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On Some EP Operators

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Abstract

Let H be Hilbert space, and let $T : H \to H$ be a bounded linear operator with closed range. In this paper, we introduce a new family of operators with generalized inverse $T^\dagger$ such that $T^\dagger T \geq TT^\dagger$, which is weaker than the case of EP. Moreover we characterize such operators and give some fundamental properties.

1 Introduction and preliminaries

Throughout this paper we assume that H_1, H_2, and H are separable complex Hilbert spaces with inner product $(\cdot, \cdot)$. Let $B(H_1, H_2)$ be the set of all bounded linear operators from H_1 into H_2. Let $B_C(H_1, H_2)$ be the subspace of all $T \in B(H_1, H_2)$ such that the range of T is closed in H_2. If $H_1 = H_2 = H$, we write $B(H) = B(H, H)$ and $B_C(H) = B_C(H, H)$. For $T \in B(H_1, H_2)$, $\ker T$ and $R(T)$ denote the kernel and the range of T, respectively.

According to Nashed [6], $T \in B_C(H_1, H_2)$ has a Moore-Penrose inverse $T^\dagger$, that is, $T^\dagger$ is the unique solution for the equations:

$$TT^\dagger T = T, \; T^\dagger TT^\dagger = T^\dagger, \; (TT^\dagger)^* = TT^\dagger, \text{ and } (T^\dagger T)^* = T^\dagger T, \qquad (1.1)$$

where T^* denotes the adjoint operator of T. Later of this, we write M-P inverse for short.

We need the following results of $T^\dagger$ and $R(T)$. See [3, 4, 5] for details.

Theorem A. (i) For any $T \in B_C(H_1, H_2)$ with M-P inverse $T^\dagger$, we have that

$$T^\dagger T = P_{R(T^\dagger)}, \; TT^\dagger = P_{R(T)}, \; (T^\dagger)^\dagger = T, \text{ and } (T^\dagger)^* = (T^*)^\dagger,$$

where P_M is the orthogonal projection from H onto M.

(ii) For any $T \in B(H)$,

(1) $R(T)$ is closed if and only if $T^\dagger$ is bounded;

(2) $R(T)$ is closed if and only if $R(T^*)$ is closed.

An operator T in $B(H)$ is said to be an *EP operator* if the range of T is equal to the range of its adjoint T^*, i.e., $R(T) = R(T^*)$. For $S, T \in B(H)$, we write

$[S, T] := ST - TS$ for the commutator of S and T. We know that T in $B(H)$ is EP if and only if $[T^\dagger, T] = 0$ (see [7, 10]). An operator $T \in B(H)$ is called *normal* if $[T^*, T] = 0$ and *hyponormal* if $[T^*, T] \geq 0$ (see e.g. [9]).

2 Characterization of hypo-EP operators

In this section, we define a new family of operators and give some properties.

Definition. For an operator $T \in B_C(H)$, if $[T^\dagger, T] \geq 0$ then T is called a *hypo-EP operator.*

The following theorems immediately follows from the definition of hypo-EP operators, Theorems 2, 3 of §29 of [1] and (1) of Theorem A.

Proposition 2.1. Let $T \in B_C(H)$ with a bounded M-P inverse $T^\dagger$. Then the following statements are equivalent:
(1) T is hypo-EP;
(2) $R(T^*) \supseteq R(T)$;
(3) $R(T^\dagger) \supseteq R(T)$;
(4) $T^\dagger T^2 T^\dagger = TT^\dagger$;
(5) $T(T^\dagger)^2 T = TT^\dagger$;
(6) $\|T^\dagger Tx\| \geq \|TT^\dagger x\|$ for all $x \in H$.

Remarks 2.2. By using the results of Douglas [2], the following statements can be proved. Here, we refer to a result, only.
Let $T \in B_C(H)$ with a bounded M-P inverse $T^\dagger$. Then the following statements are equivalent: (1) T is hypo-EP; (2) $\exists \alpha \geq 0$ s.t. $TT^* \leq \alpha T^\dagger (T^*)^\dagger$; (3) $\exists C \in B(H)$ s.t. $T = T^\dagger C$.

Theorem 2.3. Let $T \in B_C(H)$ with a bounded M-P inverse $T^\dagger$. Then T is hypo-EP if and only if

$$\|T^\dagger x\| \leq \|T^\dagger\|^2 \|Tx\| \text{ for all } x \in H.$$

Proof. Suppose that T is a hypo-EP operator. Then, from (6) of Proposition 2.1, T satisfies the following condition;

$$\|T^\dagger Tx\| \geq \|TT^\dagger x\| \text{ for all } x \in H.$$

It follows from (1.1) that for all $x \in H$,

$$\begin{aligned}\|T^\dagger x\| &= \|T^\dagger TT^\dagger x\| \leq \|T^\dagger\| \|TT^\dagger x\| \\ &\leq \|T^\dagger\| \|T^\dagger Tx\| \leq \|T^\dagger\|^2 \|Tx\|.\end{aligned}$$

Thus we have

$$\|T^\dagger x\| \le \|T^\dagger\|^2\|Tx\| \text{ for all } x \in H.$$

Conversely, we suppose that $\|T^\dagger x\| \le \|T^\dagger\|^2\|Tx\|$ $(\forall x \in H)$. Then,

$$Tx = 0 \Rightarrow T^\dagger x = 0, \text{ i.e., } \ker T \subseteq \ker T^\dagger.$$

Hence we have $(\ker T)^\perp \supseteq (\ker T^\dagger)^\perp$. Now we notice that $(\ker T)^\perp = R(T^*) = R(T^\dagger)$ and $(\ker T^\dagger)^\perp = R(T)$. Therefore

$$R(T^*) = R(T^\dagger) \supseteq R(T), \text{ i.e., } T \text{ is hypo} - \text{EP}. \qquad \square$$

Theorem 2.4. Let $T \in B_C(H)$ with a bounded M-P inverse $T^\dagger$. Then T is hypo-EP if and only if one of the following statements holds:
(1) $T^\dagger T^2 = T$;
(2) $T^*T^\dagger T = T^*$.

Proof. (1) From (4) and (5) of Proposition 2.1 we have that T is hypo-EP if and only if $R(T) \subseteq R(T^*) = R(T^\dagger)$. Thus by (1.1) we have

$$T^\dagger T^2 = (T^\dagger T)T = P_{R(T^\dagger)}T = T.$$

The converse is clear from $T^\dagger T = P_{R(T^\dagger)}$.
(2) It is clear from (1.1) and (i) of Theorem A that

$$(T^\dagger T^2)^* = ((T^\dagger T)T)^* = T^*(T^\dagger T). \qquad \square$$

Next, we give an example of hypo-EP operators. The following proposition is clear from the fact that if T is hyponormal then $R(T) \subseteq R(T^*)$.

Proposition 2.5. Let $T \in B_C(H)$ with a bounded M-P inverse $T^\dagger$. If T is hyponormal then it is hypo-EP.

The above proposition guarantees that introducing such an operator is meaningful.

3 Fundamental properties

In this section, we consider about three questions;
(i) What condition is a hypo-EP operator EP?
(ii) Is the limit of hypo-EP operators also hypo-EP?
(iii) What is a value of $\gamma(T^\dagger T - TT^\dagger)$?

Next we show fundamental results to investigate hypo-EP operators with generalized inverse.

Theorem 3.1. Let $T \in B_C(H)$ with a bounded M-P inverse $T^\dagger$. If $[T^\dagger T, T+T^\dagger] = 0$ then T is hypo-EP.

Proof. By (1.1) and Theorem A,

$$\begin{aligned}[T^\dagger T, T + T^\dagger] &= T^\dagger T(T + T^\dagger) - (T + T^\dagger)T^\dagger T \\ &= T^\dagger T^2 + T^\dagger - T - (T^\dagger)^2 T = 0.\end{aligned}$$

Multiplying by T on the left hand side, by (1.1) we have

$$TT^\dagger - (TT^\dagger)(T^\dagger T) = 0.$$

Hence by Proposition 2.1 (5) we have the required conclusion. □

Corollary 3.2. Let $T \in B_C(H)$ with a bounded M-P inverse $T^\dagger$. If $[T, T^\dagger T] = 0$ then T is hypo-EP.

Proof. Since $[T, T^\dagger T] = 0$,

$$TT^\dagger T - T^\dagger T^2 = 0.$$

Multiplying by $T^\dagger$ on the right hand side, by (1.1) we have

$$TT^\dagger = (T^\dagger T)(TT^\dagger).$$

Hence by Proposition 2.1 (5) we have the required conclusion. □

Theorem 3.3. Let $T \in B_C(H)$ with a bounded M-P inverse $T^\dagger$. Suppose that T is hypo-EP. If $[TT^\dagger, T + T^\dagger] = 0$ then T is EP.

Proof. By (1.1) and Theorem A,

$$\begin{aligned}[TT^\dagger, T + T^\dagger] &= TT^\dagger(T + T^\dagger) - (T + T^\dagger)TT^\dagger \\ &= TT^\dagger T + T(T^\dagger)^2 - T^2T^\dagger - T^\dagger TT^\dagger \\ &= T + T(T^\dagger)^2 - T^2T^\dagger - T^\dagger = 0.\end{aligned}$$

Thus, we have

$$\begin{aligned}&T^2 + T(T^\dagger)^2T - T^2T^\dagger T - T^\dagger T \\ &= (TT^\dagger)(T^\dagger T) - T^\dagger T = 0.\end{aligned}$$

Hence, we have

$$T^\dagger T \leq TT^\dagger.$$

Here $T^\dagger T \geq TT^\dagger$, by assumption.
Therefore, $T^\dagger T = TT^\dagger$, i.e., T is EP. □

Corollary 3.4. Let $T \in B_C(H)$ with a bounded M-P inverse $T^\dagger$. Suppose that T is hypo-EP. If $[T, TT^\dagger] = 0$ then T is EP.

Proof. By (1.1),

$$\begin{aligned}[T, TT^\dagger] &= T^2T^\dagger - TT^\dagger T \\ &= T^2T^\dagger - T = 0.\end{aligned}$$

Thus we have

$$(T^\dagger T)(TT^\dagger) = T^\dagger T.$$

Which means $T^\dagger T \leq TT^\dagger$. Thus T is EP by assumption. □

Wei and Chen [11] proved the following theorem, which is a theorem for the continuity of $T^\dagger$.

Theorem B (Cor.1 of [11]). Let $T \in B_C(H_1, H_2)$, and let $\{T_n\}$ be a sequence of operators in $B_C(H_1, H_2)$. Let $T_n^\dagger$ be the M-P inverse of T_n for every n. Suppose that $T_n \to T$ (with respect to the norm $||\cdot||$ on $B_C(H_1, H_2)$). Then the following conditions are equivalent:
(1) $T_n^\dagger \to T^\dagger$;
(2) $T_n^\dagger T_n \to T^\dagger T$;
(3) $\sup_n ||T_n^\dagger|| < \infty$.

By Theorem B, we get the following theorem.

Theorem 3.5. Let $T \in B_C(H)$, and let $\{T_n\}$ be a sequence of hypo-EP operators in $B_C(H)$. Let $T_n^\dagger$ be the M-P inverse of T_n for every n. Suppose that $T_n \to T$ (with respect to the norm $||\cdot||$ on $B_C(H)$). Then T is a hypo-EP operator.

Proof. It is clear from Theorem B that if $T_n \to T$ then $T_nT_n^\dagger \to TT^\dagger$. And following inequality holds,

$$||T_n^\dagger T_n x - T^\dagger T x|| \geq |||T_n^\dagger T_n x|| - ||T^\dagger T x|||.$$

Hence we have

$$||T_n^\dagger T_n x|| \to ||T^\dagger T x|| \text{ for all } x \in H.$$

Similarly, we obtain

$$||T_n T_n^\dagger x|| \to ||TT^\dagger x|| \text{ for all } x \in H.$$

Therefore, by Proposition 2.1 (7), we have

$$||T^\dagger T x|| = \lim_{n\to\infty} ||T_n^\dagger T_n x|| \geq \lim_{n\to\infty} ||T_n T_n^\dagger x|| = ||TT^\dagger x||.$$

That is, T is hypo-EP. □

According to Kato [8], for $T \in B(H_1, H_2)$, the *reduced minimum modulus* $\gamma(T)$ of T is defined as follows:

$$\gamma(T) = \inf \frac{\|Tx\|}{\operatorname{dist}(x, \ker T)},$$

where $\operatorname{dist}(x, \ker T) = \min_{y \in \ker T} \|x - y\|$.

The following statements are well known, see [8] for details.

Theorem C. For any $T \in B(H)$. Then
(1) $R(T)$ is closed if and only if $\gamma(T) > 0$;
(2) if $\gamma(T) > 0$ then $\|T^\dagger\| = \frac{1}{\gamma(T)}$.

By using of Theorem C, we show the following theorem.

Theorem 3.6. Let $T \in B_C(H)$ with a bounded M-P inverse $T^\dagger$. Suppose that T is hypo-EP but it is not EP. Then

$$\gamma(T^\dagger T - TT^\dagger) = 1.$$

To show this, we need the following two lemmas.

Lemma 3.7. Let P be a bounded projection operator. Then $P^\dagger = P$, $P^\dagger P = PP^\dagger = P$.

Proof. Since P is bounded, $P^\dagger P$ and $PP^\dagger$ are projection operators. Hence $(P^\dagger P)^* = P^\dagger P$. It follows from $(P^\dagger)^* = (P^*)^\dagger = P^\dagger$ that

$$PP^\dagger = P^*(P^\dagger)^* = (P^\dagger P)^* = P^\dagger P.$$

Thus, we have $PP^\dagger = P^\dagger P$. Using the above relation, we have

$$P = PP^\dagger P = P^2 P^\dagger = PP^\dagger.$$

Hence,

$$P^\dagger = P^\dagger P P^\dagger = P^\dagger P = P.$$

Therefore,

$$P^\dagger = P. \qquad \square$$

Theorem 3.8. Let $T \in B_C(H)$ with a bounded M-P inverse $T^\dagger$. Then

$$\gamma(T^\dagger T) = \gamma(TT^\dagger) = 1.$$

Proof. Since $T^\dagger T$ and $TT^\dagger$ are bounded operators, from Theorem C

$$\gamma(T^\dagger T) = \frac{1}{\|(T^\dagger T)^\dagger\|} \text{ and } \gamma(TT^\dagger) = \frac{1}{\|(TT^\dagger)^\dagger\|}.$$

Since $T^\dagger T$ and $TT^\dagger$ are non-trivial projection operators, from Lemma 3.7 we have

$$\|(T^\dagger T)^\dagger\| = \|T^\dagger T\| = 1 \text{ and } \|(TT^\dagger)^\dagger\| = \|TT^\dagger\| = 1.$$

Therefore,

$$\gamma(T^\dagger T) = \frac{1}{\|T^\dagger T\|} = 1 \text{ and } \gamma(TT^\dagger) = \frac{1}{\|TT^\dagger\|} = 1. \qquad \square$$

Proof of Theorem 3.6. Since T is not EP, $T^\dagger T - TT^\dagger$ is a non-trivial projection operator. Thus by Lemma 3.8 we have $\|T^\dagger T - TT^\dagger\| = 1$.
Therefore,

$$\gamma(T^\dagger T - TT^\dagger) = \frac{1}{\|(T^\dagger T - TT^\dagger)^\dagger\|} = \frac{1}{\|T^\dagger T - TT^\dagger\|} = 1. \qquad \square$$

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