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Maximal Subgroups of the Sporadic Simple Group of Rudvalis

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Abstract

The maximal subgroups of the Rudvalis sporadic simple group are completely classified up to conjugacy.

1 Introduction.

The Rudvalis group *Rud* is one of the six sporadic finite simple groups which are not involved in the Fischer-Griess Monster. The aim of this paper is to classify the maximal subgroups of *Rud*, where we use ATLAS notation to denote the isomorphism types of groups [2].

Theorem 1.1 *The Rudvalis simple group of order* $2^{14}\cdot 3^3\cdot 5^3\cdot 7\cdot 13\cdot 29$ *has exactly* 15 *conjugacy classes of maximal subgroups. The isomorphism types of the representatives are as follows:*

(*A*) *Four 2-local subgroups:*
- (1) $2\cdot 2^{4+6}{:}S_5$,
- (2) $2^{3+8}{:}L_2(7)$,
- (3) $2^6\cdot G_2(2)$ *(non-split)*,
- (4) $(2^2\times Sz(8)){:}3$.

(*B*) *One 3-local and three 5-local subgroups:*
- (5) $(3\cdot A_6)\cdot 2^2$,
- (6) $(5_+^{1+2}{:}Q_8)\cdot 4$,
- (7) $5^2{:}GL_2(5)$,
- (8) $(5{:}4)\times A_5$.

(*C*) *Seven non-local subgroups:*
- (9) ${}^2F_4(2)$,
- (10) $U_3(5).2$,
- (11) A_8,
- (12) $L_2(25){:}2^2$,
- (13) $L_2(29)$,
- (14) $PGL_2(13)$,
- (15) $A_6\cdot 2^2$.

It should be mentioned that the same result has also obtained by R. Wilson [10] by fully using computer for calculating matrices of degree 28. The original version of the present paper was written in 1984, completely independent from Wilson's work (see p. 248 in [2]). Since the methods I used in that paper are not so dramatically different from those used by Wilson, I did not submit the paper. However, I have been asked by several people where my paper appeared and some of them kindly encouraged me to publish it. Thus I decided to publish it, in order to make it easy to access and to stress a difference between my method and Wilson's: that is, in the present paper, the classification has done without using computer. In particular, the existence of a subgroup isomorphic to A_8, which was not established before Wilson's or the author's work, will be shown by a computer-free method.

We give several remarks about notation. We use standard notation in group theory. For example, Q_8 and SD_{16} mean the quarternion group of order 8 and the semidiheadral group of order 16, respectively. We use $\Omega_1(P)$ to denote the subgroup generated by elements of order p of a p-group P. As for other notation, we follow [2]. We also freely use information (structure constants, for example) obtained from the character table of the Rudvalis group, which we can see f. g. in [2] p. 127. Throughout this paper, we adopt the naming of the conjugacy classes of the Rudvalis group by [2].

We recall several papers about the Rudvalis group *Rud*, whose results will be used in this paper. Parrott [7] determined much of local structures of *Rud* as well as the conjugacy classes (see [2] p. 127). He also proved the existence of subgroups isomorphic to ${}^2F_4(2)$ and 2^6G_2, by applying the Brauer trick. Conway and Wales [4] constructed the double cover of *Rud* as the matrix group of degree 28 over $\mathbf{C}$, and proved that it has a transitive rank-3 permutation representation of degree 4060 with one point satbilizer ${}^2F_4(2)$. Afterward, Conway [3] gave a more comprehensible description of this representation as well as the existence of subgroups isomorphic to $U_3(5).2$. The character tables of the Rudvalis group and its double cover were calculated by J. S. Frame. Assa [1] and O'Nan [6] characterized *Rud* by its Sylow 2-subgroup and a 2-local subgroup isomorphic to $2^6G_2(2)$, respectively. The existence of subgroups isomorphic to $L_2(29)$ has shown by Young [11]. By observing the representation over $\mathbf{F}_2$ obtained from the 28-dimensional ordinary representation above, Mason and Smith [8] showed the existence of subgroups isomrphic to $L_2(13)$. Furthermore, the list of maximal subgroups of ${}^2F_4(2)'$ was obtained independently by Tchakerian [9] and Wilson [10].

Now we describe the outline of the proof of the theorem. We start with an observation that every maximal subgroup of a finite simple group G is the normalizer in G of a characteristically simple subgroup. A characteristically simple group is a direct product of isomorphic simple groups. In particular, each abelian characteristically simple group is an elementary abelian p-group for some prime p.

Thus the process of making a complete list of maximal subgroups of G, up to conjugacy, is divided into the following three steps. First, we make the complete list of characteristically simple groups which actually occur as subgroups of G. Second, we determine the classes and normalizers in G of each subgroup of this list. Finally, we choose maximal subgroups by examining the inclusion relations up to conjugacy among those normalizers.

First step contains constructions of some unknown simple subgroups. Usually, such subgroups are of fairly large indices in G and do not have nice geometric meanings. Thus this construction should be done by examining the generators and relations. This usually requires enough information about explicit matrix representations for individual elements of G and tough calculation which often need help of computers. However, for the Rudvalis group, the only unknown subgroup is A_8, which has a nice presentation. Thus, we can verify these relations by only observing local subgroups (see §9.1,2). This is the reason why we can avoid use of computers.

Second step is complicated and sometimes we need computer to complete it. In this paper, we will finish this step without using computer, since we can reduce the required works by mainly exploiting the following three methods. The first one is to use the argument based on enumerations of structure constants. This will be applied to subgroups isomorphic to A_5, A_6, $L_2(7)$, $L_2(13)$, $L_2(25)$, $L_3(3)$ and $Sz(8)$. Since the structure constants are rather small for these subgroups, this argument is very useful. The second one is to show that the subgroup in question should fix a non-trivial vector of a 28-dimensional vector space over $\mathbf{F}_2$ on which *Rud* acts (see

§2.2). This is applied to the subgroups isomorphic to ${}^2F_4(2)$ or $L_2(7)$. The third method is to generate groups by taking successive normalizers. The subgroups isomorphic to A_7, A_8, $L_2(29)$ and $U_3(3)$ are treated by this method.

Finally, we describe the organization of this paper. In §2, we explain notation and several arguments frequently used in this paper. In §3 and 4, we treat the first and second steps above for abelian characteristically simple subgroups. That is, we determine the maximal p-local subgroups for each prime p dividing the order of *Rud*. In §5, we will consider the first step for non-abelian subgroups. Subgroup isomorphic to A_8 will be constructed in §9.2. The other sections are devoted to complete the second step for non-abelian characteristically simple subgroups. Then, by observing the results in every sections, we immediately have the main theorem.

2 Notation and some standard arguments.

Throughout this paper, G will denote a fixed finite group isomorphic to the Rudvalis simple group. For any subgroup H of G, we adopt the ATLAS notation to denote the conjugacy classes of H; that is, the classes of elements of a given order are arranged in decending order of the orders of their stabilizers. If there is a risk of confusion, we denote these classes of H with the subscript, like $(2A)_H$. For $G = H$, the subscript G will be dropped. Elements contained in a class (nX) of H are called *$(nX)_H$-elements.* If a subgroup of H has an isomorphism type X in ATLAS notation, we call it an X-subgroup. For an $(lA)_H$-element x and an $(mB)_H$-element y with $xy \in (nC)_H$, the subgroup $\langle x, y\rangle$ of H is called of *H-type* $(lA, mB; nC)_H$. Furthermore, $C_H((nA)_H)$ and $N_H((nA)_H)$ mean the isomorphism types of the centralizer and the normalizer of the cyclic group generated by an element of a class $(nA)_H$, respectively. Finally, we say that a subgroup H of G is *nA-pure*, if any non-trivial element of H lies in a class nA (of G).

2.1 Enumeration of structure constants.

For a subgroup H of G and a fixed element $t \in (nC)_H$, we define

$$(lA, mB; t)_H := \{(x, y) \in (lA)_H \times (mB)_H | xy = t\}.$$

The subscript H is dropped when it is clear which group is meant. The *structure constant* $\#(lA, mB; t)$ can be calculated as follows, using the character table of H:

$$\#(lA, mB; t)_H = \frac{|H|}{|C_H(lA)||C_H(mB)|} \sum_\chi \frac{\chi(g)\chi(h)\overline{\chi(t)}}{\chi(1)},$$

where g and h are any elements of lA and mB respectively, χ runs over all the irreducible characters of G and $\overline{\chi(t)}$ means the complex conjugate of $\chi(t)$. In particular, this value is independent of the particular choice of $t \in (nC)_H$. Thus we also denote it by $\#(lA, mB; nC)$.

Let H be a subgroup of G of G-type $(lA, mB; nC)$, and let t be an $(nZ)_G$-element of H. For a subset K of G containing t, we define

$$(lA, mB; t) \cap K := \{(x, y) \in (lA) \times (mB) | x, y \in K, xy = t\}.$$

In non-local analyses (§5–9), we often consider the set $(lA, mB; t) \cap K$ for the set $K = \cup_{g \in N_G(t)} H^g$. We denote this important set by $H(t)$: $H(t) := \cup_{g \in N_G(t)} H^g$. Assume that $(lA, mB; t) \cap (H \cap H^g) =$

$\emptyset$ for any $g \notin N_G(t)$. Then the cardinality of this set is given by

$$|(lA, mB; t) \cap H||N_G(t) : N_G(t) \cap N_G(H)|.$$

We note that the first factor of this value can be calculated by the character table of H, if we know the fusion pattern of H in G. In fact, if the sets $(lA)_G \cap H$ and $(mB)_G \cap H$ are divided into the conjugacy classes $(lA_1)_H, \ldots, (lA_p)_H$ of H and $(lB_1)_H, \ldots, (lB_q)_H$ of H, respectively, we have

$$|(lA, mB; t) \cap H| = \sum_{i=1}^{p} \sum_{j=1}^{q} |((lA_i)_H, (mB_j)_H; t)_H)|.$$

This formula turns out to be very useful to determine the conjugacy classes of non-abelian simple subgroups of G (see §6–8).

2.2 The 28-dimensional $\mathbf{F}_2G$-module.

Mason and Smith [8] have constructed an $\mathbf{F}_2G$-module V of dimension 28 as follows. Let $\tilde{G}$ be the double cover of G and V_0 be $\mathbf{C}G$-module of dimension 28 over $\mathbf{C}$ constructed in [4]. Then there exists a $\mathbf{Z}[i]\tilde{G}$-module V_1 containing a basis of V_0. Since $\mathbf{Z}[i]/(i+1) \cong \mathbf{F}_2$, we get an $\mathbf{F}_2G$-module V of dimension 28 over $\mathbf{F}_2$ by reading the coefficients of representation matrices afforded by V_1 modulo $i+1$; that is, $V := \mathbf{Z}[i]/(i+1) \otimes V_1$.

We can find in [8] Table 8 the G-orbits on the set V^* of non-zero vectors of V and the corresponding stabilizers. For convenience, we quote it here, where the first column provides a name for a representative of each orbit, the second column shows the isomorphism type of the corresponding stabilizer, and the third column gives 1/29 times the length of the corresponding G-orbit. Note that any element of order 29 of G acts fixed point freely on V^*.

Table 1: R-orbits on V^*.

Name	Stabilizer	(Orbit length)/29	Name	Stabilizer	(Orbit length)/29
f	$^2F_4(2)$	140	s	S_7	998,400
l	$Aut(L_2(25))$	161,280	p	$PGL_2(13)$	2,304,000
t	$(2^2 \times Sz(8))3$	14,400	w	$2.2^4.2A_5$	1,310,400
v	$2.2^{4+8}S_5$	20,475	g	$2^6 \cdot G_2(2)$	6,500
a	$3.Aut(A_6)$	1,164,800	r	$2^{(8)}S_3$	3,276,000

From this table, we can calculate the dimensions of subspaces of V fixed by elements of G of odd orders (see [8] Lemma 2.3).

Lemma 2.1 *The dimensions of $C_V(3A)$, $C_V(5A)$, $C_V(5B)$, and $C_V(7A)$ are* 10, 8, 4 *and* 4, *respectively.*

The following argument (DIM in [8]) will be used in §7 and 8, as well as the above Table and Lemma. Let $g_1 = 1, g_2, \ldots, g_n$ be elements of odd order of a finite group X acting on a space W over a field k of characteristic 2. Assume that for all possible $\mathbf{F}_2X$-irreducible modules W_i $(i = 1, \ldots, k)$ of dimensions less than or equal to $\dim W$ and for each $j = 1, \ldots, n$, we know the dimensions of subspaces $C_W(g_j)$ of W and $C_{W_i}(g_j)$ of W_i fixed by g_j. Then the multiplicities m_i of W_i among the $\mathbf{F}_2X$-composition factors of W satisfy the following n linear equations

$$\sum_{i=1}^{k} m_i \dim C_{W_i}(g_j) = \dim C_W(g_j), \quad j = 1, \dots, n.$$

These equations allow us to restrict (or even solve for) the non-negative integers m_i ($i = 1, \dots, n$).

3 2-local subgroups.

Assume that elements $z, t, v, w, w_1, a, b, c, d, y, y_2, u$ are involutions and x_1, x_2 are elements of order 4 with $x_1^2 = x_2^2 = z$ satisfying the following commutator relations, where all unstated commutators are trivial:

$$\begin{aligned}
&[w_1, a] = [w, b] = [v, c] = [d, t] = [x_1, x_2] = z;\ [w_1, y] = vt,\ [w, y] = t;\ [w_1, y_2] = vz,\\
&[w, y_2] = vt;\ [w_1, u] = w,\ [v, u] = t;\ [a, x_1] = t,\ [a, x_2] = vz;\ [b, x_1] = v,\ [b, x_2] = vt;\\
&[c, x_1] = w,\ [c, x_2] = w_1wtz;\ [d, x_1] = w_1,\ [d, x_2] = wv;\ [a, b] = vt,\ [a, d] = w,\\
&[b, c] = w_1;\ [x_2, y] = z,\ [x_1, y_2] = tz;\ [x_2, y_2] = vtz,\ [u, x_1] = z,\ [u, x_2] = x_1;\ [a, y_2] = t,\\
&[b, y] = vz;\ [b, u] = ax_1wv,\ [c, y] = a;\ [d, y] = abx_2vt,\ [d, u] = c;\ [c, y_2] = abx_1x_2w_1,\\
&[d, y_2] = b;\ [y_2, u] = y.
\end{aligned}$$

In this section, we set $V = \langle z, t, v \rangle$, $E = \langle z, t, v, w, w_1 \rangle$, $J = \langle E, x_1, x_2, a, b, c, d \rangle$, $T = \langle J, y, y_2, u \rangle$ and $X = \langle z, t, w, a, c, u \rangle$ in the group generated by the above elements with the above relations.

Lemma 3.1. [7],[1] *The group T is isomorphic to a Sylow 2-subgroup of the Rudvalis group G.*

We will identify T with a Sylow 2-subgroup of G. The above relations show that $Z(T) = \langle z \rangle$. We set $H := C_G(z)$.

Lemma 3.2. [7], [1]

(1) *The group G has two classes of involutions with representatives z and yx_1x_2. The involution z is central and a square of some element of order 4, while the involution yx_1x_2 is not central nor a square.*

(2) *The subgroup H has five classes of 2A-involutions of G with representatives z, t, a, y, u.*

Parrott [7] describes some 2-local subgroups of G.

Proposition 3.3.

(1) *We have $J = O_2(H)$, $J' = \Phi(J) = E \cong 2^5$, and $H/J \cong S_5$. Furthermore $C_G(E) = \langle E, x_1, x_2 \rangle$ and $\Omega_1(C_G(E)) = \langle z \rangle$. In particular, $H = N_G(E)$. If x is an element of odd prime order of H, we have $C_J(x) \cong Q_8$. The involution u corresponds to a transposition of $H/J \cong S_5$.*

(2) *We have $C_G(V) = \langle E, x_1, x_2, a, b, y, y_2 \rangle$ and $N_G(V)/C_G(V) \cong L_3(2)$.*

(3) *We have $C_G(X) = X$ and $N_G(X)/X \cong G_2(2)$.*

(4) *For the element* $\tilde{y} = yx_1x_2$, *we have* $C_G(\tilde{y}) = \tilde{E} \times S$, *where* $\tilde{E}$ *is a* $2B$*-pure four group containing* $\tilde{y}$ *and* $S \cong Sz(8)$. *Furthermore,* $N_G(\tilde{E}) = C_G(\tilde{y}).\tilde{Q}$, *where* $\tilde{Q}$ *is a subgroup of order* 3 *with* $\tilde{E}\tilde{Q} \cong A_4$ *and* $S\tilde{Q} \cong Aut(Sz(8))$.

We will start with the 2-local analysis. First, we note the following.

Lemma 3.4. *If* W *is an elementary abelian 2-subgroup of* G *containing a* $2B$*-involution, then either* $N_G(W)$ *is conjugate to a subgroup of* $N_G(\tilde{E})$ *or* $N_G(W)$ *is contained in the normalizer of a* $2A$*-pure elementary abelian subgroup of order* 8.

Proof. We may assume that W contains $\tilde{y}$ and therefore $W \subseteq C_G(\tilde{y}) = \tilde{E}\times S$, where $S \cong Sz(8)$ by Pro. 3.3 (4). If W is contained in $\tilde{E}$, we have $O_2(C_G(W)) = \tilde{E}$ and $N_G(W) \subseteq N_G(\tilde{E})$. Assume that $W \not\subseteq \tilde{E}$. Then there is an involution $s \in S$ with $W \cap s\tilde{E} \neq \emptyset$. We have $C_G(W) \subseteq \tilde{E} \times C_S(s)$. The subgroup $C_S(s)$ is a special 2-group with $\Omega_1(C_S(s)) \cong 2^3$. We note that any involution of S lies in the class $2A$, because it commutes with a conjugate of the group $\tilde{Q}$ of order 3 (see Prop. 3.3 (4)). Since we have $\#(2A, 2A; 2B) = 0$, any product of two mutually commuting $2A$-involutions is a $2A$-involution. In particular, the involutions of $C_G(\tilde{y}) \setminus S$ are $2B$-involutions. Thus, by the above inclusion, $N_G(W)$ normalizes the $2A$-pure elementary abelian group $\Omega_1(C_S(s))$ of order 8. □

By the lemma above, in order to classify maximal 2-local subgroups, it suffices to consider normalizers of $2A$-pure subgroups. The next lemma about $2A$-pure four subgroups can be verified by strightforward computations and Prop. 3.2.

Lemma 3.5.

(1) *Any* $2A$*-pure four subgroup of* G *is conjugate to* $\langle z,t\rangle$, $\langle z,a\rangle$, $\langle z,y\rangle$ *or* $\langle z,u\rangle$.

(2) *We have* $C_G(z,t) = \langle E, x_1, x_2, a, b, c, y, y_2\rangle$ *and* $C_G(z,t)'' = V$; $C_G(z,a) = \langle z,t,v,w,a,c,x_1y_2,y,bx_2w_1y_2,u\rangle$ *and* $\langle g^2 | g \in C_G(z,a)'\rangle = \langle t\rangle$; $C_G(z,y) = \langle z,t,a,v,x_1,ww_1x_2b,y,y_2,u\rangle$ *and* $L_3(C_G(z,y)) = \langle t\rangle$, *where* $L_1(K) = K'$ *and* $L_n(K) = [L_{n-1}(K), K]$ *for a group* K; $C_G(z,u) = X(\langle u\rangle \times Q\langle y\rangle)$, *where* Q *is a Sylow* 3*-subgroup of* H *inverted by* y, *and* $O_2(C_G(z,u)) = X$.

(3) *We have* $N_G(\langle z,t\rangle) \subseteq N_G(V)$, $N_G(\langle z,a\rangle) \subseteq C_G(t)$, $N_G(\langle z,y\rangle) \subseteq C_G(t)$ *and* $N_G(\langle z,u\rangle) \subseteq N_G(X)$.

Lemma 3.6. *Let* W *be a* $2A$*-pure elementary abelian subgroup containing* z. *Then, by replacing* W *by its suitable* H*-conjugate,* W *satisfies one of the following possibilities* $(a),\ldots,(g)$, *where we set* $E_1 := \langle z,t,v,a,y\rangle \cong 2^5$, $F_0 := \langle z,t,v,w,a\rangle \cong 2^5$, $F_1 := \langle z,t,a,y,u\rangle \cong 2^5$ *and* $F_2 := \langle z,t,v,y,y_2\rangle \cong 2^5$. *Furthermore, we have* $C_G(F_i) = F_i$ *for* $i = 0,1,2$.

(a) $W \subseteq E$

(b) $\langle z,y\rangle \subseteq W \subseteq E_1$.

(c) $\langle z,u\rangle \subseteq W \subseteq X$.

(d) $\langle z,a\rangle \subseteq W \subseteq \langle z,t,w,a,c\rangle$ $(\subseteq X)$.

(e) $\langle z,a\rangle \subseteq W \subseteq F_0$.

(f) $\langle z,y\rangle \subseteq W \subseteq F_1$ *or* $\langle z,u\rangle \subseteq W \subseteq F_1$.

(g) $\langle z,y\rangle \subseteq W \subseteq F_2$.

Proof. Since the claims follow mainly from straightforward computations, we will give the proof of a part of them. Suppose $W \not\subseteq J$. Then we may assume that W contains $\langle z,y\rangle$ or $\langle z,u\rangle$.

We consider the case $\langle z,y\rangle \subseteq W$. The group $\langle z,t,y\rangle$ is a normal subgroup of $\overline{C_G(z,y)} = M\langle w_3\rangle$, where $M := \langle z,t,y,v,a,x_1,y_2,u\rangle$ and $w_3 := ww_1x_2b$. The factor group $\overline{C_G(z,y)} :=$

$C_G(z,y)/\langle z,t,y\rangle$ is generated by the image $\overline{M}$ of M, which is an elementary abelian normal 2-subgroup of index 2, and the image $\overline{w_3}$. Thus we have either $W \subseteq M$ or $W \cap Mw_3 \neq \emptyset$. Since we may verify that the centralizer in $\overline{C_G(z,y)}$ of an involution of $\overline{Mw_3}$ is $\langle v, a, x_1y_2, w_3\rangle$, we have $W \subseteq \langle z,t,y,a,x_1y_2,w_3\rangle$ in the latter case.

Assume that $W \subseteq M$. The group E_1 is an elementary abelian subgroup of M of index 8. We may verify that the coset $E_1 g$ does not contain any involutions for $g = x_1, x_1u, x_1y_2, x_1y_2u, y_2u$, and the sets of involutions of the cosets E_1u and E_1y_2 are $\langle z,t,a,y\rangle u$ and $\langle z,a,v,y\rangle y_2$, respectively. Thus we have $W \subseteq E_1$, $W \subseteq F_1$ or $W \subseteq F_2$, if $W \subseteq M$. We note that F_1 and F_2 are self-centralizing and F_1 is conjugate to E by a Lemma in [7].

All the remaining cases are treated by similar methods. □

Lemma 3.7. *For a self-centralizing 2A-pure elementary abelian subgroup W of G of order 2^5, $N_G(W)$ is conjugate to a subgroup of H, $N_G(\langle z,t\rangle)$, $N_G(\langle z,a\rangle)$, $N_G(\langle z,y\rangle)$, $N_G(\langle z,u\rangle)$, or $N_G(\langle z,t,v\rangle)$.*

Proof. The group $N_G(W)/W$ is isomorphic to a subgroup of $L_5(2)$, as $C_G(W) = W$. Then possible prime divisors of $|N_G(W)/W|$ are 2, 3 and 5, since $|G|$ is prime to 31 and no $2A$-involution commutes with an element of order 7. If $N_G(W)$ is a 2-group, it is conjugate to a subgroup of the Sylow 2-subgroup T contained in $H = C_G(z)$. Thus we may assume that there is an element ω of $N_G(W)$ of order p for $p = 3$ or 5. Then $C_W(\omega) \neq 1$, and therefore we may assume that $z \in C_W(\omega)$. By Lemma 3.6, we have $W = F_i$ for some $i = 0,1,2$ and then $|E \cap W| = 2^4, 2^2, 2^3$, respectively. If $p = 5$, then ω acts trivially on $(E \cap W)/\langle z\rangle$ and so $E \cap W \subseteq C_W(\omega)$, which contradicts the fact $C_J(\omega) \cong Q_8$ (see Lem. 2.3 (1)). Thus $p = 3$ and $N := N_G(W)$ is a $\{2,3\}$-group. Since N/W is a subgroup of $L_5(2)$, $|C_W(\omega)| = 2$ or 2^3. We have $|C_W(\omega)| = 2$ and $|N|_3 = 3$, since there is one class of elements of order 3 of G and a Sylow 2-subgroup of $C_G(3A)$ is a semi-diheadral group of order 16. Thus ω acts fixed point freely on $|E \cap W/\langle z\rangle|$, and therefore $|E \cap W| = 2^3$ and $W = F_2$. Then $N \cap H = \langle E, x_1, x_2, a, b, y, y_2, u\rangle =: S$ and $|S| = 2^{12} \leq |N|_2$. If $|N|_2 = 2^{12}$, then $N = \langle S, \omega\rangle \subseteq C_G(z)$. If $|N|_2 = 2^{13}$, a Sylow 2-subgroup S_0 of N containing S normalizes $Z(S') = \langle z,t,v\rangle$, and therefore we have $N = \langle S_0, \omega\rangle \subseteq N_G(\langle z,t,v\rangle)$. If $|N|_2 = 2^{14}$, $N/O_2(N) \cong A_3$ or S_3. Then $|O_2(N)| = 2^{13}$ or 2^{14}, and so $Z(O_2(N)) \cong 2$ or 2^2, as $Z(T) = \langle z\rangle$. Thus, in this case, N is contained in the normalizer of an elementary abelian subgroup of order at most 4. Hence the claim follows from Lemma 3.5 (1). □

Lemma 3.8. *Let W be a 2A-pure elementary abelian subgroup of G of order at least* 8. *Then, under the notation in* Lemma 3.6, *the following holds.*

(1) *If $\langle z,a\rangle \subseteq W \subseteq F_0$, by replacing W by a suitable conjugate, one of the following occurs: $\langle z,a\rangle \subseteq W \subseteq E_1$, $N_G(W) \subseteq N_G(\langle z,t\rangle)$, or $N_G(W) \subseteq N_G(X)$.*

(2) *If $\langle z,y\rangle \subseteq W \subseteq F_1$ or $\langle z,u\rangle \subseteq W \subseteq F_1$, by replacing W by a suitable conjugate, one of the following occurs: $N_G(W) \subseteq N_G(F_1)$, $\langle z,u\rangle \subseteq W \subseteq X$, or $\langle z,y\rangle \subseteq W \subseteq E_1$,.*

(3) *If $\langle z,y\rangle \subseteq W \subseteq F_2$, by replacing W by a suitable conjugate, one of the following occurs: $\langle z,y\rangle \subseteq W \subseteq E_1$ or $N_G(W) \subseteq N_G(F_0)$.*

Proof. We only prove the claim (1). The other claims are proved by similar arguments.

Since $\langle z,a,t,v\rangle$ is a maximal elementary abelian subgroup of F_0, $W \subseteq \langle z,a,t,v\rangle \subseteq E_1$ or $W \cap \langle z,a,t,v\rangle w \neq \emptyset$. In the latter case, W contains w, tw, vw or tvw, since $\langle z,a\rangle \subseteq W$. Since $a^y = a$, $w^y = tw$, $(vw)^y = tvw$ and $a^m = a$, $w^m = vw$ for $m = y_2ux_1$, we may assume that $w \in W$.

Thus, if $|W| = 2^3$, we have $W = \langle z, a, w\rangle$, $C_G(W) = \langle z, t, v, w, a, c, u\rangle$ and $C_G(W)' = \langle z, t\rangle$, and therefore $N_G(W) \subseteq N_G(\langle z, t\rangle)$. If $|W| = 2^4$, $W = \langle z, a, w, t\rangle$, $W = \langle z, a, w, v\rangle$ or $W = \langle z, a, w, v\rangle$. Since the latter two groups are conjugate under the action of u, we may assume that one of the first two cases occurs. In the first case, $C_G(W) = X$ and $N_G(W) \subseteq N_G(X)$. In the second case, $C_G(W) = F_0$ and $N_G(W) \subseteq N_G(F_0)$. □

We may also prove the following lemma by similar argument as above.

Lemma 3.9. *Let W be a 2A-pure elementary abelian subgroup of G of order at least 8.*

(1) *If $\langle z, u\rangle \subseteq W \subseteq X$, $N_G(W)$ lies in $N_G(X)$ or $N_G(\langle z, a\rangle)$.*

(2) *If $\langle z, a\rangle \subseteq W \subseteq \langle z, t, w, a, c\rangle$ and $W \not\subseteq F_0$, $N_G(W)$ is contained in $N_G(X)$, where F_0 is the subgroup in* Lemma 3.6.

By Lemmas 3.5–9, we can conclude that if W is a $2A$-pure elementary abelian 2-subgroups of G, then either W is conjugate to a subgroup of E of order at least 8, or $N_G(W)$ is conjugate to a subgroup of $C_G(z)$, $N_G(V)$ or $N_G(X)$.

Lemma 3.10. *For a subgroup W of E of order at least 8, $N_G(W)$ is conjugate to a subgroup of $N_G(V)$ or $C_G(z)$.*

Proof. Since the elements of $E \setminus \langle z\rangle$ form one conjugacy class of H, we may assume that $z, t \in W$. The set $E \setminus \langle z\rangle$ is divided into the following four T-classes: $\{t, zt\}$, $\{v, vz, vt, vtz\}$, $\{we | e \in \langle z, t, v\rangle\}$ and $\{w_1 e | e \in \langle z, t, v, w\rangle\}$. Since $\langle z, t, w\rangle^{y_2} = \langle z, t, vw\rangle$, $\langle z, t, w_1\rangle^{y_2} = \langle z, t, w_1 v\rangle$, $\langle z, t, w_1\rangle^u = \langle z, t, w_1 w\rangle$ and $\langle z, t, w_1 w\rangle^y = \langle z, t, v w_1 w\rangle$, we may assume that W contains $\langle z, t, v\rangle = V$, $\langle z, t, w\rangle =: V_0$ or $\langle z, t, w_1\rangle =: V_1$. We have $C_G(V_0) = \langle E, x_1, x_2, a, c, u\rangle$, $\langle g^2 | g \in C_G(V_0)\rangle = \langle z\rangle$; $C_G(V_1) = \langle E, x_1, x_2, b, c\rangle$, $C_G(V_1)' = E$ and $Z(C_G(V_1)) = \langle z\rangle$. Thus, if $|W| = 2^3$, then $W = V$ or $N_G(W) \subseteq C_G(z)$. Assume that $|W| = 2^4$. Then $W = \langle V, w\rangle$, $\langle V, w_1\rangle$, $\langle V, ww_1\rangle$, $\langle V_0, w_1\rangle$, $\langle V_0, vw_1\rangle$, $\langle V_1, vw\rangle$ or $\langle V_0, w_1\rangle$. Since $\langle V, w_1\rangle^u = \langle V, ww_1\rangle$, $\langle V_0, w_1\rangle^y = \langle V_0, vw_1\rangle$ and $\langle V_0, w_1\rangle^{y_2 u} = \langle V_1, vw\rangle$, we may assume that $W = \langle V, w\rangle$, $\langle V, w_1\rangle$ or $\langle V_0, w_1\rangle$. Then $C_G(W) = \langle E, x_1, x_2, a\rangle$, $\langle E, x_1, x_2, b\rangle$ or $\langle E, x_1, x_2, c\rangle$, and therefore $C_G(W)' = V$ or $\langle z, w, tw_1\rangle$. Then $N_G(W) \subseteq N_G(V)$ or $N_G(\langle z, w, tw_1\rangle)$. The latter group is conjugate to a subgroup $N_G(V)$ or $C_G(z)$ by the argument above. If $|W| = 2^5$, we have $W = E$ and $N_G(W) \subseteq C_G(z)$. The lemma has proved. □

Hence, we get the following conclusions.

Proposition 3.11. *For a 2A-pure elementary abelian 2-subgroup W of G, $N_G(W)$ is conjugate to a subgroup of $C_G(z)$, $N_G(V)$ or $N_G(X)$.*

Proposition 3.12. *Any subgroup of G with a non-trival normal 2-subgroup is conjugate to one of the subgroups in the list below:*

(1) $C_G(z) \cong 2.2^{4+6}{:}S_5$,	(3) $N_G(V) \cong 2^{3+8}{:}L_3(2)$,
(2) $N_G(\tilde{E}) \cong (2^2 \times Sz(8)){:}3$,	(4) $N_G(X) \cong 2^6 \cdot G_2(2)$.

Remark. Since G has a subgroup isomorphic to S_7 and the centralizer of a transposition of this group is of type $2 \times S_5$, the extension $C_G(2A)/O_2(C_G(2A))$ splits. In [6], O'Nan showed that the extension $N_G(V)/O_2(N_G(V))$ splits, but $N_G(X)/X$ does not split.

4 Odd local subgroups.

The structures of centralizers of elements of odd prime orders are determined by Parrott [7]. In this section, we use this information to determine maximal p-local subgroups for an odd prime p dividing $|G|$.

Proposition 4.1. *Any subgroup of G with a normal 3-group is conjugate to a subgroup of $N_G(3A) \cong 3Aut(A_6)$ or $N_G(V_3) \cong 3^2{:}GL_2(3)$, where V_3 is an elementary abelian group of order 9. Furthermore, G has one class of elementary abelian subgroups of order 9, and $N_G(V_3)$ is contained in a subgroup isomorphic to ${}^2F_4(2)'$.*

Proof. A Sylow 3-subgroup P of G is an extra special group of order 3^3 and exponent 3. Since any elements of order 3 of $N_G(Z(P))/P \cong Aut(A_6)$ are conjugate to each other, the elementary abelian subgroups of order 9 of G form one class. We note that P is contained in a subgroup T of G isomorphic to ${}^2F_4(2)'$. Thus, by [9], P is contained in a subgroup L of T isomorphic to $Aut(L_3(3))$. Since there is no element of order prime to 3 centralizing a subgroup of G of order 9, an elementary abelian subgroup of order 9 is self-centralizing. Thus its normalizer in G coincides with that in L, which is isomorphic to $3^2{:}GL_2(3)$. □

Proposition 4.2. *Any subgroup of G with a normal 5-group is conjugate to a subgroup of $N_G(5A) \cong (5^{1+2}{:}Q_8)4$, $N_G(5B) \cong (5{:}4) \times A_5$ or $N_G(V_5) \cong 5^2{:}GL_2(5)$, where V_5 is a $5A$-pure elementary abelian group of order 25.*

Proof. As is shown in [4], G has a subgroup M isomorphic to $Aut(L_2(25))$. Let V be a Sylow 5-subgroup of M and P a Sylow 5-subgroup of G containing V. By [7] Lemma 16, P is an extra special group of order 5^3 and exponent 5, $N_G(P) = N_G(Z(P)) \supseteq C_G(Z(P)) \cong 5^{1+2}{:}Q_8$, and $P \setminus Z(P)$ consists of 40 $5A$-elements and 80 $5B$-elements. It suffices to determine classes and normalizers of elementary abelian 5-subgroups of order 25 of P.

The group V is $5A$-pure, since $Z(P)$ is contained in V and $N_M(V) \cong (5^2{:}24)2$ acts transitively on $V^\#$. As $|C_G(2A)|_5 = 5$, the group $C_G(Z(P))/P \cong Q_8$ acts fixed-point-freely on $P/Z(P)$ and it does not normalize $V/Z(P)$. Then $C_G(V) = V$ and $V \cap V^g = Z(P)$ for some $g \in C_G(Z(P))$. Thus $N_G(V)/V$ is isomorphic to a subgroup of $GL_2(5)$ of index at most 2, and $V \cap V^g \setminus Z(P)$ coincides with the set of 40 $5A$-elements of $P \setminus Z(P)$. Furthermore, the latter fact shows that for an elementary abelian subgroup E ($\neq V, V^g$) of order 25 of P, we have $Z(P) = \langle E \cap 5A \rangle$ and $N_G(E) \subseteq N_G(Z(P))$.

Suppose $N_G(V)/V$ is a subgroup of $GL_2(5)$ of index 2. Then it is isomorphic to the central product $4 * SL_2(5)$ and has no element of order 8, which is a contradiction. Thus $N_G(V)/V \cong GL_2(5)$, and therefore $N_G(V) \cong 5^2{:}GL_2(5)$ by the theorem of Gashütz. □

Remark about $N_G(5B)$. For a $5B$-element γ_1, we have $C_G(\gamma_1) = \langle \gamma_1 \rangle \times A \cong 5 \times A_5$, where $A = (C_G(\gamma_1))'$. Since any involution centralizing a $5B$-element lies in the class $2B$, a Sylow 2-subgroup of A is a $2B$-pure four subgroup. Thus $C_G(E) = E \times S \cong 2^2 \times Sz(8)$ and γ_1 is contained in S. Let Q be a complement of $C_G(E)$ in $N_G(E)$. Since $Aut(Sz(8)) \setminus Sz(8)$ contains an element of order 3 centralizing a Sylow 5-group of $Sz(8)$, we may assume that Q centralizes a Sylow 5-normalizer $\langle \gamma_1, g \rangle \cong 5:4$ of S. Then $E:Q$ is a subgroup of A centralizing $\langle \gamma_1, g \rangle$. Since g is an element of order 4 acting on A and centralizing $E{:}Q$, g centralizes $A \cong A_5$. Thus

$N_G(\gamma_1) = \langle C_G(\gamma_1), g\rangle = \langle \gamma_1, g\rangle \times A \cong 5{:}4 \times A_5$. In particular, A is of type $(2A, 3A, 5A)$, since an element of order 5 of A centralizes a $2A$-involution g^2.

For $p = 7$, 13 or 29, a Sylow subgroup of G is of order p. Thus a p-local subgroup is a subgroup of a Sylow p-normalizer, whose structure is determined by [7].

Lemma 4.3 (1) *We have* $N_G(7A) \cong (2^2 \times 7{:}2){:}3$. $N_G(7A)$ *is a subgroup of the normalizer of the* $2B$*-pure four subgroup* $O_2(C_G(7A))$.

(2) *We have* $N_G(13A) \cong (2^2 \times 13 : 4) : 3$. $N_G(13A)$ *is a subgroup of the normalizer of the* $2B$*-pure four subgroup* $O_2(C_G(13A))$.

(3) *We have* $N_G(29A) = N_G(29B) \cong 29{:}14$.

5 Non-abelian characteristically simple subgroups.

Assuming the classification of finite simple groups (though it is enough to quote only partial results), we may determine the following list of isomorphism types of non-abelian characteristically simple groups of orders dividing $|G| = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$:

A_n for $n = 5, \dots, 13$; $A_5 \times A_5$, $A_6 \times A_6$;
$L_2(q)$ for $q = 7$, 8, 13, 25, 27, 29, 64;
$L_3(3)$, $L_3(4)$, $U_3(3)$, $U_3(4)$, $U_3(5)$; $Sz(8)$, $G_2(4)$, ${}^2F_4(2)'$; J_2.

Among the above candidates, neither $L_2(8)$ nor $L_2(64)$ could be an isomorphism type of a subgroup of G, since both of them contain elements of order 9. Since a Sylow 3-subgroup of G is not abelian, G does not have a subgroup isomorphic to $L_2(27)$. We can eliminate subgroups isomorphic to $U_3(4)$, J_2 or $G_2(4)$ as follows. If any of these groups is isomorphic to a subgroup H of G, involutions of H are of class $2A$ of G, since they are square elements or commute with elements of order 3 of H. However, an element of order 5 of H centeralizes both involutions and elements of order 3. This contradicts the fact that elements of order 5 of G centralizing $2A$-involutions are of class $5A$, while those centralizing elements of order 3 are of class $5B$. This fact also shows that G has no subgroup isomorphic to $A_5 \times A_5$, and so no subgroup isomorphic to $A_6 \times A_6$.

Lemma 5.1. *G has no subgroup isomorphic to* $L_3(4)$.

Proof. Let L be a subgroup of G isomorphic to $L_3(4)$. Any element of order 4 of L is contained in a subgroup of L isomorphic to A_6. Thus they are of class $4D$ of G by Prop. 6.4. Then the fusion of elements of L in G is determined as follows except elements of order 5.

L	$1A$	$2A$	$3A$	$4A$	$4B$	$4C$	$5A$	$5B^*$	$7A$	$7B^*$
G	$1A$	$2A$	$3A$	$4D$	$4D$	$4D$	$5A$ or $5B^*$		$7A$	$7A$

Then we have a contradiction by computing the multiplicity $(\chi|_L, 1_L)$ for the irreducible character χ of G of degree 406. □

As for the remaining groups, it is known that every groups except A_8 is isomorphic to a subgroup of G: The ${}^2F_4(2)$-subgroups of G appeared in the original construction of the Rudvalis group (see [4]), and they contain subgroups isomorphic to A_5, A_6 $L_2(25)$ and $L_3(3)$ (see [9],[10]). Conway [3] proved the existences of subgroups of G isomorphic to $U_3(5)$ and A_7. Mason and Smith [8] established the existence of an $L_2(13)$-subgroup of G as a stabilizer of a vector of the

28-dimensional G-module over $\mathbf{F}_2$ (see §2). The groups $Sz(8)$, $U_3(3)$ and $L_2(7)$ are isomorphic to subgroups of local subgroups of G. K.-C. Young [11] proved the existence of an $L_2(29)$-subgroup of G.

In §9, we will construct a subgroup of G isomorphic to A_8 by means of generators and relations.

Thus we have shown the following result.

Proposition 5.2 *The following is the complete list of the isomorphism types of non-abelian characteristically simple subgroups of the Rudvalis group G:*

A_5, A_6, A_7, A_8; $L_2(7)$, $L_2(13)$, $L_2(25)$, $L_2(29)$;
$L_3(3)$, $U_3(3)$, $U_3(5)$, $Sz(8)$, ${}^2F_4(2)$.

The reminder of this paper is mainly devoted to the much more complicated problem; that is, the determination of the conjugacy classes of the subgroups above and their normalizers.

6 Subgroups A_5, A_6 and $Sz(8)$.

6.1 Subgroups isomorphic to A_5.

A presentation of A_5 is $\langle a, b | a^2 = b^3 = (ab)^5 = 1\rangle$. Thus the classes of subgroups of G isomorphic to A_5 can be determined by computing the structure constants $(2X, 3A; 5Y)$ for $X, Y = A, B$.

Proposition 6.1.

(1) *There are at most two classes of A_5-subgroups of G of type $(2A, 3A; 5A)$. Their normalizers are conjugate to subgroups of the centralizer of a $2A$-involution.*

(2) *There are at most three classes of A_5-subgroups of G of type $(2B, 3A; 5A)$. The normalizer of a subgroup of one class is the normalizer of a $5B$-element, and the normlizers of subgroups of the other classes are contained in the centralizers of $2A$-involutions.*

(3) *There is a unique class of A_5-subgroups of G of type $(2A, 3A; 5B)$. Their normalizers are isomorphic to $(3 \times A_5).2$, and they are contained in the normalizers of $3A$-elements.*

(4) *There is a unique class of A_5-subgroups of G of type $(2B, 3A; 5B)$. They are self-normalizing.*

Proof. (1) The centralizer of a $5A$-element is isomorphic to $5^{1+2}{:}Q_8$ and there is no element of order 5 centralizing a $2A$-involution and a $3A$-element. Then the centralizer of a subgroup of type $(2A, 3A; 5A)$ is a cyclic group of order at most 4, and the assertion follows from the fact $\#(2A, 3A; 5A) = 500 = 1/2|C_G(5A)|$.

(2) By Lemma 3, the commutator subgroup A of the centralizer of a $5B$-element x is of type $(2B, 3A; 5A)$. Let B be a subgroup of type $(2B, 3A; 5A)$ centralizing an element y of order 5. Then y is a $5B$-element and $B = C_G(y)'$, and so B is conjugate to A and $N_G(B) = N_G(y)$.

We fix a $5A$-element γ of A. Assume that $\langle\alpha, \beta\rangle$ centralizes an element of order 5 for a pair (α, β) of $(2B, 3A; \gamma)$. Then $\langle\alpha, \beta\rangle$ is contained in $A(\gamma)$ (recall the notation in 2.1) by the above remark and Sylow's theorem. Since we have $|(2B, 3A; \gamma) \cap A(\gamma)| = 5|N_G(\gamma){:}N_G(\gamma) \cap N_G(A)| = 5 \cdot 1000 \cdot 4/|5{:}4 \times 5{:}2| = 100$, any pair of this set generates a subgroup conjugate to A.

Since $\#(2B,3A,5A) = 600$, there remain 500 pairs of $(2B,3A;\gamma)$ not centralizing any element of order 5. Since their centralizers are cyclic groups of order at moat 4, these pairs are divided into at most two orbits under $C_G(\gamma)$. Thus the result follows.

(3) Let q be an element of order 3. As $N_G(q)/\langle q\rangle \cong Aut(A_6)$, there is a subgroup T of $N_G(q)$ with $T/\langle q\rangle \cong S_5$. Since a Sylow 3-subgroup of G is isomorphic to 3_+^{1+2} and the Schur multiplier of S_5 is prime to 3, $\langle q\rangle$ has a complement S in T by applying the Gaschütz theorem. We set $A = S'$. Since q centralizes A, A is of type $(2A,3A;5B)$. The centralizer of a $5B$-element is isomorphic to $3\times A_5$ and its commutator subgroup is of type $(2B,3A;5A)$. Then $C_G(A) = \langle q\rangle$, since a $5A$-element centralizes no element of $2B\cup 3A$ and no 3^2-subgroup centralizes an element of order 5. Thus the result follows from the fact $(2A,3A;5B) = 100 = 1/3\cdot|C_G(5B)|$.

(4) Let x be a $5B$-element and A a subgroup of type $(2B,3A;5B)$ containing x. If $C_G(A)\neq 1$, $C_G(A)$ contains a $5B$-element a by the same argument as in (3). However, then $C_G(a)' = A$ is of type $(2B,3A;5A)$, which is a contradiction. Thus $C_G(A) = 1$ and A is self-normalizing, since $2B$-involutions are not squares in G. Then, under the natation in 2.1, we have $|(2B,3A;x)\cap A(x)| = 5|N_G(x):N_G(a)\cap A| = 5\cdot 2^4\cdot 3\cdot 5^2/2\cdot 5 = 600 = \#(2B,3A;5B)$, and therefore any pair of $(2B,3A;x)$ are conjugate to each other. □

6.2 Subgroups isomorphic to A_6.

The group A_6 has a presentation $\langle a,b|a^2 = b^4 = (ab)^5 = (ab^2)^5 = 1\rangle$ and involutions of A_6 are square elements. Since any element of a prime order does not commute with an A_6-subgroup, the possible types of A_6-subgroups of G are $(2A,4X;5Y)$ for $(X,Y) = (D,A)$, (D,B) or (B,B) by the following structure constants:
$\#(2A,4C;5A) = 250 = |C_G(5A)|/4$, $\#(2A,4A;5A) = 500 = |C_G(5A)|/2$, $\#(2A,4D;5A) = 2000 = 2|C_G(5A)|/4$, $\#(2A,4B;5A) = \#(2A,4C;5B) = \#(2A,4A;5B) = 0$, $\#(2A,4D;5B) = 600 = 2|C_G(5B)|$, $\#(2A,4B;5B) = 300 = |C_G(5B)|$.

The next two lemmas acount for $\#(2A,4B;5B)$ and the half of $\#(2A,4D;5B)$.

Lemma 6.2. *There is a unique class of subgroups of G of type $(2A,4B;5B)$. They are isomorphic to $Sz(8)$ and their normalizers are conjugate to the normalizer of a $2B$-pure four group.*

Proof. For a $5B$-element x, we have $N_G(x) = \langle x,y\rangle\times A\cong 5{:}4\times A_5$, where $A = C_G(x)'$. A Sylow 2-subgroup E of A is a $2B$-pure four group with $N_G(E) = (E\times S){:}Z\cong(2^2\times Sz(8))3$, where $EZ\subseteq A$ and $\langle x,y\rangle\subseteq S$. (See the remark after Prop. 4.2.) Since y is an element of order 4 centralizing a $5A$-element of A, y is a $4A$ or $4B$-element. The group $S = C_G(E)'\cong Sz(8)$ has two non-real classes $4A$ and $4B$. Since involutions and elements of order 5 of S are $2A$ and $5B$-elements, respectively, the structure constants $\#(2A,4A;5A)_S = \#(2A,4B;5A)_S = 30$ and $\#(2A,4A;5B)_G = 0$ show that the element y of S of order 4 is a $4B$-element. By observing the list of maximal subgroups of $Sz(8)$, any one of 60 pairs of $(2A,4B;5B)\cap S$ generates S. Thus, by the usual counting argument, we have $|(2A,4B;x)\cap S(x)| = 60|N_G(x):N_G(x)\cap N_G(S)| = 60\cdot 2^4\cdot 3\cdot 5^2/|EZ\times\langle x,y\rangle| = 300 = \#(2A,4B;x)$, and therefore any pair of $(2A,4B;x)\cap S(x)$ generates a subgroup conjugate to S. As $N_G(S) = N_G(O_2(C_G(S))) = N_G(E)$, the assertions have proved. □

In the proof of Prop. 6.1(3), we have shown that there is a subgroup Σ isomorphic to S_5 which normalizes a subgroup $\langle q\rangle$ of order 3. We let x be an element of Σ of order 5.

Lemma 6.3. *The group Σ is of type $(2A,4D;5B)$ and $|(2A,4D;x)\cap\Sigma(x)| = 300$.*

Proof. The group S_5 is generated by a transposition and an element of order 4 such that their product is of order 5. Since a transposition centralizes an element of order 3, $\Sigma \cong S_5$ is of type $(2A, 4X; 5B)$. By Lemma 6.2 and structure constants, Σ is of type $(2A, 4D; 5B)$, and the latter part of the claim follows from the usual counting argument. □

We will show in §9 that G has a subgroup isomorphic to A_8 containing a $5B$-element. Thus there is, in fact, an A_6-subgroup containing an $5B$-element.

Lemma 6.4.

(1) *Any subgroup of G isomorphic to A_6 is of type $(2A, 4D; 5A)$ or $(2A, 4D; 5B)$.*

(2) *There is a unique class of subgroups isomorphic to A_6 of type $(2A, 4D; 5A)$. Their normalizers are isomorphic to $Aut(A_6)$ and are contained in subgroups isomorphic to ${}^2F_4(2)$.*

(3) *There is a unique class of subgroups isomorphic to A_6 of type $(2A, 4D; 5B)$. Their normalizers are isomorphic to $Aut(A_6)$.*

Proof. The claim (1) follows from the previous remark and Lemma 6.2. Let A be a subgroup isomorphic to A_6 and x an element of A of order 5.

Assume that x is a $5B$-element and A is of type $(2A, 4D; 5B)$. The set $(2A, 4A; 5A)_{A_6}$ consists of 10 pairs and any pair of this set generates A_6. Thus we have $|(2A, 4D; x) \cap A(x)| = 10|N_G(x) : N_G(x) \cap N_G(A)|$, which is not greater than 300 by Lemma 6.3. Then $|N_G(x) \cap N_G(A)| \geq 40$, and so the equality holds and $N_G(A) \cong Aut(A_6)$. This proves the claim (3).

Assume that x is a $5A$-element and A is of type $(2A, 4D; 5A)$. There is a subgroup Σ of G isomorphic to S_7. By identifying Σ with the symmetric group on $\{1, \ldots, 7\}$, we may take $x = (12345)$. Since x centralizes a transposition commuting an element of order 3, x is a $5A$-element and the subgroup B of Σ of even permutations fixing the letter 7 is of type $(2A, 4D; 5A)$. Since $4D$-elements of B are permutations of type 124, the group C of even permutations preserving the partition $\{1, \ldots, 5\}$, $\{6, 7\}$ is also an S_5-subgroup of type $(2A, 4D; 5A)$.

If $(\alpha, \beta) \in (2A, 4D; x)$ is contained in A or B, $C_G(\alpha, \beta) = 1$ and the orbit of $C_G(x)$ containing (α, β) is of length $1000 = |C_G(x)|$. Since $\#(2A, 4D; 5A) = 2 \times 1000$ and there is a subgroup $C \cong S_5$ of type $(2A, 4D; 5A)$, we can conclude that the group A is conjugate to B and the only 1000 pairs of $(2A, 4D; x)$ generate subgroups isomorphic to A_6. Then $N_G(A)$ is isomorphic to $Aut(A_6)$, by the same computation in the case $x \in 5B$.

As we will show in Table 3 of §7, elements of order 5 of a ${}^2F_4(2)$-subgroup F are $5A$-elements and contained in a subgroup of F isomorphic to $Aut(A_6)$ (see [9]). Thus $N_G(A)$ is contained in a ${}^2F_4(2)$-subgroup. □

6.3 Subgroups isomorphic to $Sz(8)$.

As is shown in the proof of Lemma 6.2, a $Sz(8)$-subgroup is of type $(2A, 4X; 5Y)$. Since there is no $2A$-involution centralizing an element of order 7, the possible types are $(2A, 4D; 5A)$, $(2A, 4D; 5B)$, $(2A, 4B; 5B)$. In the first two cases, the usual counting arguments show that these groups are contained in local subgroups. Thus they are the commutator subgroups of centralizers of some $2B$-pure four subgroups. Then they are of type $(2A, 4B; 5B)$, which is a contradiction. Therefore, $Sz(8)$-subgroups are of type $(2A, 4B; 5B)$. Then, by Lemma 6.2, we have the following:

Proposition 6.5. *There is a unique class of subgroups of G isomorphic to $Sz(8)$. They are of type $(2A, 4B; 5B)$ and their normalizers are normalizers of $2B$-pure four groups.*

7 Subgroups $^2F_4(2)'$, $L_3(3)$, $L_2(13)$, $L_2(25)$ and $L_2(29)$.

7.1 Subgroups isomorphic to $^2F_4(2)'$.

The original construction of the Rudvalis group shows that G has a subgroup F isomorphic to $^2F_4(2)$ such that G is a permutation group of rank 3 on the cosets G/F. The character ϕ of the corresponding permutation representation is the sum of the three irreducible characters of G of degrees 1, 783 and 3276.

Table 2: Conjugacy classes of $^2F_4(2)$.

Name	Repre.	F'	$\lvert C_F(g)\rvert$	Class in G	$\lvert C_G(g)\rvert$	$\phi(g)$
$1A$	1	y	$\lvert F\rvert$	$1G$	$\lvert G\rvert$	4060
$2A$	α_{12}	y	$2^{12}5$	$2A$	$2^{14}35$	92
$2B$	α_{10}	y	$2^{10}3$	$2A$	$2^{14}35$	92
$4A$	α_5	n	2^85	$4A$	2^935	32
$4B$	$(4A)^{-1}$	n	2^85	$4A$	2^935	32
$4C$	$\alpha_5\alpha_6$	y	2^73	$4A$	2^935	32
$4D$	$\alpha_7\alpha_8$	y	2^8	$4C$	2^{10}	4
$4E$	$\alpha_2\alpha_6\alpha_8$	n	2^63	$4B$	2^835	20
$4F$	$\alpha_5\alpha_7$	n	2^7	$4D$	2^9	8
$4G$	$\alpha_5\alpha_6\alpha_8$	y	2^7	$4D$	2^9	8
$8A$	$\alpha_2\alpha_4\alpha_6\alpha_8$	y	2^6	$8B$	2^6	4
$8B$	$(8A)^3$	y	2^6	$8B$	2^6	4
$8C$	$\alpha_2\alpha_4$	n	2^5	$8B$	2^6	4
$8D$	$\alpha_2\alpha_3\alpha_4\alpha_6\alpha_7$	y	2^4	$8C$	2^5	2
$8E$	$\alpha_2\alpha_3\alpha_5$	n	2^4	$8A$	2^53	6
$16A$	$\alpha_1\alpha_3$	n	2^4	$16A$	2^4	2
$16B$	$(16A)^3$	n	2^4	$16B$	2^4	2
$16C$	$\alpha_1\alpha_3\alpha_5$	y	2^4	$16A$	2^4	2
$16D$	$(16C)^3$	y	2^4	$16B$	2^4	2
$3A$	t_4	y	2^33^3	$3A$	2^43^35	10
$6A$	$2B \times 3A$	y	2^33	$6A$	2^43	2
$12A$	$3A \times 4C$	y	2^23	$12A$	2^33	2
$12B$	$3A \times 4E$	n	2^23	$12B$	2^23	2
$12C$	$3A \times 4E$	n	2^23	$12B$	2^23	2
$5A$	t_9	y	2^25^2	$5A$	2^35^3	10
$10A$	$2A \times 5A$	y	2^25	$10A$	2^35	2
$20A$	$4A \times 5A$	n	2^25	$20A$	2^25	2
$20B$	$4B \times 5A$	n	2^25	$20A$	2^25	2
$13A$	t_{17}	y	13	$13A$	2^25	4

Since the conjugacy classes of F are known, we can determine the fusion of elements of F in G by the values of ϕ, using the formula $\phi(g) = |C_G(g)| \sum_{i=1}^{s} \frac{1}{|C_F(g_i)|}$, where $\{g_1, \ldots, g_s\}$ is a complete system of representatives of F-classes of $\{g^x | x \in G\}$. The results are found in the above table, where $\alpha_1, \ldots, \alpha_{12}$ are generators of a Sylow 2-subgroup of F satisfying the relations given in [1]. In the first and second columns of this table, F-classes of F are exhibited with thier names and representatives g. In the third column, y and n mean that $g \in F'$ and $g \notin F'$, respectively. The orders of centralizers of g in F and G are given in the fourth and sixth columns, respectively, and the fusion of g in G is shown in the fifth column. The seventh column shows the value of the permutation character ϕ.

Lemma 7.1. *There is a unique class of subgroups of G isomorphic to ${}^2F_4(2)'$. Their normalizers are conjugate to F.*

Proof. Let T be a subgroup of G isomorphic to ${}^2F_4(2)'$. Since the Schur multiplier of T is trivial, the double cover $\tilde{G}$ of G has a subgroup $\tilde{T}$ isomorphic to T. The degrees of the irreducible characters of T not exceeding 28 are 1, 26 and 27. Thus $\tilde{T}$ fixes a non-trivial subspace of the 28-dimensional space V_0 over $\mathbf{Q}(i)$ on which $\tilde{G}$ acts. By reducing modulo 2, we have a G-space V over $\mathbf{F}_2$ of dimension 28 on which T has a non-trivial fixed subspace. Thus the result follows from Table in 2.2.

7.2 Subgroups isomorphic to $L_3(3)$, $L_2(25)$ or $L_2(13)$.

Let L be a subgroup of G isomorphic to $L_3(3)$, $L_2(25)$ or $L_2(13)$. Involutions of L are $2A$-involutions, since they centralize elements of order 3. Then the structure constants of these groups show that L is of type $(2A, 3A; 13A)$. The following lemmas follow from the usual counting arguments.

Lemma 7.2. *There is a subgroup L of G isomorphic to $L_2(25)$ with $N_G(L) \cong Aut(L_2(25))$. For an element x of L of order 13, $|(2A, 3A; x) \cap L| = 2 \cdot 13$ and $|(2A, 3A; x) \cap L(x)| = 3 \cdot 4 \cdot 13$. Any pair of the set $(2A, 3A; x) \cap L$ generates L.*

Lemma 7.3. *A subgroup F of G isomorphic to ${}^2F_4(2)$ contains a subgroup M isomorphic to $L_3(3)$ with $N_G(M) = N_F(M) \cong L_2(13)$. For an element x of M of order 13, we have $|(2A, 3A; x) \cap M| = 13$ and $|(2A, 3A; x) \cap M(x)| = 3 \cdot 4 \cdot 13$.*

Lemma 7.4. *There is a subgroup P of G isomorphic to $L_2(13)$ with $N_G(P) \cong PGL_2(13)$. For an element x of P of order 13, we have $|(2A, 3A; x) \cap P| = 13$ and $|(2A, 3A; x) \cap P(x)| = 4 \cdot 13$.*

Since $\#(2A, 3A; 13A) = 7 \cdot 4 \cdot 13$, the above lemmas imply the following.

Proposition 7.5. *For $L \cong L_2(25)$, $L_2(13)$ or $L_3(3)$, there is a unique class of subgroups of G isomorphic to L. Their normalizers in G are isomorphic to $Aut(L_2(25))$, $PGL_2(13)$ and $Aut(L_3(3))$ for $L \cong L_2(25)$, $L_2(13)$ and $L_3(3)$, respectively. The normalizer isomorphic to $Aut(L_3(3))$ is contained in a subgroup isomorphic to ${}^2F_4(2)$.*

7.3 Subgroups isomorphic to $L_2(29)$.

There is an $L_2(29)$-subgroup of G (see [11]).

Proposition 7.6. *There is a unique class of subgroups of G isomorphic to $L_2(29)$. They are self-normalizing.*

Proof. Let L_i $(i = 1,2)$ be subgroups isomorphic to $L_2(29)$. By Sylow's theorem, we may assume that $L_1 \cap L_2$ contains a Sylow 29-subgroup P of G. Then $N_{L_1}(P) = N_G(P) = N_{L_2}(P) \cong 29:14$. For a complement D of P in $N_G(P)$, we have $N_{L_i}(D) \cong D_{28}$ $(i = 1,2)$. Since an involution of L_i centralizes an element of order 7, it is of class $2B$. Then for the unique involution x of the complement D, $C_G(x) = E \times S \cong 2^2 \times Sz(8)$. As $N_{L_i}(D) \subsetneq C_G(x)$, the involution u_i of $N_{L_i}(D) \setminus D$ is a product of u_i' of E and an involution u_i'' of S inverting $O_7(D)$. Since a Sylow 7-normalizer of S is isomorphic to $7:2$, we may assume that $u_1'' = u_2'' =: v$. Then $u_i' = 1$ and $u_i = x$, since an involution of S is of class $2A$ and $N_{L_i}(D)$ does not contain $v = x \cdot xv$. Thus $u_1' \cdot u_2' \in \langle x \rangle$ and $N_{L_1}(D) = \langle x, O_7(D), u_1' \rangle = \langle x, O_7(D), u_2' \rangle = N_{L_2}(D)$, and then $L_1 = \langle N_G(P), N_{L_1}(D) \rangle = \langle N_G(P), N_{L_2}(D) \rangle = L_2$.

Since $2B$-involutions are not squares, L_1 is not contained in a subgroup isomorphic to $PGL_2(29)$. Thus L_1 is self-normalizing.

8 Subgroups $L_2(7)$ and $U_3(3)$.

8.1 Subgroups isomorphic to $L_2(7)$.

Let $X = \langle z, t, w, a, c, u \rangle$ be the elementary abelian subgroup of order 2^6 in §3. The normalizer $N_G(X)$ is a non-split extension of $G_2(2)$ by X, but X has a complement isomorphic to $U_3(3)$ in $N_G(X)'$ (see [6] Lemma 3.4). In order to compute the structure constant $|(2A, 3A; x) \cap N_G(X)|$ for an element x of order 7 of $N_G(X)$, we recall the following explicit definition of $G_2(2)$ given by Dickson (see [5] and [6]).

Let $V = \mathbf{F}_2^7$ be the vector space of row vectors of length 7 with coefficients in $\mathbf{F}_2$ and e_i $(i = 0, \ldots, 6)$ its natural basis. We define a non-singular quadratic form q on V by $q(\mathrm{x}) = x_0^2 + x_1x_4 + x_2x_5 + x_3x_6$ for $\mathrm{x} = (x_i)_{i=0}^6 \in V$, and identify $O_7(2)$ with the group of matrices A of $GL_2(7)$ preserving q: $q(\mathrm{x}) = q(\mathrm{x}A)$ for any $\mathrm{x} \in V$. We denote by n the matrix of $GL_7(2)$ with (i,j)-entry 1 for $(i,j) = (2,6)$, $(3,5)$, $(4,0)$, $(4,1)$ and $i = j = 0, \ldots, 6$ and (i,j)-entry 0 otherwise. We also define matrices m and $t(Y)$ for $Y \in L_3(2)$ as follows, where I denotes the identity matrix of size 3:

$$m := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}, \; t(Y) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & {}^tY^{-1} \end{pmatrix}.$$

We may verify that m, n and $t(Y)$ for $Y \in L_3(2)$ are contained in $O_7(2)$. By [5], the subgroup $K := \langle m, n, t(Y) | Y \in L_3(2) \rangle$ is isomorphic to $G_2(2)$. Since K acts trivially on $\mathbf{F}_2\mathbf{1}$, where $\mathbf{1} = \sum_{i=0}^6 \mathrm{e}_i$, K acts on the 6-dimensional space $W := V/\mathbf{F}_2\mathbf{1}$. The image of a subset X of V in W is denoted by $\overline{X}$. Note that K' is isomorphic to $U_3(3)$ and contains the group $L := \{t(Y) | Y \in L_3(2)\} \cong L_3(2)$.

We also set $x = t(X)$, $\alpha = t(A)$ and $\beta = t(B)$ for

$$X := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \; A := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \text{ and } B := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix};$$

and $\mathrm{x}_1 := \overline{0}$, $\mathrm{x}_2 := (0;0,0,0;0,0,1)$, $\mathrm{x}_3 := (0;1,0,1;0,0,0)$ and $\mathrm{x}_4 := \mathrm{x}_2 + \mathrm{x}_3$. In the semidirect product $W:L$ with respect to the natural action of L on the space W, we define $L_i := \langle \mathrm{x}_i\alpha, \mathrm{x}_i\beta \rangle$ for $i = 1, \ldots, 4$. Finally, we set $V_+ := \langle \mathrm{e}_i | i = 0,1,2,3 \rangle$ and $V_- := \langle \mathrm{e}_i | i = 0,4,5,6 \rangle$.

Then we can verify the following lemma by straightforward calculation.

Lemma 8.1. *Under the notation above, we have the following results:*

(1) *The elements* α, β *and* x *are of order* 2, 3 *and* 7, *respectively. We have* $\alpha\beta = x$ *and* $C_W(\alpha) \cap [W, \beta] = \{\mathbf{x}_i | i = 1, \ldots, 4\}$.

(2) *The groups* L_i *are isomorhpic to* $L_3(2)$ *for* $i = 1, 2, 3, 4$. *In the semidirect product* $W : L$, *we have* $\langle L_1, L_2 \rangle = \overline{V_+} : L$ *and* $\langle L_3, L_4 \rangle = \overline{V_-} : L$.

In the rest of this subsection, we consider $(2A, 3A; x)$ for an element x of order 7 of $N_G(X)$. We note that $C_G(x) \cong 2^2 \times 7$, $N_G(x) \cong (2^2 \times 7)3$ and $O_2(C_G(x))$ is a $2B$-pure four subgroup.

Lemma 8.2. *For an element* x *of order* 7 *of* $N := N_G(X) \cong 2^6 \cdot G_2(2)$, *the following hold:*

(1) *We have* $|(2A, 3A; x) \cap N| = 4 \cdot 7$ *and* $|(2A, 3A; x) \cap N(x)| = 4 \cdot 4 \cdot 7$.

(2) *There are four subgroups* L_i $(i = 1, \ldots, 4)$ *of* N *of type* $(2A, 3A; x)$. *They are isomorphic to* $L_2(7)$.

(3) *For an involution* u *of* N *inverting* x, *we have* $(N \supseteq)\ N_G(L_i) = \langle L_i, u \rangle \cong PGL_2(7)$ *for* $i = 1, 2$, $N_G(L_i) = L_i$ *for* $i = 3, 4$, $L_3^u = L_4$, *and* $\langle L_1, L_3 \rangle^u = \langle L_1, L_4 \rangle \cong 2^3 : L_3(2)$. *Furthermore,* L_1 *is not conjugate to* L_2 *in* G.

(4) *There are two complements* U *and* V *of* X *in* N' *such that* $U \cap V$ *is a Sylow* 3*-normalizer in both* U *and* V. *The groups* U *and* V *are self-normalizing and are conjugate in* N, *but not in* N'.

Proof. Set $\mathcal{X} := (2A, 3A; x) \cap N$ and $M := N' \cong 2^6 : U_3(3)$. We let U be a complement of X in M containing x. Since groups of type $(2A, 3A; 7A)$ are perfect, we have $\mathcal{X} = \mathcal{X} \cap M$. Then for any $(\alpha, \beta) \in \mathcal{X}$, we may uniquely write $\alpha = \alpha_1\alpha_2$ and $\beta = \beta_1\beta_2$, where $\alpha_1, \beta_1 \in X$ and $\alpha_2, \beta_2 \in U$. We may verify that $(\alpha, \beta) \in \mathcal{X}$ if and only if $(\alpha_2, \beta_2) \in \mathcal{X} \cap U$ and $\alpha_1 = \beta_1 \in C_X(\alpha_2) \cap [X, \beta_2]$. Note that all faithful representations of $U_3(3)$ of dimension 6 over $\mathbf{F}_2$ are equivalent to each other (see [6], Lemma 15). Then we may take as X and U the groups W and K' in Lemma 8.1, respectively. Since x acts regularly on $\mathcal{X} \cap U$, it follows from Lemma 8.1 that for each $(\alpha_2, \beta_2) \in \mathcal{X} \cap U$ there are four elements $\alpha_1 \in X$ with $(\alpha_1\alpha_2, \alpha_1\beta_2) \in \mathcal{X}$. Thus $|\mathcal{X} \cap U| = 4 \cdot 7$ and any pair of $\mathcal{X} \cap U$ generates an $L_2(7)$-subgroup. There are four such $L_2(7)$-subgroups in total.

As $N_N(x) \cong 7:6$, the set $\{N^g | g \in N_G(x)\}$ consists of four conjugates on which $O_2(C_G(x))$ acts regularly. We will show that $\mathcal{X} \cap N^g = \emptyset$ for any $1 \neq g \in O_2(C_G(x))$. Suppose (α, β) lies in $\mathcal{X} \cap N^g$ for some $1 \neq g \in O_2(C_G(x))$. Then $L := \langle \alpha, \beta \rangle$ is an $L_2(7)$-subgroup of $M \cap M^g$. Since $N_L(x)$, $N_M(x)$ and $N_{M^g}(x)$ are isomorphic to $7:3$, we have $N_L(x) = N_M(x) = N_{M \cap M^g}(x)$, and therefore the involution g acts on $N_L(x)$. On the other hand, since g lies in the normal subgroup $O_2(C_G(x))$ of $N_G(x)$, an element q of order 3 of $N_L(x)$ acts on $O_2(C_G(x))$. Then q centralizes an $2B$-involution g, which is a contradiction. Thus we proved the above claim, and then we have $|(2A, 3A; x) \cap N(x)| = 4|\mathcal{X}| = 4 \cdot 4 \cdot 7$. The assertions (1) and (2) are verified.

To show the assertions (3) and (4), we will first observe that the involution u normalizes $L = \langle \alpha_2, \beta_2 \rangle$ for any $(\alpha_2, \beta_2) \in \mathcal{X} \cap U$. Note that the maximal subgroups of $U_3(3)$ are of indices 28, 36 and 63 and those of index 36 are isomorphic to $L_2(7)$. Thus U is transitive on $X \setminus \{1\}$, and $U \cap U^h = C_U(h)$ is a subgroup of U of index 36 for any $1 \neq h \in X$. We set $\Omega := \{U^g | g \in M\} = \{U^g | g \in X\}$ and $V := U^u$. If $U = V$, the group $\langle U, u \rangle$ $(\cong G_2(2))$ would be a complement of X in N. Thus $U \neq V$. As $x \in U \cap V$, $V \notin \Omega$. Since U and V contain Sylow 3-subgroups of M, $V^k \cap U$ contains $N_U(R)$ for some $k \in M$, where R is a Sylow 3-subgroup of U. We have $V^k \cap U = N_U(R)$, since R is maximal in U and $V \notin \Omega$. Then the V-orbit on Ω containing $U^{k^{-1}}$ is of length $|V : V \cap U^{k^{-1}}| = 28$. In particular, U is not contained in this orbit and the V-orbit containing U is of length at most 36. Then this length is $36 = |V:U \cap V|$, and therefore $U \cap V \cong L_2(7)$. Thus for any pair $(\alpha_2, \beta_2) \in \mathcal{X} \cap U$, we have $L = \langle \alpha_2, \beta_2 \rangle = U \cap V$ and L is normalized by the involution u.

For $L_1 := L$, we have $N_G(L_1) = \langle L_1, u \rangle \subseteq N$, as $C_G(L_1) = 1$. By Lemma 8.1, we may assume that $\langle L_1, L_3 \rangle \cong \langle L_1, L_4 \rangle \cong 2^3 : L_2(7)$. Suppose L_i is normalized by u for $i = 3$ or 4. Then $\langle L_1, u \rangle \cong PGL_2(7)$ acts on $O_2(\langle L_1, L_i \rangle) \cong 2^3$, and therefore L_1 centralizes this 2^3-subgroup, which is a contradiction. Thus $\langle L_1, L_3 \rangle^u = \langle L_1, L_4 \rangle$. Since u acts on the set $\{L_i | i = 1, \ldots, 4\}$ of $L_2(7)$-subgroups of N containing x, we have $L_3^u = L_4$, $L_2^u = L_2$, and $N_G(L_2) = \langle L_2, u \rangle \subseteq N$. Suppose $N_G(L_i)$ properly contains L_i for $i = 3$ or 4. Then $N_G(L_i) \cong PGL_2(7)$ and there is an involution u' of $N_G(L_i)$ inverting x. Such an involution u' lies in the coset $uO_2(C_G(x))\langle x \rangle$. As is shown in the proof of the assertion (1), the group $M \cap M^g$ does not contain an $L_2(7)$-subgroup for any $1 \neq g \in O_2(C_G(x))$. Then $u' \in u\langle x \rangle$, which is a contradiction. Thus L_i are self-normalizing for $i = 3, 4$. Since the hypothesis $U \neq N_G(U)$ implies the existence of an involution u' of $N_G(U)$ inverting x, we may show that the $U_3(3)$-subgroup U is self-normalizing, by the same reasoning as above. Furthermore, the same reasoning can be used to derive a contradiction from the assumption that $L_1^g = L_2$ for some $g \in G$, because it implies that we may take $g \in N_G(x)$. Hence assertions (3) and (4) have been proved. □

Lemma 8.3. *For an S_7-subgroup Σ of G containing an element x of order 7, $|(2A, 3A; x) \cap \Sigma| = 2 \cdot 7$ and $|(2A, 3A; x) \cap \Sigma(x)| = 4 \cdot 2 \cdot 7$ hold. There are two subgroups of Σ of type $(2A, 3A; x)$ containing x, which are conjugate $L_2(7)$-subgroups and are self-normalizing.*

Proof. Involutions of Σ are $2A$-involutions, since they commute with elements of order 3. By the character table of S_7, $|(2A, 3A; x) \cap \Sigma| = 2 \cdot 7$ and there are two subgroups of Σ of type $(2A, 3A; x)$ containing x. Since Σ is self-normalizing and $N_\Sigma(x) \cong 7:6$, $\{\Sigma^g | g \in N_G(x)\} = \{\Sigma^g | g \in O_2(C_G(x))\}$. Suppose the set $(2A, 3A; x) \cap \Sigma \cap \Sigma^g$ contains a pair (α, β) for $1 \neq g \in O_2(C_G(x))$. Then, since g acts on $\langle \alpha, \beta \rangle \cap N_G(x) = O^2(N_\Sigma(x)) = (\Sigma \cap \Sigma^g) \cap N_G(x)$, the $2B$-involution g centralizes an element of order 3, which is a contradiction. Thus $|(2A, 3A; x) \cap \Sigma(x)| = 2 \cdot 7 |N_G(x) : N_G(x) \cap \Sigma| = 4 \cdot 2 \cdot 7$. □

Lemma 8.4. *For an S_7-subgroup Σ of G containing an element x of order 7 of $N := N_G(X) \cong 2^6 \cdot G_2(2)$, we have $(2A, 3A; x) \cap \Sigma(x) \cap N(x) = \emptyset$ and $|(2A, 3A; x) \cap (\Sigma(x) \cup N(x))| = 6 \cdot 4 \cdot 7$.*

Proof. Suppose there is a pair (α, β) of $(2A, 3A; x) \cap \Sigma \cap N$. We will observe that this assumption leads to a contradiction. Then the lemma immediately follows from usual counting arguments (see §2.1). We set $L := \langle \alpha, \beta \rangle$ and consider the G-space V of dimension 28 over $\mathbf{F}_2$ (see §2.2). Note that any irreducible $L_3(2)$-module over $\mathbf{F}_2$ is equivalent to one of the following: trivial module V_1, the natural module V_3 of dimension 3, its contragradient V_3^*, and the 8-dimensional module V_8 contained in $V_3 \otimes V_3^*$. We know dimensions of subspaces of these modules fixed by elements of odd prime orders of $L_3(2)$, along with dimensions of subspaces of V fixed by elements of G of odd prime orders. Thus, by DIM-argument in §2.2, we can determine the multiplicity of the trivial modules among the L-composition factors of V as an L-module over $\mathbf{F}_2$. It is 2.

We use the notation in Table in §2.2. A vector $0 \neq z \in V$ is of type g or s if and only if the stabilizer G_z contains an $L_2(7)$-group. There are vectors x and y of V of type g and s with the stabilizers $G_x = N$ and $G_y = \Sigma$ in G, respectively. Since $L \cong L_2(7)$ fixes vectors x and y, $C_V(L) = \langle x, y \rangle$ by the remark above. Any vector z $(\neq 0, x, y)$ of $\langle x, y \rangle$ is of type g or s. If it is of type g, there is a conjugate $N^m(\neq N)$ with $L \subset N \cap N^m$, which is a contradiction by the proof of Lemma 8.2 (1). If z is of type s, we also have a contradiction by Lemma 8.3 (1). □

Lemma 8.5. *We have* $|(2A, 3A; x) \cap L| = 2 \cdot 7$ *and* $|(2A, 3A; x) \cap L(x)| = 3 \cdot 4 \cdot 7$ *for an* $L_2(13)$-*subgroup* L *containing an element* x *of order* 7. *Any pair* $(\alpha, \beta) \in (2A, 3A; x) \cap L$ *generates* L.

Proof. By the character table of $L_2(13)$, we have $|(2A, 3A; x) \cap L| = 2 \cdot 7$. Since any proper subgroup of $L_2(13)$ are solvable and subgroups of type $(2A, 3A; 7X)$ are non-solvable, $L = \langle \alpha, \beta \rangle$ for any $(\alpha, \beta) \in (2A, 3A; x) \cap L$. In particular, $(2A, 3A; x) \cap L \cap L^g = \emptyset$ for $g \in N_G(x)$ with $L \neq L^g$. Since $N_G(L) \cong PGL_2(13)$ by Lemma 7.4 and $N_G(x) \cap N_G(L) \cong 2 \times D_{14}$, we have $|(2A, 3A; x) \cap L(g)| = 2 \cdot 7 \cdot |N_G(x) : N_G(L)| = 3 \cdot 4 \cdot 7$. □

As $\#(2A, 3A; 7A) = 9 \cdot 4 \cdot 7$, we have the following conclusion about conjugacy classes of $L_2(7)$-subgroups by Lemmas 8.2–5.

Proposition 8.6. *There are four classes of subgroups of* G *isomorphic to* $L_2(7)$, *with representatives* L_i $(i = 1, \ldots, 4)$ *satisfying the following properties: The normalizers of* L_1 *and* L_2 *in* G *are isomorphic to* $PGL_2(7)$ *and* L_3 *and* L_4 *are self-normalizing. The normalizers of* L_i $(i = 1, 2, 3)$ *are contained in* $N_G(X) \cong 2^6 \cdot G_2(2)$ *and* L_4 *is contained in a subgroup of* G *isomorphic to* S_7.

8.2 Subgroups isomorphic to $U_3(3)$.

Proposition 8.7. *There is a unique class of subgroups of* G *isomorphic to* $U_3(3)$. *They are self-normalizing and are conjugate to a subgroup of* $N_G(X) \cong 2^6 \cdot G_2(2)$.

Proof. Let U be a $U_3(3)$-subgroup of $N := N_G(X)$ and R a Sylow 3-subgroup of U. The group R is a Sylow 3-subgroup of G with normalizers $N_U(R) \cong 3^{1+2}{:}8$ in U and $N_G(R) \cong 3^{1+2}{:}SD_{16}$. Let h be a generator of a complement of R in $N_U(R)$. Since h^2 is an element of order 4 centralizing $Z(R)$, h^2 is a $4A$-element of G and so $h \in (8A)$. We recall that there is a conjugate V of U in N with $U \cap V = N_U(R)$ by Lemma 8.2 (4). Take any $U_3(3)$-subgroup M of G. To show that M is conjugate to U, we may assume that $R \subseteq M$ by Sylow's theorem, and therefore that $N_U(R) = N_M(R) \subseteq M$. We will show that $M = U$ or V.

First, we will observe that there is an involution g of M satisfying $h^g = h^{-3}$, $[g, q]$ is of order 8 for any $1 \neq q \in Z(R)$, and $M = \langle R, h, g \rangle$. To show this claim, we identify M with the matrix group $\{A \in GL_3(\mathbf{F}_9) | AJ{}^t\overline{A} = J\}$, where $\overline{A} = (a_{ij}^3)$ for $A = (a_{ij})$ and J is the matrix of size 3 with (i, j)-entry 1 for $i + j = 4$ and 0 otherwise. We may assume that R consists of matrices $\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & -a^3 & 1 \end{pmatrix}$ for $a, b \in \mathbf{F}_9$ with $a + a^3 + b^4 = 0$, h is the diagonal matrix with (i, i)-entries η^{-3}, η^2, η for $i = 1, 2, 3$, respectively, where η is a generator of $\mathbf{F}_9^*$. Let v be the matrix with (i, j)-entry $1, -1$, and 1 for $(i, j) = (1, 3), (2, 2)$ and $(3, 1)$, respectively, and 0 otherwise. Then we may verify that v is an involution satisfying the required properties. (Note that $\langle R, h \rangle$ is maximal in M.) In particular, U and V contain involutions g_1 and g_2 with the above properties, respectively. We have $\langle h, g_1 \rangle \neq \langle h, g_2 \rangle$.

We take a ${}^2F_4(2)$-subgroup F of G containing R. Then $N_G(R) = N_F(R)$. We will observe that there are involutions g_3 and g_4 satisfying the following properties: $h^{g_3} = h^{g_4} = h^{-3}$, $g_3 g_4$ is an involution, $\langle h, g_3 \rangle \neq \langle h, g_4 \rangle$ and $\langle R, h, g_3 \rangle = \langle R, h, g_4 \rangle = F$. To show this, we identify

$h \in (8A)_G \cap F$ with the element $\alpha_2\alpha_3\alpha_5$ in Table in §7.1. Then, by using the commutator formulas in [1], we may verify that $g_3 =$ and $g_4 = \alpha_8\alpha_9\alpha_{11}\alpha_{12}$ satisfies the former three properties in the above. (Note that $h^4 = \alpha_{10}$.) Since $N_F(R)$ does not contain $\langle h, g_3\rangle$ or $\langle h, g_4\rangle$, we have $\langle R, h, g_3\rangle = F = \langle R, h, g_4\rangle$ by observing the list of maximal subgroups of ${}^2F_4(2)'$ (see [9],[10]).

Next we will consider the group $\langle h, g\rangle$ for an involution g with $h^g = h^{-3}$. In the rest of this proof, we use notation in §3. Note that x_2u is of class $(8A)$ (see [7] Table 2) and $x_2uy = (x_2u)^{vay_2}$. By using commutator formulas in §2, we can verify that $C_G(x_2u) = \langle t, w, x_2u, q_1\rangle$ and $N_G(x_2u) = C_G(x_2u)\langle y, u\rangle$, where q_1 is an element of order 3 with $[q_1, u] = 1$ and $q_1^y = q_1^{-1}$. The element q_1 acts on $\langle t, w\rangle$. We may assume that $t^{q_1} = tw$ and $(tw)^{q_1} = w$. Now, taking suitable conjugates of R, U and N, we may assume that $Z(R) = \langle q_0\rangle$, where $q_0 := q_1^{y_2av}$. Then $x_2u = (x_2u \cdot y)^{y_2av}$ inverts q_1, and so $\langle h\rangle$ and $\langle x_2u\rangle$ are two cyclic subgroups of order 8 of $N_G(q_0) \setminus C_G(q_0)$. Since $N_G(q_0)/\langle q_0\rangle \cong Aut(A_6)$, the group $\langle x_2u, u, y\rangle^{y_2av}$ is a Sylow 2-subgroup of $N_G(q_0)$. Then $\{x_2uy, (x_2uy)^3, (x_2uy)^{-3}\}^{y_2av}$ coincides with the set of elements of order 8 of $\langle x_2u, u, y\rangle^{y_2av} \setminus C_G(q_0)$, and so $\langle h\rangle$ is conjugate to $\langle x_2u\rangle$ in $N_G(q_0)$. Thus, by taking suitable conjugates of R, U and N under $N_G(q_0)$, we may assume that $Z(R) = \langle q_0\rangle$ and $\langle h\rangle = \langle x_2u\rangle$. Then if g is an involution with $h^g = h^{-3}$, we have $g \in N_G(x_2u)$. By observing this explicitly known group $N_G(x_2u)$, we may conclude that $\langle h, g\rangle$ is one of the following six subgroups:

$$A_1 := \langle h, y\rangle,\ A_2 := \langle h, ty\rangle,\ A_3 := \langle h, q_1y\rangle,$$
$$A_4 := \langle h, tw, q_1y\rangle,\ A_5 := \langle h, q_1^{-1}y\rangle \text{ or } A_6 := \langle h, wq_1^{-1}y\rangle.$$

Hence, by the preceeding remark, the $U_3(3)$-subgroup M containing $R\langle h\rangle = N_U(R)$ coincides with $\langle R, A_i\rangle$ for some $i = 1, \ldots, 6$ and $F = \langle R, A_{j_1}\rangle = \langle R, A_{j_2}\rangle$ for some $j_1, j_2 = 1, \ldots, 6$.

Finally, we will show that $\langle R, A_i\rangle$ is not isomorphic to $U_3(3)$ nor ${}^2F_4(2)$ for $i = 1, 2$. Since y centralizes y_2av, y acts on $Z(R) = \langle q_0\rangle$. Then $\langle R, h, y\rangle \subseteq N_G(q_0)$ and the claim has proved for $i = 1$. As $q_0, ty, h \in H = C_G(z)$, we have $[q_0, ty] \equiv [q_0, kty] \equiv q_0 \pmod{O_2(H)}$ for any element k of A_2. Then $[q_0, k]$ is not of order 8 for any $k \in A_2$, and therefore $\langle R, A_2\rangle \not\cong U_3(3)$ by the preceeding remark about $U_3(3)$. Furthermore, we can show that $\langle R, A_2\rangle \not\cong {}^2F_4(2)$ as follows. If $\langle R, A_2\rangle \cong {}^2F_4(2)$, by the preceeding remark about ${}^2F_4(2)$, there is an involution g of A_2 such that $(ty)g$ is an involution and $\langle R, h, g\rangle \cong {}^2F_4(2)$. By computing modulo $O_2(H)$ as above, we have $\langle h, g\rangle = A_2$, which is a contradiction.

Hence we conclude that $\langle R, A_i\rangle \cong {}^2F_4(2)$ for exactly two indices i of $\{3, \ldots, 6\}$ and $\langle R, A_j\rangle \cong U_3(3)$ for the other indices j. Thus the latter two groups must coincide with U and V. Thus $M = U$ or V, which proved the uniqueness of conjugacy class of $U_3(3)$-subgroups. Then the claim about normalizers follows from Lemma 8.2 (4). □

9 Subgroups A_7, A_8 and $U_3(5)$.

9.1 Subgroups isomorphic to A_7.

There is a subgroup U of G isomorphic to $U_3(3)$ with $N_G(U) \cong U_3(5).2$ (see [3]). The group $U_3(5)$ has two classes of elements of order 5. By suitably numbering classes $(5X)_U$ for $X = B, C, D$, we may assume that $(5A)_U$ and $(5B)_U$ are contained in $(5A)_G$ and the other two classes are contained in $(5B)_G$. There is an A_7-subgroup A of U containing an element of $(5X)_U$ for each $X = B, C, D$. Then they are A_7-subgroups containing both $5A$ and $5B$-elements of G. The normalizer in U of an A_7-subgroup containing $(5B)_U$-elements is isomorphic to S_7 (see [3]).

Lemma 9.1. *For an element a of order 3 and an element h of order 4 with $[a, a^h] = 1$ and $a^{h^2} = a^{-1}$, we denote by $\mathcal{S}(a,h)$ the set of $2A$-involutions u of G satisfying the relations $a^u = a^{-1}$, $[a, h^2] = 1$, and that $a^h \cdot u$ is of order 4. Then $\mathcal{S}(a,h)$ consists of 9 involutions and is divided into three cosets by $\langle a \rangle$.*

Proof. Since $N_G(a)/\langle a \rangle \cong Aut(A_6)$, there are subgroups $M := C_G(a)$, L and Σ with $M/\langle a \rangle \cong M_{10}$, $L/\langle a \rangle \cong PGL_2(9)$ and $\Sigma/\langle a \rangle \cong S_6$. Involutions of $L \setminus L'$ are $(2B)_G$-involutions (see [7]). We identify $\overline{\Sigma} := \Sigma/\langle a \rangle$ with the symmetric group on six letters $\{1, \ldots, 6\}$ and use the bar notation. Since u and h^2 are $(2A)_G$-involutions of $N_G(a)$, they correspond to odd permutations of $\overline{\Sigma}$. Then we may assume that $\overline{h}^2 = (12)$. Since $\overline{a}^{\overline{h}}$ is an element of order 3 inverted by $\overline{h}^2$, we may assume that $\overline{a}^{\overline{h}} = (123)$. Thus the odd permutation $\overline{u}$ commuting with $\overline{h}^2$ is (34), (35) or (36), and then the claim follows. □

Proposition 9.2.

(1) *There is a unique class of subgroups of G isomorphic to A_7 containing $(5A)_G$-elements. Their normalizers are isomorphic to S_7 and are conjugate to a subgroup of a $U_3(5)$-subgroup.*

(2) *There is a unique class of subgroups of G isomorphic to A_7 containing $(5B)_G$-elements. They are self-normalizing.*

Proof. Let A be an A_7-subgroup of G and R a Sylow 3-subgroup of A. By Prop. 6.4 (2)(3), $N_G(B) \cong Aut(A_6)$ for any A_6-subgroup B of A containing R. Then $N_G(B) \cap N_G(R)$ is a Frobenius group isomorphic to $3^2{:}8$. Let q be a generator of a complement of R in this group and let $h := q^2$. Then $N_A(R) = N_B(R) = R\langle h \rangle$ and there is an element $a \in R$ with $[a, a^h] = 1$ and $a^{h^2} = a^{-1}$. By observing A_7, we may conclude that there is an involution $u \in \mathcal{S}(a,h)$ (see Lemma 9.1) with $\langle a, h, u \rangle = A_7$. (If we identify A with the alternating group on $\{1, \ldots, 7\}$, we may take $R = \langle (123), (456) \rangle$ and $h = (14)(2536)$. Then $a = (123)$ and $u = (23)(47)$.) Note that such an involution u is a $(2A)_G$-involution, since involutions of A_7 are square elements.

Take another A_7-subgroup C of G. We will observe that there is an involution $v \in \mathcal{S}(a,h)$ such that $\langle a, h, v \rangle$ is a conjugate of C. Since 3^2-subgroups of G are conjugate to each other, we may assume that $R \subseteq C$. Since Sylow 2-subgroups of $N_G(R)$ are isomorphic to SD_{16}, $N_G(R)$ acts transitively on the set of elements of order 4 with square roots in $N_G(R)$. Thus we may assume that $R\langle h \rangle \subseteq C$. Thus if we let $\mathcal{S}(a,h) = u_1\langle a \rangle \cup u_2\langle a \rangle \cup u_3\langle a \rangle$ (see Lemma 9.1), C coincides with one of $\langle a, h, u_i \rangle$ $(i = 1,2,3)$ by Lemma 9.1.

Since there is an S_7-subgroup of G, we may assume that $\langle a, h, u_1 \rangle \cong A_7$ and this subgroup is contained in an S_7-subgroup of G. By observing S_7, we may conclude that there is an involution m of $N_G(\langle a, h, u_1 \rangle)$ with $a^m = a^{-1}$, $h^m = h^{-1}$ and $[u_1, m] = 1$. (In the identification of S_7 above, we may take $m = (23)$.) Since there is an A_7-subgroup of G containing a $(5B)_G$-element, we may assume that $\langle a, h, u_2 \rangle$ is such an A_7-subgroup. If its normalizer is isomorphic to S_7, then a transposition of this S_7-subgroup centralizes both $3A$ and $5B$-elements, which is a contradiction. Thus $\langle a, h, u_2 \rangle$ is self-normalizing. Since the above involution m acts on the set $\mathcal{S}(a,h)$, we have $\langle a, h, u_3 \rangle = \langle a, h, u_2 \rangle$.

Summarizing the arguments in the paragraphs above, the lemma has proved. □

9.2 Subgroups isomorphic to A_8.

Proposition 9.3. *There is a subgroup of G isomorphic to A_8.*

Proof. Let B be an A_7-subgroup of G containing a $(5B)_G$-element. We identify B with the alternating group on letters $\{1,\dots,7\}$. We define elements a, h, u and b of B by $a := (123)$, $h := (14)(2536)$, $u := (23)(47)$ and $b := a^h = (456)$ (see the proof of 9.2). Recall that the alternating group A_{n+2} has a presentation $\{a_1,\dots,a_n|\mathcal{R}\}$, where $\mathcal{R}$ is the set of the following relations: $a_1^3 = a_i^2 = 1$ for $i = 2,\dots,n$, $(a_ia_{i+1})^3 = 1$ for $i = 1,\dots,n-1$, and $(a_ia_j)^2 = 1$ for any $i,j = 1,\dots,n$ with $|i-j| > 1$. We set elements a_i $(i = 1,\dots,5)$ of B by $a_1 := a = (123)$, $a_2 := (uhuhu)^{b^{-1}} = (23)(16)$, $a_3 := h^2 = (23)(56)$, $a_4 := (h^2)^{b^{-1}} = (23)(45)$, and $a_5 := u = (23)(47)$. We may verify that these a_i $(i = 1,\dots,5)$ satisfy the above relations for $n = 5$. Thus $A_7 \cong \langle a_i|i = 1,\dots,5\rangle = \langle a,h,u\rangle = B$. As $N_G(\langle a,b\rangle) \cong 3^2 : GL_2(3)$, there is an involution m of $N_G(\langle a,b\rangle)$ with $a^m = a^{-1}$, $b^m = b$ and $h^m = h^{-1}$. Since m commutes with the element b of order 3, m is a $(2B)_G$-involution. We will observe that the elements a_i $(i = 1,\dots,5)$ above and $a_6 := m$ satisfy the above presentation of A_{n+2} for $n = 6$.

The elements b, h^2, u and m are contained in $N_G(a)$. Then they are contained in a subgroup Σ of $N_G(a)$ with $\overline{\Sigma} := \Sigma/\langle a\rangle \cong S_6$. By identifying $\overline{\Sigma}$ with the symmetric group of six letters $\{1,\dots,6\}$, we may assume that $\overline{b} = (123)$, $\overline{h^2} = (12)$ and $\overline{u} = (35)$ by arguments in Lemma 9.1. Since m is an odd permutation commuting with $\overline{b}$ and $\overline{h^2}$, $m = (45)$, (56) or (46). Since $B = \langle a,h,u\rangle$ is self-normalizing, we have $m = (56)$ or (46). Thus $\overline{um}$ and $um = a_5a_6$ are of order 3.

Since $[h^2,m] = 1 = [b,m]$, the elements $a_3a_6 = h^2m$ and $a_4a_6 = (h^2m)^{b^{-1}}$ are involutions. Furthermore, $a_1a_6 = am$ is of order 2, since m inverts a.

The final relation we have to establish is $(a_2a_6)^2 = 1$. Note that a_2a_6 is an involution if and only if $[uhuhu,m] = 1$. We set $\overline{N_G(h^2)} = N_G(h^2)/\langle h^2\rangle$ and will use the bar convention. The elements u, h and m lie in $N_G(h^2)$. Since $[\overline{h},\overline{m}] = 1$ and um is of order 3, the elements $\overline{um}$ and $\overline{u^h m}$ are of order 3. Since $u^hu = h^2\cdot(147)$ lies in B, the element $\overline{u^h u}$ is of order 3. As $|C_G(h^2)|_3 = 3$, we may verify that the subgroup $\langle\overline{u},\overline{u^h},\overline{m}\rangle$ generated by three involutions is isomorphic to S_3 and the relation $(\overline{u^{hu}})^{\overline{m}} = \overline{u^{hu}}$. Thus $(uhuhu)^m = uhuhu$ or $(uhuhu)\cdot h^2$. If the latter case occurs, the product of $(uhuhu)^m = (14)(56)$ and $a_4 = (23)(45)$ is $(1564)(23)$ and is of order 4. However, this product is conjugate to $uhuhu\cdot a_4 = (154)$, since $[a_4,m] = 1$, which is a contradiction. Thus we have $(uhuhu)^m = uhuhu$, and so the final relation $(a_2a_6)^2 = 1$.

Since A_8 is simple and the elements a_i $(i = 1,\dots,6)$ are not identity elements, we have $\langle a_i|i = 1,\dots,6\rangle \cong A_8$. Thus this is an A_8-subgroup of G. □

Proposition 9.4. *There is a unique class of subgroups of G isomorphic to A_8. They are self-normalizing.*

Proof. Take any A_8-subgroup A of G. Since any element of order 5 of A commutes with elements of order 3, it is of class $(5B)_G$. If $N_G(A) \neq A$, $N_G(A) \cong S_7$ contains a transposition centralizing $(5B)_G$-elements, which is a contradiction. Thus A is self-normalizing.

There are A_7-subgroups $B_2 \cap B_3 \cong A_6$. Then, using the notation in the proof of Prop. 9.2, we may assume that $B_2 = \langle a,h,u_2\rangle$ and $B_3 = \langle a,h,u_3\rangle$. We have $A = \langle B_2,B_3\rangle$. Since any A_8-subgroup C has a conjugate C^g containing $\langle a,h\rangle$, we have $A = C^g$. □

9.3 Subgroups isomorphic to $U_3(5)$.

Proposition 9.5. *There is a unique class of subgroups of G isomorphic to $U_3(5)$. Their normalizers are isomorphic to $U_3(5).\,2$, the extension of $U_3(5)$ by the field automorphism.*

Proof. Let $E = \langle a, b\rangle$ be a 3^2-subgroup of G and C a complement ($\cong GL_2(3)$) of E in $N_G(E)$. We identify a and b with the natural basis of $\mathbf{F}_3^2$, the vector space of row vectors of length 2 with entries in $\mathbf{F}_3$, and identify each element of C with the matrix representing its action on E. We take elements $c := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $h := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $m := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. By Lemma 9.1, the set $\mathcal{S}(a,h)$ is divided into three cosets $u_i\langle a\rangle$ ($i = 1,2,3$). We may assume that $A = \langle a, h, u_1\rangle$ and $B = \langle a, h, u_2\rangle$ are A_7-subgroups containing $(5A)_G$ and $(5B)_G$-elements, respectively, by the proof of Prop. 9.2. Note that $B^m = \langle a, h, u_3\rangle$ and $A^m = A$, since m acts on $\mathcal{S}(a,h)$.

We take any $U_3(5)$-subgroup U of G. We may assume that $A \subseteq U$. As $N_U(E) \cong 3^2{:}Q_8$, we have $N_U(E) = E\langle h, h^c\rangle$. By observing $U_3(5)$, we may find an involution $v \in \mathcal{S}(a, h^c) = \mathcal{S}(a,h)^c$ such that $\langle a, h^c, u\rangle$ is an A_7-subgroup of U. The group $\langle a, h^c, u\rangle$ contains an element of order 5 which is conjugate to no element of order 5 of A, and therefore it is B^c or B^{mc} by the remark above. Since B^c and B^{mc} do not contain h and $h^{c^{-1}}$, respectively, the maximality of A_7 in $U_3(5)$ implies that $U = \langle B^c, h\rangle$ or $U = \langle B^{mc}, h^{c^{-1}}\rangle = \langle B^c, h\rangle^{c^{-1}mc}$. (Note that m inverts c and h.) Thus $U_3(5)$-subgroups containing A are conjugate to each other, and the conjugacy of $U_3(5)$-subgroups of G has proved. Suppose $N_G(U) \setminus U$ contains an element of order 3. Then it centralizes an central element of order 5 of U, which contradicts the fact that central elements of order 5 of U are $(5A)_G$-elements. Thus $N_G(U) \cong U_3(5).\,2$. □

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