_2+1)!\ R(Y_{i_1+j_1}(t), Y_{i_2+j_2+1}(t)) \left(\hat{\nabla}^{n-k-1+j_3}_{s^{n-k-1+j_3}} V \right)_{(t,0,0)}$.

(3) $\left(\hat{\nabla}_{s^m \lambda^3 s^n} V \right)_{(t,0,0)} = \left(\hat{\nabla}^{m+n+3}_{s^{m+n}\lambda^3} V \right)_{(t,0,0)}$

$+ \sum_{k=0}^{n-1} \left(\sum_{\ell=0}^{n-k-1} C^{k+\ell}_{k\ \ell} \right) \Big[\sum_{\substack{i_1+i_2=k \\ j_1+j_2+j_3=m}} C^k_{i_1 i_2} C^m_{j_1 j_2 j_3} 3\cdot(i_1+j_1)!$

$\cdot (i_2+j_2+1)!\ R(Y_{i_1+j_1}(t), Y_{i_2+j_2+1}(t)) \left(\hat{\nabla}^{n-k+j_3}_{s^{n-k+j_3-1}\lambda} V \right)_{(t,0,0)}$

$+ \sum_{\substack{i_1+i_2+i_3=k \\ j_1+j_2+j_3+j_4=m}} C^k_{i_1 i_2 i_3} C^m_{j_1 j_2 j_3 j_4} 2\cdot(i_1+j_1)!(i_2+j_2)!(i_3+j_3+1)!$

$\cdot (\nabla_{Y_{i_1+j_1}(t)} R)(Y_{i_2+j_2}(t), Y_{i_3+j_3+1}(t)) \left(\hat{\nabla}^{n-k+j_4-1}_{s^{n-k+j_4-1}} V \right)_{(t,0,0)} \Big]$.

Proof. Since $\hat{\psi}_\lambda(t,s,\lambda) = Y(t,s)^* \phi(t,s,\lambda)$,

$$(\hat{\nabla}_{s^k} \hat{\psi}_\lambda)_{(t,0,0)} = k!\ Y_k(t) \qquad (k=1,2,\ldots).$$

Since $\hat{\nabla}_{s^k} \hat{\psi}_s$ is divisible by λ, $(\hat{\nabla}_{s^k} \hat{\psi}_s)_{(t,0,0)} = 0$ $(k=1, 2\ldots)$. If V_1, V_2 and V_3 are smooth vector fields on U_δ, then

$$\left((\hat{\nabla}^{k}_{s^k} R)(V_1,V_2)V_3 \right)_{(t,0,0)} = 0 \qquad (k=1,2,\ldots).$$

Combining those equations and Lemma 2.1 we can prove Lemma 2.2.

Let $\tau(M)$ denote the tangent bundle of M. Let $Z \in \tau_{c(t)}(M)$. Using Lemma 2.2 we have the following lemmas.

Lemma 2.3. (1) $\left(\nabla_s Z^* \right)_{(t,0,0)} = 0.$

(2) $\left(\nabla_s 2\, Z^* \right)_{(t,0,0)} = 0.$

(3) $\left(\nabla_s 3\, Z^* \right)_{(t,0,0)} = R(Y_0(t),Y_1(t))Z.$

(4) $\left(\nabla_s 4\, Z^* \right)_{(t,0,0)} = 6R(Y_0(t),Y_2(t))Z$
$+ (\nabla_{Y_0(t)}R)(Y_0(t),Y_1(t))Z.$

(5) $\left(\nabla_s 5\, Z^* \right)_{(t,0,0)} = 3(\nabla^2_{Y_0(t)}R)(Y_0(t),Y_1(t))Z$
$+ 8(\nabla_{Y_1(t)}R)(Y_0(t),Y_1(t))Z + 16(\nabla_{Y_0(t)}R)(Y_0(t),Y_2(t))Z$
$+ 12R(Y_1(t),Y_2(t))Z + 36R(Y_0(t),Y_3(t))Z$
$+ R(Y_0(t),R(Y_0(t),Y_1(t))Y_0(t))Z.$

Lemma 2.4. (1) $(\nabla_s \psi_t(t,s))_{(t,0,0)} = \nabla_t Y_0(t).$

(2) $(\nabla_s 2\, \psi_t(t,s))_{(t,0,0)} = 2\nabla_t Y_1(t) + R(Y_0(t),c'(t))Y_0(t).$

(3) $(\nabla_s 3\, \psi_t(t,s))_{(t,0,0)} = \Big[6\nabla_t Y_2 + 4R(Y_0,c')Y_1$
$+ 2R(Y_1,c')Y_0 + \varepsilon_1 R(Y_0,c')Y_0 + (\nabla_{Y_0}R)(Y_0,c')Y_0 \Big]_{c(t)}.$

(4) $(\nabla_s 4\, \psi_t(t,s))_{(t,0,0)} = \Big[24\nabla_t Y_3 + 2(\nabla_{c'}R)(Y_0,Y_1)Y_0$
$+ 4\varepsilon_2 R(Y_0,c')Y_0 + 18R(Y_0,c')Y_2 + 6(\nabla_{Y_0}R)(Y_0,c')Y_1 + 12R(Y_1,c')Y_1$
$+ 8\varepsilon_1 R(Y_0,c')Y_1 - 4(\nabla_{Y_0}R)(c',Y_1)Y_0 - (\nabla^2_{Y_0} R)(c',Y_0)Y_0$
$- 2\varepsilon_1(\nabla_{Y_0}R)(c',Y_0)Y_0 - 6R(c',Y_2)Y_0 + R(Y_0,R(c',Y_0)Y_0)Y_0 \Big]_{c(t)}.$

(5) $(\nabla_s 5\, \psi_t(t,s))_{(t,0,0)} = \Big[(\nabla^3_{Y_0} R)(Y_0,c')Y_0$
$+ 6(\nabla^2_{Y_0} R)(Y_1,c')Y_0 + 2(\nabla^2_{Y_0} R)(Y_0,c')Y_1 + 3\varepsilon_1(\nabla^2_{Y_0} R)(Y_0,c')Y_0$

$$+ 6(\nabla_{Y_0}^2 R)(Y_0,c')Y_1 + 4(\nabla_{c'}\nabla_{Y_0}R)(Y_0,Y_1)Y_0 - 10\varepsilon_2(\nabla_{Y_0}R)(c',Y_0)Y_0$$
$$+ 4\varepsilon_1(\nabla_{Y_0}R)(Y_0,Y_1)c' + 8\varepsilon_1(\nabla_{Y_0}R)(Y_1,c')Y_0 + 16\varepsilon_1(\nabla_{Y_0}R)(Y_0,c')Y_1$$
$$+ 36(\nabla_{Y_0}R)(Y_0,c')Y_2 + 18(\nabla_{Y_0}R)(Y_2,c')Y_0 + 32(\nabla_{Y_0}R)(Y_1,c')Y_1$$
$$+ 2(\nabla_{Y_0}R)(Y_0,R(c',Y_0)Y_0)Y_0 + 12(\nabla_{c'}R)(Y_0,Y_1)Y_1$$
$$+ 14(\nabla_{c'}R)(Y_0,Y_2)Y_0 + 7/3\ R(Y_0,(\nabla_{Y_0}R)(c',Y_0)Y_0)Y_0 + 120\ \nabla_t Y_4$$
$$+ 20\varepsilon_3 R(Y_0,c')Y_0 + 40\varepsilon_2 R(Y_0,c')Y_1 + 10\varepsilon_1 R(c',Y_2)Y_0$$
$$+ 50\varepsilon_1 R(Y_1,c')Y_2 + 20\varepsilon_1 R(Y_1,c')Y_1 + 7/3\ \varepsilon_1 R(Y_0,R(c',Y_0)Y_0)Y_0$$
$$+ 96R(Y_0,c')Y_3 + 24R(Y_3,c')Y_0 + 48R(Y_2,c')Y_1 + 72R(Y_1,c')Y_2$$
$$+ 8R(Y_0,R(c',Y_1)Y_0)Y_0 + 6R(Y_0,R(c',Y_0)Y_1)Y_0$$
$$+ 12R(Y_0,R(c',Y_0)Y_0)Y_1 - 6R(Y_1,R(c',Y_0)Y_0)Y_0$$
$$- 10R(c',R(Y_0,Y_1)Y_0)Y_0 + 8R(Y_0,R(Y_0,Y_1)Y_0)c'\Big]_{c(t)}.$$

From (2.1)

$$\Big[\ \nabla_s{}^k\ \psi_t(t,s)\ \Big]_{(t,0,0)} = \Big[\ \nabla_s{}^k\ (\varepsilon(t,s)c'(t)^*)\ \Big]_{(t,0,0)}$$

($k=0,1,2\ldots$). Combining Lemma 2.3 and Lemma 2.4 we can solve ε_k and Y_k inductively.

Theorem 2.5. (1) $\varepsilon_0(t) = 1$.

(2) $\nabla_t Y_0(t) = \varepsilon_1(t)c'(t)$.

(3) $2\nabla_t Y_1(t) + R(Y_0(t),c'(t))Y_0(t) = 2\varepsilon_2(t)c'(t)$.

(4) $\nabla_t Y_2(t) + 1/2\ R(Y_0(t),c'(t))Y_1(t)$
$+ 1/2\ R(Y_1(t),c'(t))Y_0(t) + 1/6\ \varepsilon_1(t)R(Y_0(t),c'(t))Y_0(t)$
$+ 1/6\ (\nabla_{Y_0(t)}R)(Y_0(t),c'(t))Y_0(t) = \varepsilon_3(t)c'(t)$.

(5) $\Big[\ 24\nabla_t Y_3 + 2(\nabla_{c'}R)(Y_0,Y_1)Y_0 + 6(\nabla_{Y_0}R)(Y_0,c')Y_1$
$- (\nabla_{Y_0}^2 R)(c',Y_0)Y_0 - 2\varepsilon_1(\nabla_{Y_0}R)(c',Y_0)Y_0 - 2(\nabla_{Y_0}R)(Y_0,Y_1)c'$

$$- 4(\nabla_{Y_0}c')(c',Y_1)Y_0 + 4\varepsilon_2 R(Y_0,c')Y_0 + 12R(Y_0,c')Y_2$$
$$+ 12R(Y_1,c')Y_1 + 4\varepsilon_1 R(Y_0,c')Y_1 + 4\varepsilon_1 R(Y_1,c')Y_0 - 12R(c',Y_2)Y_0$$
$$+ R(Y_0,R(c',Y_0)Y_0)Y_0 \Big]_{c(t)} = 24\varepsilon_4(t)c'(t).$$

(6) $$\Big[120\nabla_t Y_4 + (\nabla^3_{Y_0} R)(Y_0,c')Y_0 + 6(\nabla^2_{Y_0} R)(Y_1,c')Y_0$$
$$+ 2(\nabla^2_{Y_0} R)(Y_0,c')Y_1 + 6\varepsilon_2(\nabla^2_{Y_0} R)(Y_0,c')Y_0 + 18(\nabla^2_{Y_0} R)(Y_0,c')Y_2$$
$$- 3(\nabla^2_{Y_0} R)(Y_0,Y_1)c' - 8(\nabla_{Y_1} R)(Y_0,Y_1)C' + 4(\nabla_{c'},\nabla_{Y_0} R)(Y_0,Y_1)Y_0$$
$$+ 10\varepsilon_2(\nabla_{Y_0} R)(Y_0,c')Y_0 + 4\varepsilon_1(\nabla_{Y_0} R)(Y_1,c')Y_0 + 20\varepsilon_1(\nabla_{Y_0} R)(Y_0,c')Y_1$$
$$+ 36(\nabla_{Y_0} R)(Y_0,c')Y_2 + 18(\nabla_{Y_0} R)(Y_2,c')Y_0 + 32(\nabla_{Y_0} R)(Y_1,c')Y_1$$
$$- 10(\nabla_{Y_0} R)(Y_0,Y_1)c' - 16(\nabla_{Y_0} R)(Y_0,Y_2)c'$$
$$- 2(\nabla_{Y_0} R)(Y_0,(Y_0,c')Y_0)Y_0 + 12(\nabla_{c'},R)(Y_0,Y_1)Y_1$$
$$+ 14(\nabla_{c'},R)(Y_0,Y_2)Y_0 - 7/3\ R(Y_0,(\nabla_{Y_0} R)(Y_0,c')Y_0)Y_0$$
$$+ 20\varepsilon_3 R(Y_0,c')Y_0 + 20\varepsilon_2 R(Y_0,c')Y_1 + 20\varepsilon_2 R(Y_1,c')Y_0$$
$$+ 20\varepsilon_1 R(Y_2,c')Y_0 + 20\varepsilon_1 R(Y_0,c')Y_2 + 20\varepsilon_1 R(Y_1,c')Y_1$$
$$- 7/3\ \varepsilon_1 R(Y_0,R(Y_0,c')Y_0)Y_0 + 60R(Y_0,c')Y_3 + 60R(Y_3,c')Y_0$$
$$+ 60R(Y_2,c')Y_1 + 60R(Y_1,c')Y_2 - 8R(Y_0,R(Y_1,c')Y_0)Y_0$$
$$-6R(Y_0,R(Y_0,c')Y_1)Y_0 - 12R(Y_0,R(Y_0,c')Y_0)Y_1$$
$$+ 6R(Y_1,R(Y_0,c')Y_0)Y_0 - 10R(c',R(Y_0,Y_1)Y_0)Y_0$$
$$+ 7R(Y_0,R(Y_0,Y_1)Y_0)c' \Big]_{c(t)} = 120\varepsilon_5(t)c'(t).$$

§ 3. Symmetric spaces.

In this section we consider the equation (2.1) in the case of symmetric spaces.

Let M be a symmetric space. Then there exists a symmetric pair (G,H,σ) such that $M = G/H$ and σ is an involutive automorphism of G satisfying $(G_\sigma)_0 \subset H \subset G_\sigma$, where G_σ denote the closed subgroup of G consisting of all the elements fixed by σ. Let $q: G \longrightarrow G/H$ be the natural projection. We have a symmetric algebra $(\mathfrak{g},\mathfrak{h},\sigma)$ and a canonical decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ of (G,H,σ). Let $C = \{c(t)\}$ be a closed regular curve in M. There exists a horizontal lift $\hat{C} = \{\hat{c}(t)\}$ of C. Let L_g and R_g be a left and a right translation of G by g, respectively. We use the same notations as in §2. Put

$$w(t) = (dL_{\hat{c}(t)^{-1}})_{\hat{c}(t)}\ \hat{c}'(t)\ ,$$

$$Z(t,s) = (dL_{\hat{c}(t)^{-1}})_{\hat{c}(t)}\ Y(t,s).$$

$$\hat{\psi}(t,s) = \hat{c}(t)\ \exp sZ(t,s).$$

Here we identify the tangent space $\tau_0(G/H)$ of G/H at 0=1H with $\mathfrak{m}$. Then $q(\hat{\psi}(t,s)) = \psi(t,s)$. By Helgason [7], Chapter IV, Theorem 3.3,

$$((dq)_e\ w(t))^*_{\exp sZ(t,s)\cdot 0} = (d\tau_{\exp sZ(t,s)})_0\ (dq)_e\ w(t).$$

Hence by(2.1) we have

$$(3.1)\quad (dq)_{\hat{\psi}(t,s)} \frac{\partial\hat{\psi}(t,s)}{\partial t} = \varepsilon(t,s)\ (d\tau_{\exp sZ(t,s)})_0\ (dq)_e\ w(t).$$

From Helgason [7], Chapter II, Theorem 1.7 we have

$$\frac{\partial \exp sZ(t,s)}{\partial t} = (dL_{\exp sZ(t,s)})_e \frac{1 - \exp(-\mathrm{ad}\ sZ(t,s))}{\mathrm{ad}\ sZ(t,s)} s \frac{\partial Z(t,s)}{\partial t}.$$

Note that

$$(dR_{\exp sZ(t,s)})_e\ w(t) = (dL)_{\hat{\psi}(t,s)}\ (\exp \mathrm{ad}(-sZ(t,s)))w(t).$$

Therefore from (3.1) we have the following

Theorem 3.1.

$$(dq)_e\Big[(\exp \mathrm{ad}(-sZ(t,s)))w(t) + s\, \frac{1 - \exp(-\mathrm{ad}\; sZ(t,s))}{\mathrm{ad}\; sZ(t,s)}\, \frac{\partial Z(t,s)}{\partial t} \Big]$$
$$= \quad (dq)_e\; \varepsilon(t,s)w(t).$$

Let $Z(t,s) = \Sigma_{k=0}^{\infty} s^k Z_k(t)$ be the formal Taylor expansion of $Z(t,s)$. Combining the relations

$$[\,\mathcal{H},\ \mathcal{M}\,] \subset \mathcal{M}, \qquad [\,\mathcal{M},\ \mathcal{M}\,] \subset \mathcal{H}$$

and Theorem 3.1 we obtain the following

Theorem 3.2. (1) $Z_0'(t) = \varepsilon_1(t)w(t)$.

(2) $2Z_1'(t) + [Z_0(t),[w(t),Z_0(t)]] \;=\; 2\varepsilon_2(t)w(t)$.

(3) $6Z_2'(t) + 3[Z_0(t),[Z_1(t),w(t)]] + 3[Z_1(t),[Z_0(t),w(t)]]$
$\quad + \varepsilon_1(t)[Z_0(t),[Z_0(t),w(t)]] \;=\; \varepsilon_3(t)w(t)$.

(4) $\Big[\, 24Z_3' + 12[Z_0,[Z_2,w]] + 12[Z_1,[Z_1,w]] + 12[Z_2,[Z_0,w]]$
$\quad - (\mathrm{ad}\ Z_0)^4 w + 4\varepsilon_1[Z_0,[Z_1,w]] + 4\varepsilon_1[Z_1,[Z_0,w]]$
$\quad + \varepsilon_2[Z_0,[Z_0,w]]\,\Big]_{c(t)} \;=\; \varepsilon_4(t)w(t)$.

(5) $\Big[\, 120Z_4' + 20\varepsilon_3[Z_0,[Z_0,w]] + 20\varepsilon_2[Z_1,[Z_0,w]]$
$\quad + 20\varepsilon_2[Z_0,[Z_1,w]] + 20\varepsilon_1[Z_2,[Z_0,w]] + 20\varepsilon_1[Z_0,[Z_2,w]]$
$\quad + 20\varepsilon_1[Z_1,[Z_1,w]] - 7/3\ \varepsilon_1(\mathrm{ad}\ Z_0)^4 w + 60[Z_0,[Z_3,w]]$
$\quad + 60[Z_3,[Z_0,w]] + 60[Z_1,[Z_2,w]] + 60[Z_2,[Z_1,w]]$
$\quad - 5(\mathrm{ad}\ Z_0)^3(\mathrm{ad}\ Z_1)w - 5(\mathrm{ad}\ Z_1)(\mathrm{ad} Z_0)^3 w$
$\quad - 5(\mathrm{ad}\ Z_0)(\mathrm{ad}\ Z_1)(\mathrm{ad}\ Z_0)^2 w - 5(\mathrm{ad}\ Z_0)^2(\mathrm{ad}\ Z_1)(\mathrm{ad}\ Z_0)w\,\Big]_{c(t)}$
$\quad = \;120\varepsilon_5(t)w(t)$.

Now consider the case that C is a geodesic in an m-dimensional symmetric space G/H. Let $\{x_1, \ldots, x_m\}$ be a Fermi coordinates along the geodesic C such that $\left(\frac{\partial}{\partial x_1}\right)_{c(t)} = c'(t)$. Let $\langle , \rangle$ be a G-invariant Riemannian metric on G/H. Since $\langle Y(t,s), c'(t)\rangle = 0$, $Y_0(t)$ is a parallel vector field along C. Thus we can assume that $\left(\frac{\partial}{\partial x_2}\right)_{c(t)} = Y_0(t)$. Put

$$e_i(t) = \left(\frac{\partial}{\partial x_i}\right)_{c(t)} \quad (i = 1, \ldots, m),$$

$$R_{ijk\ell} = \langle R(e_i(0), e_j(0))e_k(0),\ e_\ell(0) \rangle \quad (i,j,k,\ell = 1, \ldots, m).$$

From Theorem 3.2 we have the following

Theorem 3.3. (1) $\varepsilon_1(t) = 0$,

$$Y_1(t) = -\ t/2\ \Sigma_{i=2}^{m}\ R_{212i}.$$

(2) $\varepsilon_2(t) = 1/2\ R_{2121}$,

$$Y_2(t) = t^2/8\ \Sigma_{i,j=2}^{m}\ (\ R_{212j}R_{21ji} + R_{211j}R_{j12i}\ )e_i(t).$$

(3) $\varepsilon_3(t) = t/2\ \Sigma_{i=3}^{m}\ R_{122i}R_{121i}$,

$$\begin{aligned} Y_3(t) = -1/24\ \Sigma_{k=2}^{m}\Big[& 2tR_{2121}R_{212k} + t^3/2\ \Sigma_{i,j=2}^{m}(\ R_{212j}R_{21ji} \\ & + R_{211j}R_{j12i}\ )R_{21ik} + t^3\Sigma_{i,j=2}^{m}\ R_{212j}R_{212i}R_{j1ik} \\ & - t^3/2\ \Sigma_{i,j=2}^{m}(\ R_{212j}R_{21ji} + R_{211j}R_{j12i}\ )R_{1i2k} \\ & + t\Sigma_{i=2}^{m}\ R_{122i}R_{2i2k}\ \Big]\ e_k(t). \end{aligned}$$

(4)
$$\begin{aligned} \varepsilon_4(t) = & 1/12\ R_{2121}^{\ 2} + 1/8\ t^2\Sigma_{i,j=2}^{m}(\ R_{212j}R_{21ji} \\ & + R_{211j}R_{j12i}\ )R_{21i1} + 1/8\ t^2\ \Sigma_{i,j=2}^{m}\ R_{212j}R_{212i}R_{j1i1} \\ & - 1/24\ \Sigma_{i=2}^{m}\ R_{122i}^{\ 2}. \end{aligned}$$

§ 4. Total torsions.

In this section we consider in the case that M is a Riemannian manifold of constant curvature.

Theorem 4.1. Let M be an m-dimensional Riemannian manifold of constant curvature K. Then Y does not depend on s and is determined by the following equations.

(1) If $K=0$, then $$c'(t) + s\,\nabla_t Y(t) = \varepsilon(t,s)\,c'(t).$$

(2) If $K>0$, then

$$(\cos s/r)\,c'(t) + (r \sin s/r)\,\nabla_t Y(t) = \varepsilon(t,s)\,c'(t) \quad (r = \sqrt{K}\,).$$

(3) If $K<0$, then

$$(\cosh s/r)\,c'(t) + (r \sinh s/r)\,\nabla_t Y(t) = \varepsilon(t,s)\,c'(t) \quad (r = \sqrt{-K}).$$

Proof. It is clear that (1) follows from Theorem 3.1. Assume that $K>0$. Since

$$\langle c'(t), Y(t,s)\rangle = 0 \text{ and } \langle Y(t,s), \nabla_t Y(t,s)\rangle = 0,$$

it follows from Wolf [9], Chapter 2, §3 that

$$R(Y(t,s),c'(t))Y(t,s) = -\,1/r^2\,c'(t),$$

$$R(Y(t,s),\nabla_t Y(t,s))Y(t,s) = -\,1/r^2\,\nabla_t Y(t,s).$$

By Theorem 3.1 we have

$$(\cos s/r)\,c'(t) + (r \sin s/r)\,\nabla_t Y(t,s) = \varepsilon(t,s)\,c'(t).$$

Since $\langle c'(t), Y(t,s)\rangle = 0$, Y does not depend on s. Then (2) follows. Similarly we can prove (3).

Now assume that dim M = 3 and M is orientable. Let C(M) be the set of regular curves in M with C^2 topology and let F(M) denote the set of regular curves with nonzero curvature κ. Let $C = \{c(t)\} \in F(M)$ and $\{e_1, e_2, e_3\}$ be the Frenet frame of C. Let $\gamma_i(t)$ $(-\infty<t<\infty,\ i=2,3)$ be smooth functions such that

$$Y(t) = \gamma_2(t)e_2(t)+\gamma_3(t)e_3(t).$$

Combining Theorem 4.1 and Frenet-Serret equations we have

$$\begin{pmatrix} \gamma_2'(t) \\ \gamma_3'(t) \end{pmatrix} = \begin{pmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{pmatrix} \begin{pmatrix} \gamma_2(t) \\ \gamma_3(t) \end{pmatrix}. \tag{4.1}$$

Here ω is the torsion of C. Put $\xi(t) = \int_0^t \omega(t)\,dt$. Then the differential equation (4.1) has the following fundamental matrix

$$\Phi(t) = \begin{pmatrix} \cos \xi(t) & -\sin \xi(t) \\ \sin \xi(t) & \cos \xi(t) \end{pmatrix}.$$

Let $\tau(C)$ denote the total torsion of C. Put

$$\hat{\tau}(C) = \tau(C)/2\pi \bmod \mathbb{Z} \in S^1 = \mathbb{R}/\mathbb{Z}.$$

From Proposition 1.2, we have the following.

Lemma 4.2. If $C_i \in F(M)$ $(i=1,2)$, then C_1 and C_2 are equivalent if and only if $\hat{\tau}(C_1) = \hat{\tau}(C_2)$.

Let $C \in C(M)$. By Feldman [6], Theorem 6.2, we can take a sequence $C_i \in F(M)$ $(i=1,2,...)$ such that $\lim_{i\to\infty} C_i = C$. Put

$$\hat{\tau}(C) = \lim_{i\to\infty} \hat{\tau}(C_i).$$

By [2], Propopsition 4.1 $\hat{\tau}$ is well defined. Let $\bar{C}(M)$ be the set of equivalence classes of C(M). Then $\hat{\tau}$ induces a map $\bar{\tau}$: $\bar{C}(M) \longrightarrow S^1$.

Theorem 4.3. If M is an orientable three dimensional Riemannian manifolds of constant curvature. Then $\bar{\tau}$ is bijective.

References

[1] Abe K., On G-vector bundles with bracket operations and an algebra with universal mapping property, preprint.

[2] ______ , On the total torsion and a generic property of closed regular curves in Riemannian manifolds, preprint.

[3] ______ , Regular curves and the fundamental forms of projective spaces, in preparation.

[4] Amemiya, I., Lie algebra of vector fields and complex strcture, Jour. Math. Soc. Japan, 27, (1975), 545-549.

[5] Cheeger, J. and Ebin, D., Comparison Theorems in Riemannian Geometry, North-Holland (1975).

[6] Feldman, E., Geometry of immersions. I, Trans. Amer. Math. Soc., 120, (1965), 185-224.

[7] Helgason, S., Differential Geometry and Symmetric Spaces, Academic Press, (1962).

[8] Irwin, M., Smooth Dynamical Systems, Academic Press, (1980).

[9] Wolf, J., Spaces of Constant Curvature, McGraw-Hill, New York, (1967).

Faculty of Liberal Arts, Shinshu University,
Matsumoto, 390, Japan.

Received December 4, 1990