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When does the equality of a generalized Selberg inequality hold?

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ABSTRACT. The original Selberg inequality is very useful result in the prime number theory. We shall give an extended Selberg type inequality and also we shall scrutinize the conditions under which the equality of this inequality holds.

1. Statement of the results. An operator means a bounded linear operator on a Hilbert space and also N(S) means the kernel of an operator S.

Theorem 1. If $x_1, x_2, \ldots, x_n$ and x are nonzero vectors in a Hilbert space H, then

$$(I_1) \qquad \sum_{i=1}^{n} \frac{|(x, x_i)|^2}{\sum_{j=1}^{n} |(x_i, x_j)|} \le \|x\|^2 .$$

The equality in (I_1) holds iff $x = \sum_{i=1}^{n} a_i x_i$ for some complex scalars $a_1, a_2, \ldots, a_n$ such that for arbitrary $i \neq j$, $(x_i, x_j) = 0$ or $|a_i| = |a_j|$ with $(a_i x_i, a_j x_j) \ge 0$.

Theorem 2. For any operator T on a Hilbert space H and nonzero vectors $x_1, x_2, \ldots, x_n \notin N(T^*)$

$$(I_2) \qquad \sum_{i=1}^{n} \frac{|(Tx, x_i)|^2}{\sum_{j=1}^{n} |(|T^*|^{2(1-\alpha)} x_i, x_j)|} \le (|T|^{2\alpha} x, x)$$

holds for any vector $x \notin N(T)$ and for any real number α with $0 \le \alpha \le 1$.

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(i) $0<\alpha<1$. The equality in (I_2) holds iff $Tx = \sum_{i=1}^{n} a_i |T^*|^{2(1-\alpha)} x_i$ (equivalently $|T|^{2\alpha}x = \sum_{i=1}^{n} a_i T^* x_i$) for some complex scalars $a_1, a_2, \ldots, a_n$ such that for arbitrary $i \neq j$, $(|T^*|^{2(1-\alpha)} x_i, x_j) = 0$ or $|a_i| = |a_j|$ with $(a_i |T^*|^{2(1-\alpha)} x_i, a_j x_j) \geq 0$.

(ii) $\alpha = 1$. The equality in (I_2) holds iff $Tx = \sum_{i=1}^{n} a_i x_i$ for some complex scalars $a_1, a_2, \ldots, a_n$ such that for arbitrary $i \neq j$, $(x_i, x_j) = 0$ or $|a_i| = |a_j|$ with $(a_i x_i, a_j x_j) \geq 0$.

(iii) $\alpha = 0$. The equality in (I_2) holds iff $x = \sum_{i=1}^{n} a_i T^* x_i$ for some complex scalars $a_1, a_2, \ldots, a_n$ such that for arbitrary $i \neq j$, $(T^* x_i, T^* x_j) = 0$ or $|a_i| = |a_j|$ with $(a_i T^* x_i, a_j T^* x_j) \geq 0$.

Remark 1. In Theorem 1, the following (C_1) and (C_2) are sufficient conditions for the equality in (I_1);

(C_1) $x = \sum_{i=1}^{n} a_i x_i$ for some complex scalars $a_1, a_2, \ldots, a_n$ such that $(x_i, x_j) = 0$ for all $i \neq j$,

(C_2) $x = \sum_{i=1}^{n} a_i x_i$ for some complex scalars $a_1, a_2, \ldots, a_n$ such that $|a_i|$ is a constant for all i and $(a_i x_i, a_j x_j) \geq 0$ for all i and j.

It is easily seen that (C_1) and (C_2) are not always necessary conditions for the equality in (I_1) as follows. Take x_1, x_2 and x_3 such that $x_1 = (1,0,0)$, $x_2 = (0,1,0)$ and $x_3 = (1,0,1)$. Put $x = 1 \cdot x_1 + 2 \cdot X_2 + 1 \cdot x_3$. This case is neither (C_1) nor (C_2), but the equality in (I_1) surely holds. So to speak, the necessary and sufficient condition for the equality in (I_1) of Theorem 1 is "mixed type" of (C_1) and (C_2).

2. Proofs of results.

Proof of Theorem 1. (I_1) in Theorem 1 has shown by Selberg [1,§2, p.14] and recently by K. Kubo and F. Kubo [3] using diagonal matrix which dominates a positive semidefinite matrix. Here we scrutinize the conditions under which the equality in (I_1) holds.

$$0 \leq \|x - \sum_{i=1}^{n} a_i x_i\|^2 = \|x\|^2 - 2\mathrm{Re}\sum_{i=1}^{n} \overline{a_i}(x,x_i) + \sum_{i,j}^{n} a_i\overline{a_j}(x_i,x_j)$$

$$\leq \|x\|^2 - 2\mathrm{Re}\sum_{i=1}^{n} \overline{a_i}(x,x_i) + 1/2\sum_{i,j}^{n}(|a_i|^2 + |a_j|^2)|(x_i,x_j)|$$

$$= \|x\|^2 - 2\mathrm{Re}\sum_{i=1}^{n} \overline{a_i}(x,x_i) + \sum_{i=1}^{n}\{|a_i|^2\sum_{j=1}^{n}|(x_i,x_j)|\} ,$$

then we put $a_i = (x,x_i)/\sum_{j=1}^{n}|(x_i,x_j)|$, so we have the desired result (I_1). The equality in (I_1) holds iff the following (1) and (2),

(1) $$x = \sum_{i=1}^{n} a_i x_i$$

(2) $$\sum_{i,j}^{n} a_i\overline{a_j}(x_i,x_j) = 1/2\sum_{i,j}^{n}(|a_i|^2 + |a_j|^2)|(x_i,x_j)|.$$

The condition (2) is equivalent to the following (3)

(3) $$\sum_{i,j}^{n} 2\mathrm{Re}\{a_i\overline{a_j}(x_i,x_j)\} = \sum_{i,j}^{n}(|a_i|^2 + |a_j|^2)|(x_i,x_j)|.$$

On the other hand, the following inequality (4) is always valid for all i and j,

(4) $$2\mathrm{Re}\{(a_i x_i,a_j x_j)\} \leq 2|a_i||a_j||(x_i,x_j)| \leq (|a_i|^2 + |a_j|^2)|(x_i,x_j)|$$

so (3) is equivalent to the following (5) or (6) for arbitrary i and j because comparing (3) with (4)

(5) $$(x_i,x_j) = 0 \quad \text{for } i \neq j$$

(6) $$(a_i x_i,a_j x_j) = |(a_i x_i,a_j x_j)| \text{ and } |a_i| = |a_j|.$$

Whence the proof of Theorem 1 is complete.

Proof of Theorem 2.

In case $\alpha = 1$ or 0, the result is obvious by Theorem 1, so we have only to consider the case $0 < \alpha < 1$. Let $T = U|T|$ be the polar decomposition of T where U means the partial isometry and $|T| = (T^*T)^{1/2}$ with $N(U) = N(|T|) = N(T)$. We state the following obvious but important relation in order to show a proof of Theorem 2;

(*) $N(S^q) = N(S)$ for any positive operator S and for any positive number q.

Also we state the following well known result (7) [cf. [2]]

$$|T^*|^q = U|T|^q U^* \quad \text{for any positive number } q. \tag{7}$$

In Theorem 1 we replace x by $|T|^{\alpha}x$ and also x_i by $|T|^{\beta}U^*x_i$ for all $i = 1,2,\ldots,n$ where $\beta = 1 - \alpha$.

$(|T|^{\beta}U^*x_i, |T|^{\beta}U^*x_j) = (U|T|^{2\beta}U^*x_i, x_j) = (|T^*|^{2\beta}x_i, x_j)$ by (7) and

$x_1, x_2, \ldots, x_n \notin N(|T^*|^{\beta}) = N(|T^*|) = N(T^*)$ by (*), so we have (I_2) by (I_1) in Theorem 1. In this case,

$$|T|^{\alpha}x = \sum_{i=1}^{n} a_i |T|^{\beta}U^*x_i \quad \text{iff} \quad |T|^{2\alpha}x = \sum_{i=1}^{n} a_i |T|U^*x_i = \sum_{i=1}^{n} a_i T^*x_i$$

by (*) for $|T|$, on the other hand

$$|T|^{\alpha}x = \sum_{i=1}^{n} a_i |T|^{\beta}U^*x_i \quad \text{iff} \quad |T|x = |T|^{\alpha+\beta}x = \sum_{i=1}^{n} a_i |T|^{2\beta}U^*x_i$$

by (*) for $|T|$, equivalently $U|T|x = \sum_{i=1}^{n} a_i U|T|^{2\beta}U^*x_i$ by $N(U) = N(|T|)$,

iff $Tx = \sum_{i=1}^{n} a_i |T^*|^{2\beta}x_i$ by (7).

Whence the proof of (i) is complete.

Remark 2. Selberg inequality reduces to the Bessel one and the equality in this Bessel one is just Parseval identity, and (C_1) is necessary and sufficient condition for this Parseval one, that is, the cases of the equality in Selberg inequality are more than the case (C_1) for Parseval identity and this is natural and agreeable, because Selberg inequality is an extension of Bessel one.

Also in the following inequality (**) in case $0<\alpha<1$,

(**) $$|(Tx,y)|^2 \le (|T|^{2\alpha}x,x)(|T^*|^{2(1-\alpha)}y,y),$$

the equality in (**) holds iff $|T|^{2\alpha}x$ and T^*y are linearly dependent iff Tx and $|T^*|^{2(1-\alpha)}y$ are linearly dependent [2], and the case of equality in Theorem 2 reduces to this result and this is natural and agreeable because Theorem 2 is Selberg type extension of inequality (**).

After reading the first version of our manuscript, Professor F.Kubo has kindly informed us that (I_2) in Theorem 2 has been obtained independently by them.

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References

[1] E. Bombieri, Le Grand Grible dans la Théorie Analytique des Nombres, Astérisque 18, Société Mathématique de France, 1974.

[2] T. Furuta, A simplified proof of Heinz inequality and scrutiny of its equality, Proc. Amer. Math. Soc., 97(1986),751-753.

[3] K. Kubo and F. Kubo, Diagonal matrix dominates a positive semidefinite matrix and Selberg's inequality, preprint.

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