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An example of a totally geodesic foliation which is perpendicular to a certain non-singular Killing field on an arbitrary three-dimensional Lorentzian lens space

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Abstract

We construct a totally geodesic foliation which is perpendicular to a certain non-singular Killing field on an arbitrary three-dimensional Lorentzian lens space.

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1 Introduction

Totally geodesic foliations on Lorentzian manifolds are studied by several authors ([BMT], [CR], [M], [Y1], [Y2], [Y3], [Z2], [Z3], [Z4]).

An example of a codimension-1 totally geodesic foliation containing spacelike, timelike, and lightlike leaves appeared first in [Y1], and it was obtained as $\ker g(X, \cdot)$, where X is a non-singular Killing field for a Lorentzian metric g on the 2-torus T^2. So it seemed a "typical" example of a codimension-1 totally geodesic foliation. These typical examples, i.e., codimension-1 totally geodesic foliations perpendicular to non-singular Killing fields, were treated and classified in [Y3].

In [Y2], we constructed Lorentzian geodesible foliations of closed 3-manifolds having Heegaard splittings of genus one, i.e., lens spaces $L(p,q)$ of type (p,q), the 3-sphere $S^3 \cong L(1,0)$, and $S^2 \times S^1 \cong L(0,1)$. Here a Lorentzian geodesible foliation means a totally geodesic foliation for some, in general incomplete, Lorentzian metric. However, the constructed example of a totally geodesic foliation $\mathcal{F}$ was not a typical example, that is, $\mathcal{F}$ was not obtained as $\ker g(X, \cdot)$ for some non-singular Killing field X. So the natural question concerning the existence problem of typical examples arises. More precisely, we have

Question 1 *Can we give a non-singular Killing field X for some Lorentzian metric of a 3-manifold such that the distribution* $\ker g(X, \cdot)$ *is completely integrable?*

A natural idea to solve Question 1 is using a non-singular Killing field X of a Riemannian manifold (M, g) such that $\ker g(X, \cdot)$ is completely integrable. In this setting, we can solve Question 1 by the following theorem.

Theorem 4 *Let X be a non-singular vector field on a closed manifold M. Then X is a Killing field for some Riemannian metric on M if and only if X is a timelike Killing field for some Lorentzian metric on M. Moreover we can choose the exchange between the Riemannian metric and the Lorentzian metric so that the orthogonal distribution to X is coincide.*

By Theorem 4, we can easily solve Question 1 for the 3-manifolds admitting codimension-1 totally geodesic foliations perpendicular to non-singular Killing fields. However $L(p, q)$ except $L(0, 1) \cong S^2 \times S^1$ does not admit a codimension-1 totally geodesic foliation by [BH]. So we need another idea to construct examples on $L(p, q)$. Fortunately a careful usage of the tricks stated in [Y2] works well on $L(p, q)$. Hence we have the following.

Theorem 5 *Let $L(p, q)$ denote a 3-dimensional lens space of type (p, q). (we allow $(p, q) = (0, 1), (1, 0)$.) Then there exists a Lorentzian metric g on $L(p, q)$ and a non-singular Killing field X for g such that the distribution $\ker g(X, \cdot)$ is completely integrable.*

In Section 4, we consider 3-manifolds admitting totally geodesic foliations perpendicular to non-singular Killing fields. If a totally geodesic foliation contains more than one kind of leaves among spacelike, timelike, and lightlike leaves, we have the following.

Theorem 10 *Let (M, g) be a Lorentzian manifold and X a non-singular Killing field for g such that the distribution $\ker g(X, \cdot)$ is completely integrable. Denote the foliation defined by $\ker g(X, \cdot)$ by $\mathcal{F}$. Assume that $\mathcal{F}$ contains more than one kind of leaves among spacelike, timelike, and lightlike leaves. Then M is a Seifert fibered space.*

2 Killing fields for Riemannian metrics and Lorentzian metrics

In this section, we refer to relations between non-singular Killing fields for Riemannian metrics and those for Lorentzian metrics.

First we consider a modification of a Riemannian metric into a certain Lorentzian metric as follows.

Proposition 2 *Let (M, g) be a Riemannian manifold and X a non-singular Killing field for g. Assume that there exists a constant $k > 0$ such that $g(X_x, X_x) > 1/k$ for all $x \in M$. Then $h = g - kg(X, \cdot) \otimes g(X, \cdot)$ is a Lorentzian metric on M and X is a Killing field for h. Furthermore the orthogonal complement of X with respect to g is perpendicular to X with respect to h.*

Proof. It is easy to prove that h is a Lorentzian metric on M. So it is sufficient to prove that $\mathcal{L}_X(g(X,\cdot)) = 0$. Put $\omega = g(X,\cdot)$. By straight computation, we have $(\mathcal{L}_X\omega)(Y) = X(g(X,Y)) - g(X,[X,Y])$. If $Y \in \Gamma(\ker g(X,\cdot))$, then we have $g(X,Y) = 0$ and $[X,Y] \in \Gamma(\ker g(X,\cdot))$, since the distribution $\ker g(X,\cdot)$ is preserved by the flow generated by X by [Y3]. If $Y = X$, then we have $X(g(X,Y)) = 0$ and $[X,Y] = 0$. Therefore we have $\mathcal{L}_X\omega = 0$. This proves the proposition. □

Second we consider a kind of a converse of Proposition 2. We can prove it in the same way as above.

Proposition 3 *Let (M,h) be a Lorentzian manifold and X a non-singular Killing field. Assume that X is timelike and there exists a constant $k > 0$ such that $h(X_x,X_x) < -1/k$ for all $x \in M$. Then $g = h + kh(X,\cdot)\otimes h(X,\cdot)$ is a Riemannian metric and X is a Killing field for g. Furthermore the orthogonal complement of X with respect to h is perpendicular to X with respect to g.*

By putting Proposition 2 and 3 together, we have the following.

Theorem 4 *Let X be a non-singular vector field on a closed manifold M. Then X is a Killing field for some Riemannian metric on M if and only if X is a timelike Killing field for some Lorentzian metric on M. Moreover we can choose the exchange between the Riemannian metric and the Lorentzian metric so that the orthogonal distribution to X is coincide.*

By Theorem 4, we can easily solve Question 1 for the 3-manifolds admitting codimension-1 totally geodesic foliations perpendicular to non-singular Killing fields, for example, a surface bundle over S^1 whose monodromy is isotopic to a periodic map [CG].

3 A construction of a totally geodesic foliation which is perpendicular to a certain non-singular Killing field

In this section, we prove the following.

Theorem 5 *Let $L(p,q)$ denote a 3-dimensional lens space of type (p,q). (we allow $(p,q) = (0,1),(1,0)$.) Then there exists a Lorentzian metric g on $L(p,q)$ and a non-singular Killing field X for g such that the distribution $\ker g(X,\cdot)$ is completely integrable.*

The proof of this theorem is essentially similar to the proof in [Y2].

Proof of Theorem 5. If $p = 0$, that is, $L(0,1) \cong S^2 \times S^1$, the Lorentzian metric $ds^2|_{S^2} - dt^2$ and the Killing field $\partial/\partial t$ satisfy the desired conditions. Hereafter we assume that $p \neq 0$.

Let V_i denote an oriented $D^2 \times S^1$, and let m_i (resp. l_i) be a meridian (resp. longitude) in V_i $(i = 1, 2)$. Put

$$A = \begin{pmatrix} q & r \\ p & s \end{pmatrix}, \quad p, q, r, s \in \mathbf{Z}, \quad qs - pr = -1.$$

Let $f : \partial V_2 \to \partial V_1$ be the orientation reversing diffeomorphism defined by

$$f : \begin{pmatrix} \theta_2 \\ t_2 \end{pmatrix} \mapsto A \begin{pmatrix} \theta_2 \\ t_2 \end{pmatrix},$$

where $(\theta_2, t_2) \in \partial V_2$ denotes the coordinate defined by

$$(\theta_2, t_2) \mapsto (\cos\theta_2, \sin\theta_2, t_2) \in \partial V_2.$$

Note that $V_1 \bigcup_f V_2$ is diffeomorphic to the lens space $L(p, q)$ of type (p, q). Let E denote the negative eigenvalue of A, that is,

$$E = (q + s - \sqrt{(q-s)^2 + 4pr})/2,$$

and put

$$R = (q - s - \sqrt{(q-s)^2 + 4pr})/2p.$$

Step 1. We can construct a Lorentzian metric g_i on V_i and a non-singular Killing field X_i for g_i which are suitable for us as follows.

Lemma 6 *There exist a Lorentzian metric g_i on V_i, a non-singular Killing field X_i for g_i and a codimension-1 Reeb foliation $\mathcal{F}_i$ on V_i which satisfy the following conditions.*

(1) *The foliation $\mathcal{F}_i$ is obtained as* $\ker g(X_i, \cdot)$.

(2) *(Note that $\partial V_i \in \mathcal{F}_i$ is lightlike by the result of [Y2].) The linear foliation defined by the lightlike vectors on the boundary leaf $\partial V_i \in \mathcal{F}_i$ is equal to the eigenspace corresponding to the negative eigenvalue of A.*

(3) *The metric g_i satisfies the assumptions of Proposition 3.6 in [Y2].*

(4) *The gluing map f is an "isometry" from $(\partial V_2, g_2|_{\partial V_2})$ to $(\partial V_1, g_1|_{\partial V_1})$, that is,*

$$f^*(g_1|_{\partial V_1}) = g_2|_{\partial V_2}.$$

(5) *The gluing map f maps $X_2|_{\partial V_2}$ to $X_1|_{\partial V_1}$.*

Proof. Let (x, y, t) be coordinates of $D^2 \times \mathbf{R}$, where (x, y) and (t) are the canonical coordinates of $\mathbf{R}^2$ and $\mathbf{R}$, respectively. Define the diffeomorphism $\varphi : D^2 \times \mathbf{R} \to D^2 \times \mathbf{R}$ by

$$(x, y, t) \mapsto (x\cos(Rt) + y\sin(Rt), -x\sin(Rt) + y\cos(Rt), t).$$

Consider $\varphi^* g_0$, where g_0 is the Lorentzian metric on $D^2 \times \mathbf{R}$ in Example 3.5 in [Y2]. By straight computation, $\varphi^* g_0$ is given by

$$\begin{pmatrix} \dfrac{G'_{11}}{2(a^2-2)(x^2+y^2)^2} & \dfrac{G'_{12}}{2(a^2-2)(x^2+y^2)^2} & \dfrac{ax}{\sqrt{x^2+y^2}} + \dfrac{a^2Ry}{2(x^2+y^2)} \\ \dfrac{G'_{12}}{2(a^2-2)(x^2+y^2)^2} & \dfrac{G'_{22}}{2(a^2-2)(x^2+y^2)^2} & \dfrac{ay}{\sqrt{x^2+y^2}} - \dfrac{a^2Rx}{2(x^2+y^2)} \\ \dfrac{ax}{\sqrt{x^2+y^2}} + \dfrac{a^2Ry}{2(x^2+y^2)} & \dfrac{ay}{\sqrt{x^2+y^2}} - \dfrac{a^2Rx}{2(x^2+y^2)} & \dfrac{a^2R^2}{2} + a^2 - 1 \end{pmatrix},$$

$$\begin{aligned} G'_{11} &= 2(a^2-1)(x^2+y^2)x^2 + a^2(a^2-2)y^2, \\ G'_{12} &= 2(a^2-1)(x^2+y^2)xy - a^2(a^2-2)xy, \\ G'_{22} &= 2(a^2-1)(x^2+y^2)y^2 + a^2(a^2-2)x^2. \end{aligned}$$

Define the vector field X_1 by

$$R\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial t}.$$

Since $\varphi_* X_1 = \partial/\partial t$, the vector field X_1 is a non-singular Killing field for $\varphi^* g_0$. The distribution defined by $\ker \varphi^* g_0(X_1, \cdot)$ is completely integrable. Since the metric $\varphi^* g_0$ on $D^2 \times \mathbf{R}$ is invariant by $\partial/\partial t$, it defines the metric on $D^2 \times \mathbf{R}/2\pi\mathbf{Z}$.

Let V_1 and V_2 be two copies of an oriented $D^2 \times S^1$. Let (x_i, y_i, t_i) denote the coordinate of $V_i = D^2 \times S^1$ $(i = 1, 2)$. Put

$$g_1 = \varphi^* g_0,\ X_1 = R\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right) + \frac{\partial}{\partial t_1} \text{ on } V_1,$$
$$g_2 = \frac{1}{E^2}\varphi^* g_0,\ X_2 = \frac{1}{E}\left(R\left(x_2\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_2}\right) + \frac{\partial}{\partial t_2}\right) \text{ on } V_2.$$

These g_i, X_i satisfy conditions (1), (2), and (5). We will see that they satisfy conditions (3) and (4) in Step 2. □

We change coordinates from $(x_i, y_i, t_i) \in V_i$ to (r_i, θ_i, t_i), where $x_i = r_i \cos\theta_i$ and $y_i = r_i \sin\theta_i$. The metric g_1 is represented by

$$\begin{pmatrix} (a^2-1)/(a^2-2) & 0 & a \\ 0 & a^2/2 & -a^2R/2 \\ a & -a^2R/2 & a^2R^2/2 + a^2 - 1 \end{pmatrix},$$

with respect to (r_1, θ_1, t_1). Define the collar neighborhood by

$$h_i : \partial V_i \times [0, \varepsilon] \to V_i, \quad (\theta_i, t_i, u_i) \mapsto (1 - u_i, \theta_i, t_i).$$

Recall that the gluing map $f : \partial V_2 \cong \mathbf{R}^2/2\pi\mathbf{Z}^2 \to \partial V_1 \cong \mathbf{R}^2/2\pi\mathbf{Z}^2$ is defined by

$$f : \begin{pmatrix} \theta_2 \\ t_2 \end{pmatrix} \mapsto \begin{pmatrix} q & r \\ p & s \end{pmatrix} \begin{pmatrix} \theta_2 \\ t_2 \end{pmatrix}.$$

Step 2. Denote coordinates of $\partial V_1 \times [0,1]$ by (θ, t, u), where $\theta = \theta_1$ and $t = t_1$. Consider the glued manifold $V_1 \cup_{\mathrm{id}} (\partial V_1 \times [0,1]) \cup_f V_2$.

We prove a lemma similar to Lemma 3.8 in [Y2].

Lemma 7 *There exists a Lorentzian metric g' on $\partial V_1 \times [0,1]$ which is the extension of the metric $g_1 \cup g_2$ restricted on $\partial V_1 \times \{0\}$ to the metric $g_1 \cup g_2$ on $\partial V_1 \times \{1\}$ and satisfies the following conditions:*

(1) *All the components of g' with respect to (θ, t, u) depend on only $u \in [0,1]$.*
(2) *The foliation $\{\partial V_1 \times \{*\}\}$ is perpendicular to a non-singular lightlike Killing field for g', hence, the foliation $\{\partial V_1 \times \{*\}\}$ is totally geodesic with respect to g'.*

Proof. Recall that

$$g_1 = \begin{pmatrix} (a^2-1)/(a^2-2) & 0 & a \\ 0 & a^2/2 & -a^2R/2 \\ a & -a^2R/2 & a^2R^2/2 + a^2 - 1 \end{pmatrix},$$

where the right hand side is the matrix of components of g_1 with respect to $(r_1, \theta_1, t_1) \in V_1$. When we use the collar coordinates $(\theta_1, t_1, u_1) \in \partial V_1 \times [0,1]$, we have another expression of g_1 as

$$g_1 = \begin{pmatrix} a^2/2 & -a^2R/2 & 0 \\ -a^2R/2 & a^2R^2/2 + a^2 - 1 & -a \\ 0 & -a & (a^2-1)(a^2-2) \end{pmatrix}.$$

By restricting g_1 on $\partial V_1 \times \{0\} \subset \partial V_1 \times [0, \varepsilon]$, we have

$$g_1 = \begin{pmatrix} 1/2 & -R/2 & 0 \\ -R/2 & R^2/2 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Hence the metric on $\partial V_1 \times \{0\} \subset \partial V_1 \times [0,1]$ is represented by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1/2 & -R/2 & 0 \\ -R/2 & R^2/2 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1/2 & -R/2 & 0 \\ -R/2 & R^2/2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

with respect to the coordinates $(\theta, t, u) \in \partial V_1 \times [0,1]$. Since $X_1 = R\partial/\partial\theta_1 + \partial/\partial t_1$ on V_1, we have

$$X_1 = R\frac{\partial}{\partial\theta} + \frac{\partial}{\partial t}$$

on $\partial V_1 \times \{0\} \subset \partial V_1 \times [0,1]$. Note that the inverse map $f^{-1} : \partial V_1 \times \{1\} \to \partial V_2$ is represented by

$$\begin{pmatrix} -s & r \\ p & -q \end{pmatrix},$$

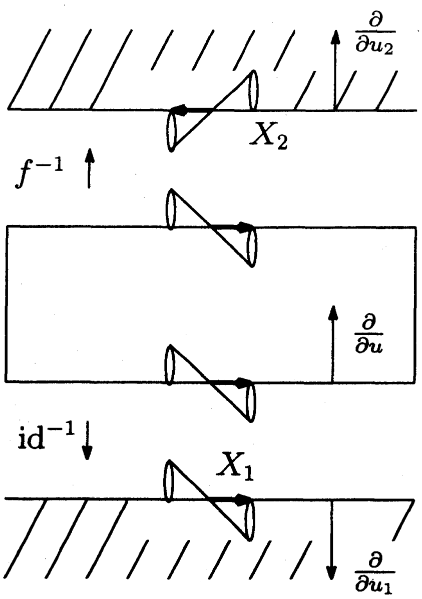

Figure : f and id

and

$$g_2 = \frac{1}{E^2}\begin{pmatrix} a^2/2 & -a^2R/2 & 0 \\ -a^2R/2 & a^2R^2/2 + a^2 - 1 & -a \\ 0 & -a & (a^2-1)/(a^2-2) \end{pmatrix}$$

with respect to the collar coordinates $(\theta_2, t_2, u_2) \in \partial V_2 \times [0, \varepsilon]$. These expressions imply that

$$\begin{pmatrix} -s & p & 0 \\ r & -q & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \frac{1}{E^2}\begin{pmatrix} 1/2 & -R/2 & 0 \\ -R/2 & R^2/2 & -1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} -s & r & 0 \\ p & -q & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \frac{1}{E^2}\begin{pmatrix} (s+pR)^2/2 & -(s+pR)(r+qR)/2 & -p \\ -(s+pR)(r+qR)/2 & (r+qR)^2/2 & q \\ -p & q & 0 \end{pmatrix}$$

on $\partial V_1 \times \{1\} \subset \partial V_1 \times [0,1]$ with respect to $(\theta, t, u) \in \partial V_1 \times [0,1]$ (Figure). By the definitions of R and E, we have

$$(s + pR)/E = 1, \ (r + qR)/E = R.$$

By substituting these, g_2 is expressed as

$$g_2 = \begin{pmatrix} 1/2 & -R/2 & -p/E^2 \\ -R/2 & R^2/2 & q/E^2 \\ -p/E^2 & q/E^2 & 0 \end{pmatrix},$$

with respect to $(\theta, t, u) \in \partial V_1 \times [0,1]$. By the definition of X_2, we have $f_* X_2 = R\,\partial/\partial\theta + \partial/\partial t$. Define the Lorentzian metric g' by

$$g'|_{(\theta,t,u)} = \begin{pmatrix} 1/2 & -R/2 & -up/E^2 \\ -R/2 & R^2/2 & uq/E^2 + (1-u) \\ -up/E^2 & uq/E^2 + (1-u) & 0 \end{pmatrix}$$

with respect to (θ, t, u). By the straight computation, we have that

$$\det g' = -\frac{1}{2}\left\{ \left(\frac{q - Rp}{E^2} \right) u + (1-u) \right\}^2 .$$

Let E' denote the positive eigenvalue of A, that is,

$$E' = (q + s + \sqrt{(q-s)^2 + 4pr})/2.$$

We have that $q - Rp = E'$. Since $EE' = -1$, we have $E'/E^2 = (E')^3 > 0$. Therefore $\det g' < 0$ for all $u \in [0,1]$.

Note that manifolds $\partial V_1 \times \{u\}$ is lightlike. Since all the components of g' with respect to (θ, t, u) depend on only u and all the components of $R\partial/\partial\theta + \partial/\partial t$ are constant, the vector field $R\,\partial/\partial\theta + \partial/\partial t$ is a non-singular Killing field for g'. The distribution $\ker g'(R\,\partial/\partial\theta + \partial/\partial t, \cdot)$ is equal to $\mathrm{Span}\{\partial/\partial\theta, \partial/\partial t\}$, hence it defines the foliation $\{\partial V_1 \times \{*\}\}$. This proves Lemma 7. □

Step 3. We change the parameter u of each component of g' to $w(u)$, where w is a function which satisfies the following:

(1) the function $w : [0,1] \to [0,1]$ is a C^∞ monotone increasing function.
(2) $\frac{d^n}{ds^n} w(0) = \frac{d^n}{ds^n} w(1) = 0$ for all integer $n > 0$.

We denote a new metric by the same symbol g'.

Put

$$g = \begin{cases} g_1 & \text{on} \quad V_1, \\ g' & \text{on} \quad \partial V_1 \times [0,1], \\ g_2 & \text{on} \quad V_2. \end{cases}$$

Note that g is a C^∞ Lorentzian metric on $V_1 \cup_{\mathrm{id}} (\partial V_1 \times [0,1]) \cup_f V_2$ by Proposition 3.6 in [Y2]. We define the vector field X by

$$X = \begin{cases} X_1 & \text{on} \quad V_1, \\ R\,\partial/\partial\theta + \partial/\partial t & \text{on} \quad \partial V_1 \times [0,1], \\ X_2 & \text{on} \quad V_2. \end{cases}$$

Note that X is a smooth non-singular Killing field for g and the distribution $\ker g(X, \cdot)$ is completely integrable. This completes the proof. □

Remark 8 We wanted to construct a totally geodesic foliation perpendicular to a Killing field on $V_1 \bigcup_{\mathrm{id}}(\partial V_1 \times [0,1]) \bigcup_f V_2$. So we cannot rotate the one-dimensional lightlike subfoliation $\mathcal{L}$ on the lightlike totally geodesic foliation $\{\partial V_1 \times \{*\}\}$. Hence $\mathcal{L}$ must coincide with an eigenspace of the matrix A. If we use the negative eigenvalue of A, the directions of the lightcones on ∂V_2 and the Killing field X_2 are reversed by the gluing map f (see Figure). So we can use the only one model $(\varphi^* g_0, \varphi^* \partial/\partial t)$. If we use the positive eigenvalue of A, the directions of the lightcones and X_2 are preserved by f. So we must use two models. This is the reason why we use the negative eigenvalue of A.

4 Manifolds admitting totally geodesic foliations perpendicular to Killing fields

In this section, we consider 3-manifolds admitting totally geodesic foliations perpendicular to Killing fields.

First we quote Zeghib's theorem concerning Killing fields on Lorentzian 3-manifolds.

Theorem 9 ([Z1] Theorem 0) *Let $(M, <,>)$ be a compact Lorentz 3-manifold and ϕ^t an isometric flow on it, which is not equicontinuous (a flow ϕ^t is equicontinuous iff the closure of $\{\phi^t\}$ in* Homeo M *is compact). Then exactly one of the following two possibilities can occur.*
i) The flow is (everywhere) spacelike and Anosov.
ii) The flow is (everywhere) lightlike and preserves a complete Lorentz metric of constant negative curvature on M.

By using the above theorem, we have the following.

Theorem 10 *Let (M, g) be a Lorentzian manifold and X a non-singular Killing field for g such that the distribution* $\ker g(X, \cdot)$ *is completely integrable. Denote the foliation defined by* $\ker g(X, \cdot)$ *by $\mathcal{F}$. Assume that $\mathcal{F}$ contains more than one kind of leaves among spacelike, timelike, and lightlike leaves. Then M is a Seifert fibered space.*

Proof. Let ϕ^t denote the one-parameter group generated by X. Since X is a non-singular Killing field, each orbit of X is spacelike, timelike, or lightlike. By the assumption that $\mathcal{F}$ contains more than one kind of leaves, there exist two orbits of X such that they have distinct types each other. By Zeghib's theorem, the closure $\mathrm{Cl}\{\phi^t\}$ in Homeo M is compact. Since $\{\phi^t\}$ is abelian, so is $\mathrm{Cl}\{\phi^t\}$. Hence $\mathrm{Cl}\{\phi^t\}$ is a torus $\mathbf{T}$ of some dimension. Take a compact one-parameter subgroup $\{a^t\}$ sufficiently near $\{\phi^t\}$ in $\mathrm{Cl}\{\phi^t\}$ so that $\{a^t\}$ defines a locally free action on M. Therefore M is a Seifert fibered space. □

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