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WANDERING VECTORS OF FINITE SUBDIAGONAL ALGEBRAS

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ABSTRACT. In this note we consider wandering vectors and their multipliers for finite subdiagonal algebras. We prove that the set of completely wandering vectors of a finite subdiagonal algebra is connected and is closed if and only if the finite subdiagonal algebra is antisymmetric. We also prove that the set of all wandering vector multipliers for an antisymmetric finite subdiagonal algebra forms a group.

1. INTRODUCTION

The notion of wandering vectors was introduced in [1] by Arveson to study factorization in finite subdiagonal algebras. This notion is very useful in the study of analytic operator algebras (cf. [1, 3, 6, 10, 11, 12] and so on). On the other hand, wandering vectors and their multipliers for unitary systems are systemically studied by several authors (cf. [2, 4, 5, 7]). It is noted that the structure of wandering vectors of both subdiagonal algebras and unitary systems is very interesting. In this note we consider wandering vectors of a finite subdiagonal algebra.

Arveson introduced the notion of subdiagonal algebras to study the analyticity in operator algebras in [1]. Let $\mathcal{M}$ be a σ-finite von Neumann algebra on $\mathcal{H}$ and $\mathfrak{D}$ a von Neumann subalgebra of $\mathcal{M}$. Let Φ be a faithful normal conditional expectation from $\mathcal{M}$ onto $\mathfrak{D}$. A subalgebra $\mathfrak{A}$ of $\mathcal{M}$, containing $\mathfrak{D}$, is called a subdiagonal algebra of $\mathcal{M}$ with respect to Φ if

(i) $\mathfrak{A} \cap \mathfrak{A}^* = \mathfrak{D}$,
(ii) Φ is multiplicative on $\mathfrak{A}$, and
(iii) $\mathfrak{A} + \mathfrak{A}^*$ is σ-weakly dense in $\mathcal{M}$.

The algebra $\mathfrak{D}$ is called the diagonal of $\mathfrak{A}$. Although subdiagonal algebras are not assumed to be σ-weakly closed in [1], the σ-weak closure of a subdiagonal algebra is again a subdiagonal algebra of $\mathcal{M}$ with respect to Φ (Remark 2.1.2 in [1]). Thus we assume that our subdiagonal algebras are always σ-weakly closed. We say that $\mathfrak{A}$ is a

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maximal subdiagonal algebra in $\mathcal{M}$ with respect to Φ in case that $\mathfrak{A}$ is not properly contained in any other subalgebra of $\mathcal{M}$ which is subdiagonal with respect to Φ. If there is a faithful normal finite trace τ on $\mathcal{M}$ such that $\tau \circ \Phi = \tau$, then we say that $\mathfrak{A}$ is a finite subdiagonal algebra with respect to Φ. Exel in [3] proved that a finite subdiagonal algebra is maximal subdiagonal. We call a finite subdiagonal algebra $\mathfrak{A}$ is antisymmetric if $\mathfrak{D} = \mathbb{C}I$.

We now assume that $\mathfrak{A}$ is a finite subdiagonal algebra with respect to Φ in $\mathcal{M}$ and let τ be the faithful normal finite trace on $\mathcal{M}$ such that $\tau \circ \Phi = \tau$. Let $\mathfrak{A}_0 = \{X \in \mathfrak{A} : \Phi(X) = 0\}$. Then $\mathfrak{A}_0$ is a two sided ideal of $\mathfrak{A}$ (cf. [1]). Let $L^p(\mathcal{M})$ $(1 \leq p \leq +\infty)$ be the non-commutative Lebesgue space associated with τ (cf. [13]). Then $L^2(\mathcal{M})$ is a Hilbert space with the inner product $\langle x, y\rangle = \tau(y^*x)$. For $T \in \mathcal{M}$ and $x \in L^2(\mathcal{M})$, let $L_T(x) = Tx$ (resp. $R_T(x) = xT$). We have that $\mathcal{L} = \{L_T : T \in \mathcal{M}\}$ (resp. $\mathcal{R} = \{R_T : T \in \mathcal{M}\}$) is a von Neumann algebra and $\mathcal{L}' = \mathcal{R}$. Furthermore, the map $T \to L_T$ (resp. $T \to R_T$) is a $*$-isomorphism (resp. $*$-anti-isomorphism) of $\mathcal{M}$ onto $\mathcal{L}$ (resp. $\mathcal{R}$) and the identity I is a cyclic and separating vector for $\mathcal{L}$ (resp. $\mathcal{R}$). For a subset E of $L^2(\mathcal{M})$, we denote by $[E]$ the closed subspace of $L^2(\mathcal{M})$ generated by E. Let $H^2 = [\mathfrak{A}]$ and $H_0^2 = [\mathfrak{A}_0]$. We easily have $L^2(\mathcal{M}) = H^2 \oplus (H_0^2)^* = H_0^2 \oplus [\mathfrak{D}] \oplus (H_0^2)^*$ (cf. [6]). We also have $\mathfrak{A}H^2 \subseteq H^2$ and $\mathfrak{A}H_0^2 \subseteq H_0^2$.

In this note we consider wandering vectors of finite subdiagonal algebras in $L^2(\mathcal{M})$ and their multipliers. We prove that the set of completely wandering vectors of a finite subdiagonal algebra is connected. It is closed if and only if the finite subdiagonal algebra is antisymmetric. We also prove that the set of all wandering vector multipliers for an antisymmetric finite subdiagonal algebra forms a group.

2. Wandering vectors of $\mathfrak{A}$

Let $\xi \in L^2(\mathcal{M})$ be a non-zero vector. The vector ξ is called to be a right (resp. left) wandering vector of $\mathfrak{A}$ if $\langle \xi A, \xi\rangle = 0$ (resp. $\langle A\xi, \xi\rangle = 0$) for any $A \in \mathfrak{A}_0$. Note that right and left wandering vectors are symmetric for $\mathfrak{A}$, so we consider the right wandering vectors only and call them wandering vectors. The following lemma is proved in [1].

Lemma 1. *([1, Lemma 4.4.2]) If ξ is a wandering vector of $\mathfrak{A}$, then there exists a partial isometry U in $\mathcal{M}$ such that $L_U\xi \in [\mathfrak{D}]$, and L_{U^*U} is the projection on $[\xi\mathcal{M}]$.*

A wandering vector ξ is called to be completely wandering if it is also right cyclic (i.e. cyclic for $\mathcal{R}$). It is known that a vector is right cyclic if and only if it is left separating. Put $W(\mathfrak{A}) = \{\xi \in L^2(\mathcal{M}) : \xi$ is a completely wandering vector of $\mathfrak{A}\}$. A unitary operator U on $L^2(\mathcal{M})$ is called to be a wandering vector multiplier of $\mathfrak{A}$ if

$UW(\mathfrak{A}) \subseteq W(\mathfrak{A})$. We denote by $M_{\mathfrak{A}}$ the set of all wandering vector multipliers of $\mathfrak{A}$. Note that for any unitary operator $U \in \mathcal{M}$, we have $L_U \in M_{\mathfrak{A}}$. The following proposition is elementary.

Proposition 1. *Let $\xi \in W(\mathfrak{A})$. Then $L^2(\mathcal{M}) = [\xi\mathfrak{A}_0] \oplus [\xi\mathfrak{D}] \oplus [\xi\mathfrak{A}_0^*]$.*

Proof. Note that $\langle \xi A, \xi \rangle = 0$ for all $A \in \mathfrak{A}$, $\mathfrak{D}\mathfrak{A}_0 \subseteq \mathfrak{A}_0$ and $\mathfrak{D}\mathfrak{A}_0 \subseteq \mathfrak{A}_0$, we easily have three subspaces $[\xi\mathfrak{A}_0]$, $[\xi\mathfrak{D}]$ and $[\xi\mathfrak{A}_0^*]$ are orthogonal each other. On the other hand, $\mathfrak{A}_0 + \mathfrak{D} + \mathfrak{A}_0^*$ is σ-weakly dense in $\mathcal{M}$. We thus have the proposition since ξ is right cyclic. The proof is complete. $\square$

Lemma 2. *Let $\xi \in W(\mathfrak{A})$ and let $\xi = V|\xi|$ be the polar decomposition of ξ. Then V is unitary and $|\xi| \in W(\mathfrak{A})$.*

Proof. We know that $V \in \mathcal{M}$ is a partial isometry. For any $A \in \mathfrak{A}_0$, we have $\langle |\xi|A, |\xi| \rangle = \tau(|\xi|^2 A) = \tau(\xi^*\xi A) = \langle \xi A, \xi \rangle = 0$, that is, ξ is wandering. Thus we have $[|\xi|\mathfrak{A}_0]$, $[|\xi|\mathfrak{D}]$ and $[|\xi|\mathfrak{A}_0^*]$ are orthogonal each other. On the other hand, $V|\xi|A = \xi A$ for all $A \in \mathcal{M}$. Then $V[|\xi|\mathfrak{A}] = [\xi\mathfrak{A}]$, $V[|\xi|\mathfrak{D}] = [\xi\mathfrak{D}]$ and $V[|\xi|\mathfrak{A}_0^*] = [\xi\mathfrak{A}_0^*]$, which implies that L_V is surjective. Thus V is a co-isometry. Note that $\mathcal{M}$ is finite, we have V is unitary and $[|\xi|\mathfrak{A}] \oplus [|\xi|\mathfrak{D}] \oplus [|\xi|\mathfrak{A}_0^*] = L^2(\mathcal{M})$. It follows that $|\xi|$ is right cyclic and therofore $|\xi| \in W(\mathfrak{A})$. The proof is complete. $\square$

Theorem 1. *Let $\mathfrak{A}$ be a finite subdiagonal algebra with respect to Φ in $\mathcal{M}$. Then $W(\mathfrak{A})$ is connected in $\mathbb{L}^2(\mathcal{M})$.*

Proof. Let $\xi \in W(\mathfrak{A})$. Then by Lemma 1, there is a partial isometry U in $\mathcal{A}$ such that $L_U\xi \in [\mathfrak{D}]$, and L_{U^*U} is the projection on $[\xi\mathcal{M}]$. We note that ξ is right cyclic, which implies that U is an isometry. Therefore U is a unitary operator in $\mathcal{M}$ since $\mathcal{M}$ is finite. Let $\eta = L_U\xi = U\xi \in [\mathfrak{D}]$. Then η is also a completely wandering vector for $\mathfrak{A}$, that is, $\eta \in W(\mathfrak{A})$. In fact, for any $A \in \mathfrak{A}_0$, $\langle \eta A, A \rangle = \langle L_U\xi A, L_U\xi \rangle = \langle \xi A, \xi \rangle = 0$ and $[\eta\mathcal{M}] = [L_U\xi\mathcal{M}] = L_U[\xi\mathcal{M}] = L^2(\mathcal{M})$. Let $\eta = V|\eta|$ be the polar decomposition of η. Then we have V is unitary and $|\eta| \in [\mathfrak{D}]$ is also a completely wandering vector by Lemma 2. Note that $\xi = L_{U^*V}|\eta| = U^*V|\eta|$. Thus without loss of generality, we assume that η itself is positive. So we have for any $\xi \in W(\mathfrak{D})$, there is a unitary element $T \in \mathcal{M}$ and a positive completely wandering vector $\eta \in [\mathfrak{A}]$ such that $\xi = T\eta$. It is known that the set of all unitary elements in $\mathcal{M}$ is connected and $L_T \in M_{\mathfrak{A}}$ for any unitary element $T \in \mathcal{M}$, which implies that ξ connects with a positive completely wandering vector in $[\mathfrak{D}]$. Thus it is enough to show that the set of all positive completely wandering vectors in $[\mathfrak{D}]$ is connected. Note that $I \in [\mathfrak{D}]$ is a positive completely wandering vector. If

$\eta \in [\mathfrak{D}]$ is also a positive completely wandering vector, then for any $\lambda \in (0,1)$, so is $\eta(\lambda) = \frac{\lambda I+(1-\lambda)\eta}{\|\lambda I+(1-\lambda)\eta\|}$. In fact, $\eta(\lambda)$ is right wandering. To show it is complete, it is enough to prove that $\eta(\lambda)$ is left separating. Let $T \in \mathcal{M}$ be a positive element such that $T\eta(\lambda) = 0$. Then we have $\tau(T\eta(\lambda)) = 0$ since $L^2(\mathcal{M}) \subset L^1(\mathcal{M})$. It follows that

$$\tau(\lambda T + (1-\lambda)(T\eta)) = \lambda\tau(T) + (1-\lambda)\tau(T^{\frac{1}{2}}\eta T^{\frac{1}{2}}) = 0.$$

Both $\tau(T)$ and $\tau(T^{\frac{1}{2}}\eta T^{\frac{1}{2}})$ are positive and τ is faithful, so $T = 0$. Thus, $\eta(\lambda) \in W(\mathfrak{A})$. It is trivial that $\eta(\lambda)$ is continuous on $[0,1]$ connects I and η. We have $W(\mathfrak{A})$ is connected. The proof is complete. □

We recall that a finite subdiagonal algebra $\mathfrak{A}$ is antisymmetric if $\mathfrak{D} = \mathbb{C}I$.

Theorem 2. *$W(\mathfrak{A})$ is closed if and only if $\mathfrak{A}$ is antisymmetric.*

Proof. If $\mathfrak{D} = \mathbb{C}I$, then by the proof of Theorem 1, for any $\xi \in W(\mathfrak{A})$, there is a unitary operator $U \in \mathcal{M}$ such that $\xi = UI = U$. Thus $W(\mathfrak{A}) = \{U \in \mathcal{M} : U \text{ is unitary}\}$ in $\mathbb{L}^2(\mathcal{M})$. Let V_n be a sequence in $W(\mathfrak{A})$ converging to an element U in $L^2(\mathcal{M})$. It follows that there is a subsequence U_{n_i} of $\{V_n\}$ such that $\lim_{i\to\infty} U_{n_i} = U_0$ in the weak operator topology for some operator $U_0 \in \mathcal{M}$. In particular, we have $\lim_{i\to\infty} \|U_{n_i} - U\|_2 \to 0$ in $L^2(\mathcal{M})$. It follows that for any $x \in L^2(\mathcal{M})$,

$$\lim_{i\to\infty} \langle (U_{n_i} - U), x\rangle = \lim_{i\to\infty} \langle (U_{n_i} - U)I, x\rangle = \langle U_0 - U, x\rangle = 0.$$

Thus we have $U = U_0$. Now

$$\lim_{i\to\infty} |\langle U_{n_i}^*(U_{n_i} - U_0)I, x\rangle| \le \lim_{i\to\infty} \|U_{n_i} - U_0\|_2 \|x\|_2 = 0$$

and

$$\lim_{i\to\infty} \langle U_0, U_{n_i}x\rangle = \langle U_0, U_0 x\rangle = \langle U_0^* U_0 I, x\rangle.$$

It follows that $\langle I, x\rangle = \langle U_0^* U_0, x\rangle$ for any $x \in \mathbb{L}^2(\mathcal{M})$, which implies that $U_0^* U_0 = I$ and therefore $U_0 = U$ is a unitary element. Thus $W(\mathfrak{A})$ is closed.

Conversely, assume that $W(\mathfrak{A})$ is closed. If there is a non trivial projection $P \in \mathfrak{D}$, then for any $\lambda > 0$, as proved above, we similarly have $\frac{P+\lambda I}{\|P+\lambda I\|} \in W(\mathfrak{A})$. It follows that $P \in W(\mathfrak{A})$ by letting $\lambda \to 0$. However, since P is nontrivial, P is not left separating. This contradiction implies that $\mathfrak{D} = \mathbb{C}I$. The proof is complete. □

By the proof of Theorem 2, we have the following corollary.

Corollary 1. *If $\mathfrak{A}$ is an antisymmetric finite subdiagonal algebra, then $W(\mathfrak{A})$ consists of all unitary elements of $\mathcal{M}$.*

The first author and Saito in [7] proved that the wandering vector multipliers for a unitary group forms a group. We note that the proof of Theorem 1 in [7] relies on two key facts. One is that the wandering vectors of the unitary group consists of all unitary elements in the associated von Neumann algebra $\mathcal{M}$. Another is a well-known theorem of Kadison in [8]. By Corollary 1 and Kadison's theorem, we similarly have the following theorem without proof.

Theorem 3. *If $\mathfrak{A}$ is an antisymmetric subdiagonal finite algebra, then $M_{\mathfrak{A}}$ forms a group.*

We do not know whether $M_{\mathfrak{A}}$ forms a group for a finite subdiagonal algebra $\mathfrak{A}$. It may be an interesting problem.

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References

[1] W. B. Arveson, *Analyticity in operator algebras*, Amer. J. Math., **89** (1967), 578–642.
[2] X. Dai and D. Larson, *Wandering vectors for unitary systems and orthogonal wavelets*, Memoirs Amer. Math. Soc., **134** (1998), No.640.
[3] R. Exel, *Maximal subdiagonal algebras*, Amer. J. Math., **110** (1988) 775–782.
[4] D. Han, *Wandering vectors for irrational rotation unitary systems*, Trans. Amer. Math. Soc., **350** (1998), 309–320.
[5] D. Han and D. Larson, *Wandering vector multipliers for unitary groups*, Trans. Amer. Math. Soc., **353** (2001), 3347–3370.
[6] G. Ji and K.-S. Saito, *Factorization in subdiagonal algebras*, J. Funct. Anal., **159** (1998), 191–202.
[7] G. Ji and K.-S. Saito, *On wandering vector multipliers for unitary groups*, Proc. Amer. Math. Soc., **133** (2005), 3263–3269.
[8] R. V. Kadison, *Isometries of operator algebras*, Ann. of Math., **54** (1951), 325–338.
[9] R. V. Kadison and J. R. Ringorse, *Fundamentals of the theory of operator algebras*, Vol. II, Academic Press, Orlando, 1986.
[10] N. Kamei, *Simple invariant subspace theorems for antisymmetric finite subdiagonal algebras*, Tohoku Math. J., **21** (1969), 467–473.
[11] K.-S. Saito, *A note on invariant subspaces for finite maximal subdiagonal algebras*, Proc. Amer. Math Soc., **77** (1979), 348–352.
[12] K.-S. Saito, *Invariant subspaces for finite maximal subdiagonal algebras*, Pacific J. Math., **93** (1981), 431–434.
[13] I. E. Segal, *A non-commutative extension of abstract integration*, Ann. of Math., **57** (1953), 401–457.

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