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Notes on minimal normal compactifications of $\mathbf{C}^2/G$

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0 Introduction

Throughout the present article, we work over the field of complex numbers.

Definition 0.1 Let S be a normal affine surface and let (X, C) be a pair of a normal compact analytic surface X and a compact (analytic) curve C on X.

(1) We call the pair (X, C) a *minimal normal compactification* of S if the following conditions are satisfied:

(i) X is smooth along C.

(ii) Any singular point of C is an ordinary double point.

(iii) $X \setminus C$ is biholomorphic to S.

(iv) For any (-1)-curve $E \subset C$, we have $(E \cdot C - E) \geq 3$.

(2) Assume that (X, C) is a minimal normal compactification of S. Then (X, C) is said to be *algebraic* if X is algebraic, C is an algebraic subvariety of X and $X \setminus C$ is isomorphic to S as an algebraic variety.

For some smooth affine surfaces, their minimal normal compacitifications have been studied by several authors. In [10], Morrow gave a list of all minimal normal compactifications of the complex affine plane $\mathbf{C}^2$ by using a result of Ramanujam [12]. Ueda [14] and Suzuki [13] studied compactifications of $\mathbf{C} \times \mathbf{C}^*$ and $(\mathbf{C}^*)^2$, where $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. In particular, Suzuki [13] gave a list of all minimal normal compactifications of $\mathbf{C} \times \mathbf{C}^*$ and $(\mathbf{C}^*)^2$.

Recently, Abe, Furushima and Yamasaki [1] studied minimal normal compactifications of $S = \mathbf{C}^2/G$, where G is a small non-trivial finite subgroup of $GL(2, \mathbf{C})$, by using the theory of the cluster sets of holomorphic mappings due to Nishino and Suzuki [11]. They gave a rough classification of the weighted dual graphs of the boundary divisors of the minimal normal compactifications of S. In most cases, the singularity type of the unique singular point of S determines the weighted dual graph of the boundary divisor. However, in the case where the singular point of S

is cyclic or is of type D (for the definition, see [8, p. 91]), they did not determine the weighted dual graph of the boundary divisor.

In this note, we shall give some results on minimal normal compactifications of $\mathbf{C}^2/G$, where G is a finite subgroup of $GL(2,\mathbf{C})$. In §2, we give a characterization of $\mathbf{C}^2$ as a homology plane (cf. Theorem 2.1). In §3, we give a complete list of the dual graphs of the boundary divisors of the minimal normal compactifications of $\mathbf{C}^2/G$ in the case where G is non-trivial and non-cyclic (cf. Theorem 3.3).

By a $(-n)$-curve $(n \geq 1)$ we mean a smooth complete rational curve with self-intersection number $-n$. A reduced effective divisor D on a smooth surface is called an *SNC-divisor* (resp. an *NC-divisor*) if D has only simple normal crossings (resp. normal crossings). Let $f : V_1 \to V_2$ be a birational morphism between smooth algebraic surfaces V_1 and V_2 and let D_i $(i = 1, 2)$ be a divisor on V_i. Then we denote the direct image of D_1 on V_2 (resp. the total transform of D_2 on V_1, the proper transform of D_2 on V_1) by $f_*(D_1)$ (resp. $f^*(D_2)$, $f'(D_2)$).

1 Preliminaries

In this section, we prove some preliminary results which are used in §§2 and 3.

Definition 1.1 Let (V_1, D_1) and (V_2, D_2) be (minimal) normal algebraic compactifications of a normal affine surface S. Then we say that (V_1, D_1) is isomorphic to (V_2, D_2) if there exists an isomorphism $\varphi : V_1 \to V_2$ such that $\varphi(D_1) \subset D_2$ and $\varphi|_{D_1} : D_1 \to D_2$ is an isomorphism.

Let S be a normal affine surface and (V, D) a minimal normal algebraic compactification of S. In Lemmas 1.2 and 1.3, we retain this situation.

Lemma 1.2 *Assume that the following two conditions* (i) *and* (ii) *are satisfied:*

(i) *For any irreducible component E of D such that $E \cong \mathbf{P}^1$ and $(E^2) \geq -1$, we have $(E \cdot D - E) \geq 3$.*

(ii) *For any irreducible component F of D such that F is a rational curve with one node and $(F^2) \geq 3$, we have $(F \cdot D - F) \geq 1$.*

Then the pair (V, D) is the unique minimal normal algebraic compactification of S, up to isomorphisms.

Proof. Suppose to the contrary that S has another minimal normal algebraic compactification (V', D') which is not isomorphic to (V, D). Then there exists a birational map $f : V \cdots\to V'$ such that $f|_{V-D} : S \to S$ is an isomorphism. We have a composite of blowing-ups $g : W \to V$ such that $h = f \circ g : W \to V'$ becomes a birational morphism. Since (V, D) and (V', D') are minimal normal algebraic compactifications of S, f cannot be a morphism. So, $g \neq \mathrm{id}$. We may assume that g is the shortest among such birational morphisms.

Put $\tilde{D} := g^*(D)_{\text{red}}$. Then $\tilde{D}$ is an NC-divisor and the birational morphism h begins with the contraction of a (-1)-curve $E' \subset \tilde{D}$. Since $D' = h_*(\tilde{D})$ is an NC-divisor, $(E' \cdot \tilde{D} - E') \leq 2$. Put $E := g_*(E')$. By the assumption on g, E is not a zero divisor. Further, since D is an NC-divisor, either $E \cong \mathbf{P}^1$ or E is a rational curve with one node as singularities. If $E \cong \mathbf{P}^1$, then $(E^2) \geq -1$ and

$$(E \cdot D - E) \leq (E' \cdot \tilde{D} - E') \leq 2,$$

which contradicts the condition (i). If E is a rational curve with one node, then $(E^2) \geq 3$. If $(E \cdot D - E) \geq 1$, then the contraction of E' makes the direct image $h_*(\tilde{D}) = D'$ a non NC-divisor. So, $(E \cdot D - E) = 0$, which contradicts the condition (ii). □

Lemma 1.3 *Assume that $\kappa(V) \geq 0$, where $\kappa(V)$ denotes the Kodaira dimension of a smooth model of V. Then (V, D) is the unique minimal normal algebraic compactification of S, up to isomorphisms.*

Proof. Let $\tilde{V}$ be a smooth model of V. Since $\kappa(\tilde{V}) = \kappa(V) \geq 0$, $\tilde{V}$ has the unique minimal model, up to isomorphisms. Since V is smooth along D, we know that D contains no smooth rational curves ℓ with $(\ell^2) \geq 0$ and no rational curves F with one nodes and with $(F^2) \geq 3$. Hence the assertion follows from Lemma 1.2. Here we note that D has no (-1)-curves E with $(E \cdot D - E) \leq 2$ because (V, D) is a minimal normal algebraic compactification of S. □

Definition 1.4 Let S be a normal affine surface and let $\pi : \tilde{S} \to S$ be a resolution of singularities of S. We define the *logarithmic Kodaira dimension* $\overline{\kappa}(S)$ by $\overline{\kappa}(S) = \overline{\kappa}(\tilde{S})$, where $\overline{\kappa}(\tilde{S})$ denotes the logarithmic Kodaira dimension of $\tilde{S}$ (cf. [6]).

Lemma 1.5 *Let V be a smooth projective rational surface and D an irreducible rational curve with one node and with $(D^2) \geq 3$. Then $\overline{\kappa}(V \setminus D) \leq 1$.*

Proof. We may assume that $V \setminus D$ contains no (-1)-curves.

Let P be the node on D. Let $\pi : \tilde{V} \to V$ be the blowing-up with the center at P and let E be the exceptional curve. Put $\tilde{D} := \pi'(D) + E$. Then $\tilde{D}$ is an SNC-divisor. If $(D^2) \geq 4$, then $(\pi'(D)^2) \geq 0$. Since $\pi'(D) \cong \mathbf{P}^1$ and $(\pi'(D) \cdot E) = 2$, we can easily see that $\overline{\kappa}(V - D) = \overline{\kappa}(\tilde{V} - \tilde{D}) \leq 1$ (cf. [6]).

We treat the case where $(D^2) = 3$ (then $(\pi'(D)^2) = -1$). Suppose that $\overline{\kappa}(V - D) = \overline{\kappa}(\tilde{V} - \tilde{D}) = 2$. Assume first that $\tilde{D} + K_{\tilde{V}}$ is not nef. By using the theory of Zariski decomposition (cf. [6]), we obtain an irreducible curve F such that $(F \cdot \tilde{D} + K_{\tilde{V}}) < 0$ and $(F^2) < 0$. By the assumption that $V \setminus D$ contains no (-1)-curves, we know that F is a (-1)-curve with $(F \cdot \tilde{D}) = 1$. Let $f : \tilde{V} \to W$ be the contraction of F and put $f_*(\tilde{D}) = D_1' + D_2'$. Then $D_1' + D_2'$ is an SNC-divisor, $(D_1' \cdot D_2') = 2$ and one of D_1' and D_2' has self-intersection number zero. So $\overline{\kappa}(W - (D_1' + D_2')) = \overline{\kappa}(\tilde{V} - \tilde{D}) \leq 1$, which is a contradiction. Assume next that $\tilde{D} + K_{\tilde{V}}$ is nef. Noting that $(\tilde{D} \cdot \tilde{D} + K_{\tilde{V}}) = 0$

and $\tilde{D} + K_{\tilde{V}}$ is nef and big, we know that $(\tilde{D}^2) < 0$ by the Hodge index theorem. This is a contradiction because $(\tilde{D}^2) = 2$. □

Proposition 1.6 *Any normal affine surface S with $\overline{\kappa}(S) = 2$ (cf. Definition 1.4) has a unique minimal normal algebraic compactification, up to isomorphisms.*

Proof. Let (V, D) be a minimal normal algebraic compactification of S and $f : \tilde{V} \to V$ the minimal resolution of V. We may identify the divisor D on V with the divisor $f'(D) = f^{-1}(D)$ on $\tilde{V}$. If $\kappa(\tilde{V}) \geq 0$, then Lemma 1.3 implies that (V, D) is the unique minimal normal algebraic compactification of S, up to isomorphisms. Hence we may assume that $\kappa(\tilde{V}) = -\infty$.

To prove Proposition 1.6, it suffices to show that the pair (V, D) satisfies the conditions (i) and (ii) in Lemma 1.2 if $\overline{\kappa}(S)(= \overline{\kappa}(\tilde{V} \setminus D)) = 2$. Suppose first that D has an irreducible component E such that $E \cong \mathbf{P}^1$, $(E^2) \geq -1$ and $(E \cdot D - E) \leq 2$. Then $(E^2) \geq 0$ by the minimality of the pair (V, D). The hypothesis $(E \cdot D - E) \leq 2$ then implies that $\overline{\kappa}(S) \leq 1$, which is a contradiction. Hence the condition (i) in Lemma 1.2 is satisfied. Suppose next that D is an irreducible rational curve with one node and $(D^2) \geq 3$. It is then clear that $\tilde{V}$ is a rational surface. We infer from Lemma 1.5 that $\overline{\kappa}(S) \leq 1$, which is a contradiction. Hence, the condition (ii) in Lemma 1.2 also is satisfied. □

2 A characterization of the affine plane

A smooth affine surface S is called a *homology plane* if $H_i(S, \mathbf{Z}) = (0)$ for any integer $i > 0$. There are some characterizations of $\mathbf{C}^2$ as a homology plane. A homology plane S is isomorphic to $\mathbf{C}^2$ if and only if one of the following conditions is satisfied:

(1) $\overline{\kappa}(S) = -\infty$.

(2) S contains at least two topologically contractible algebraic curves.

For more details, see [8, Chapter 3, §4].

By using Lemma 1.2 and the results in [5], we obtain the following result.

Theorem 2.1 *Let S be a homology plane. Then $S \cong \mathbf{C}^2$ if and only if S has at least two non-isomorphic minimal normal algebraic compactifications.*

Proof. The "only if " part is clear. To prove the "if" part, it suffices to show that $\overline{\kappa}(S) = -\infty$, that is, if $\overline{\kappa}(S) \geq 0$ then S has a unique minimal normal algebraic compactification, up to isomorphisms.

Assume that $\overline{\kappa}(S) \geq 0$. Then $\overline{\kappa}(S) \geq 1$ by [4, §8] (see also [8, Theorem 4.7.1 (p. 244)]). If $\overline{\kappa}(S) = 2$, then it follows from Proposition 1.6 that S has a unique minimal normal algebraic compacitification, up to isomorphisms. So we may assume that $\overline{\kappa}(S) = 1$.

By [5, Theorems 3 and 4], there exists a $\mathbf{C}^*$-fibration $\varphi : S \to \mathbf{P}^1$ onto $\mathbf{P}^1$ such that every fiber of φ is irreducible. By using the arguments as in [5, §3], we can find a pair (V, D) of a smooth projective surface V and an SNC-divisor D on V such that the following conditions are satisfied:

(i) $V \setminus D$ is isomorphic to S.

(ii) There exists a $\mathbf{P}^1$-fibration $\Phi : V \to \mathbf{P}^1$ such that $\Phi|_S = \varphi$.

(iii) For any (-1)-curve $E \subset D$ in a fiber of Φ, we have $(E \cdot D - E) \geq 3$.

By [5, Lemma 3.2], φ is untwisted, that is, D has exactly two irreducible components D_1 and D_2 which are not contained in any fiber of Φ. By [9, Lemma 2.10 (3)], φ has exactly one fiber f_1 with $(f_1)_{\mathrm{red}} \cong \mathbf{A}^1$. Let F_1 be the fiber of Φ containing f_1. Then, by the condition (iii), we know that a fiber F of Φ different from F_1 is reducible if and only if the scheme-theoretic fiber $F|_S$ of φ is singular.

Since $\overline{\kappa}(S) = 1$ and the $\mathbf{C}^*$-fibration φ is untwisted, we know that φ has at least three singular fibers. Indeed, if not, then S contains $(\mathbf{C}^*)^2$ as a Zariski open subset. Then $1 = \overline{\kappa}(S) \leq \overline{\kappa}((\mathbf{C}^*)^2) = 0$, which is a contradiction. We can easily see that $(D_i \cdot D - D_i) \geq 3$ for $i = 1, 2$. By the condition (iii), (V, D) is a minimal normal algebraic compactification of S and satisfies the conditions (i) and (ii) in Lemma 1.2. Hence, by Lemma 1.2, (V, D) is the unique minimal normal algebraic compacitification of S, up to isomorphisms. □

For any homology plane $S = \operatorname{Spec} A$, the coordinate ring A is factorial and $A^* = \mathbf{C}^*$ (cf. [5], [9]). By Example 2.2 below, we know that Theorem 2.1 cannot be true in the case where $S = \operatorname{Spec} A$ is a smooth affine surface such that A is factorial and $A^* = \mathbf{C}^*$.

Example 2.2 Let ℓ_0, ℓ_1, ℓ_2 be non-concurrent three lines on $\mathbf{P}^2$ and let $P_1 \in \ell_1 \setminus (\ell_0 \cup \ell_2)$ and $P_2 \in \ell_2 \setminus (\ell_0 \cup \ell_1)$ be two points. Let $\sigma : V \to \mathbf{P}^2$ be the blowing-up with centers P_1 and P_2. Put $D = \ell_0' + \ell_1' + \ell_2'$, where $\ell_i' := \sigma'(\ell_i)$ $(i = 0, 1, 2)$, and $S := V - D$. Then S is a smooth affine surface such that $A = \Gamma(S, \mathcal{O}_s)$ is factorial and $A^* = \mathbf{C}^*$ (cf. [5, Theorem 2]). The pair (V, D) is a minimal normal algebraic compacitification of S. Put $Q := \ell_1' \cap \ell_2'$. Let $\sigma_1 : V_1 \to V$ be the blowing-up at Q and let $\mu : V_1 \to W$ be the contraction of $\sigma_1'(\ell_1')$ and $\sigma_1'(\ell_2')$. Then the pair (W, D_W) $(D_W = \mu_*(\sigma_1^{-1}(D)))$ is a minimal normal algebraic compactification of S and is not isomorphic to (V, D)

3 Compactifications of $\mathbf{C}^2/G$

In this section, we study minimal normal compactifications of $\mathbf{C}^2/G$.

We give some notions on weighted graphs. As for the notions on weighted graphs, the reader may consult [4].

Definition 3.1 Let A be a graph and $v_1, \ldots, v_r$ the vertices of A. Then A is a *twig* if A is a connected linear graph together with a total ordering $v_1 > v_2 > \cdots > v_r$ among its vertices such that v_j and v_{j-1} are connected by a segment for each j $(2 \leq j \leq r)$. Such a twig is denoted by $[a_1, \ldots, a_r]$, where a_j is the weight of v_j. A twig A is said to be *admissible* if $a_j \leq -2$ for every j. For an admissible twig A, we denoted the determinant of A by $d(A)$ (cf. [4, (3.3)]).

Definition 3.2 Let $A = [a_1, \ldots, a_r]$ be an admissible twig. Then the twig $[a_r, a_{r-1}, \ldots, a_1]$ is called the *transposal* of A and denoted by ${}^tA$. We define also $\overline{A} = [a_2, \ldots, a_r]$ and $\underline{A} ={}^t (\overline{{}^tA}) = [a_1, \ldots, a_{r-1}]$. If $r = 1$, we put $\overline{A} = \underline{A} = \emptyset$ (the empty set). We call $e(A) = d(\overline{A})/d(A)$ the *inductance* of A. By [4, Corollary (3.8)], e defines a one-to-one correspondence from the set of all admissible twigs to the set of rational numbers in the interval $(0, 1)$. Hence there exists uniquely an admissible twig A^* whose inductance is equal to $1 - e({}^tA)$. We call the admissible twig A^* the *adjoint* of A.

Now, let G be a small *non-cyclic* finite subgroup of $GL(2, \mathbf{C})$ and let $S = \mathbf{C}^2/G$ be the geometric quotient surface. Let $\pi : \tilde{S} \to S$ be the minimal resolution of the unique singular point of S and E the reduced exceptional divisor of π. By [3], E is an SNC-divisor and each component of E is a rational curve. Moreover, the weighted dual graph of E looks like that of Figure 1, where $b \geq 2$ and the subgraph $A_{(i)} := [-a_1^{(i)}, -a_2^{(i)}, \ldots, -a_{r_i}^{(i)}]$ $(i = 1, 2, 3)$ is an admissible twig and $\{d(A_{(1)}), d(A_{(2)}), d(A_{(3)})\}$ is one of the following Platonic triplets: $\{2, 2, n\}$ $(n \geq 2)$, $\{2, 3, 3\}$, $\{2, 3, 4\}$ and $\{2, 3, 5\}$.

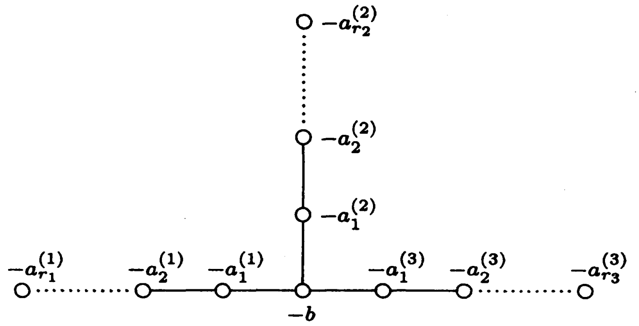

Figure 1

Now, we state the main result of this section.

Theorem 3.3 *With the same notation and assumptions as above, let (X, C) be a minimal normal compactification of S. Then we have:*

(1) *(X, C) is algebraic.*

(2) *C is an SNC-divisor, each component of C is a rational curve and the weighted dual graph of C looks like that of Figure 2, where the subgraph $B_{(i)} := [-b_1^{(i)}, -b_2^{(i)}, \ldots, -b_{s_i}^{(i)}]$ $(i = 1, 2, 3)$ is the adjoint of ${}^tA_{(i)}$.*

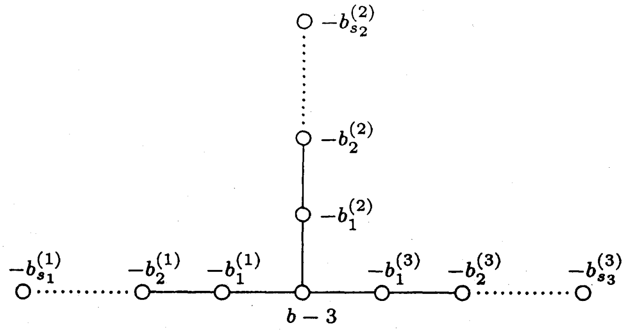

Figure 2

Remark 3.4 The assertion (1) and the assertion (2) except for the case $\{d(A_1), d(A_2), d(A_3)\} = \{2, 2, n\}$ $(n \geq 2)$ of Theorem 3.3 are proved by Abe–Furushima–Yamasaki [1].

Now, we prove Theorem 3.3. Let P be the unique singular point on $S = \mathbf{C}^2/G$ and put $T = S \setminus \{P\} (= \tilde{S} \setminus E)$. Then [7, Theorem 2 (2)] implies that T has a structure of Platonic $\mathbf{C}^*$-fiber space with respect to the $\mathbf{C}^*$-action induced by the $\mathbf{C}^*$-action on $\mathbf{C}^2$ via the center of $GL(2, \mathbf{C})$. More precisely, there exists a surjective morphism $f : T \to \mathbf{P}^1$ from T onto $\mathbf{P}^1$ such that the following four conditions are satisfied:

(1) General fibers of f are isomorphic to $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$.

(2) The generic fiber of f is isomorphic to $\mathbf{A}^1_{\mathbf{C}(t)} \setminus \{0\}$, where $\mathbf{C}(t)$ is the rational function field of one variable t.

(3) Every fiber of f is irreducible.

(4) f has exactly three singular fibers $\Delta_i = \mu_i \Gamma_i$ $(1 \leq i \leq 3)$ with $\Gamma_i \cong \mathbf{C}^*$, where $\{\mu_1, \mu_2, \mu_3\} = \{2, 2, n\}$ $(n \geq 2)$, $\{2, 3, 3\}$, $\{2, 3, 4\}$ or $\{2, 3, 5\}$.

(5) f has a normal completion $\overline{f} : \overline{T} \to \mathbf{P}^1$ (i.e., $\pi|_S = f$ and $\overline{T}$ is a smooth projective surface such that T is a Zarisiki open subset of $\overline{T}$ and $\overline{T} \setminus T$ is an NC-divisor) such that:

  (i) There exist two sections S_0 and S_1 of $\overline{f}$ such that S_0, $S_1 \subset \overline{T} \setminus T$, $S_0 \cap S_1 = \emptyset$, and other irreducible components of $\overline{T} \setminus T$ are contained in fibers of $\overline{f}$.

  (ii) Every fiber of $\overline{f}$ has a linear chain as its (weighted) dual graph.

As seen from [7, §3], we know that $\overline{T} \setminus T$ has two connected components.

Now, let $\tilde{f} : \tilde{S} \cdots\to \mathbf{P}^1$ be a rational map such that $\tilde{f}|_T = f$. We prove that:

Claim. We may assume that $\tilde{f}$ is a morphism. Moreover, $\tilde{f}$ is a $\mathbf{A}^1$-fibration onto $\mathbf{P}^1$, E has the unique component E_0 which is a section of $\tilde{f}$ and each component of $E - E_0$ is contained in a fiber of $\tilde{f}$.

Proof. By the condition (5) as above, we have birational morphisms $\tilde{g} : \tilde{V} \to \tilde{S}$ and $\tilde{h} : \tilde{V} \to \overline{T}$ such that $\tilde{g}$ is a composite of blowing-ups on E, $\tilde{g}|_{\tilde{S}\setminus E} = \mathrm{id}_T$ and $\tilde{h}$ is a contraction of curves in $\tilde{g}^{-1}(E)$ smoothly. Since E has no irreducible rational curves with self-intersecion number ≥ -1, we may assume that $\tilde{g} = \mathrm{id}_{\tilde{S}}$. Hence $\tilde{h} : \tilde{S} \to \overline{T}$ gives rise to an embedding of $\tilde{S}$ into $\overline{T}$. Hence, we may assume that $\tilde{f} = \overline{f}|_{\tilde{S}}$. Moreover, since $\overline{T} \setminus T$ has two connected components T_0 and T_1, we may assume that $E = T_0$ and $S_0 \subset E$. Thus, we know that $\tilde{f}$ is a $\mathbf{A}^1$-fibraion, that E has a unique component E_0 which is a section of $\tilde{f}$ and that each component of $E - E_0$ is contained in a fiber of $\tilde{f}$. □

By the claim as above, we know that $\tilde{S}$ contains a Zarisiki open subset isomorphic to $\mathbf{A}^1 \times C_0$, where C_0 is a smooth affine rational curve. Then [4, Theorem (9.6)] implies that every analytic compactification of $\mathbf{A}^1 \times C_0$ is algebraic. Since $\mathbf{A}^1 \times C_0$ is a Zariski open subset of $\tilde{S}$ and S has at most rational singularity, every analytic compactification of S is algebraic by [2, Theorem (2.3)]. This proves the assertion (1) of Theorem 3.3.

Let T_0 and T_1 be the connected components of $\overline{T} \setminus T$, where we assume that $S_i \subset T_i$. Put $U = T_0 + T_1$. By the condition (5) as above, we may assume further that $(E{\cdot}U{-}E) \geq 3$ for any (-1)-curve $E \subset U$. Then, a fiber F of $\overline{f}$ is reducible if and only if $F|_T$ is a multiple fiber of f. We note that every reducible fiber of $\overline{f}$ contains a unique (-1)-curve. By virtue of [4, Proposition (4.7)], the dual graph of a reducible fiber of $\overline{f}$ looks like that of Figure 3, where the subgraph $A := [-a_1, -a_2, \ldots, -a_r]$ is an admissible rational rod and $[-b_1, -b_2, \ldots, -b_s]$ the adjoint of A.

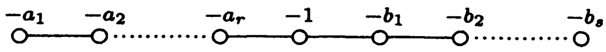

Figure 3

We can easily see that $(S_1^2) = -(S_0^2) - 3$. So, we may assume that $(S_0^2) \leq -2$. As seen from the proof of the claim as above, the weighted dual graph of T_0 is the same as that of E. The weighted dual graph of T_1 then looks like that of Figure 2.

Thus, we obtain an algebraic compactification (V, D) of $S = \mathbf{C}^2/G$ such that D is an SNC-divisor, each irreducible component of D is a rational curve and the weighted dual graph of D looks like that of Figure 2. By Lemma 1.2 and the assertion (1) of Theorem 3.3, (V, D) is the unique minimal normal compactification of S, up to isomorphisms. This proves the assertion (2) of Theorem 3.3.

The proof of Theorem 3.3 is thus completed.

References

[1] M. Abe, M. Furushima and M. Yamasaki, Analytic compactifications of $\mathbf{C}^2/G$, Kyushu J. Math., **54** (2000), 87–101.

[2] M. Artin, Some numerical criteria for contractability of curves on algebraic surfaces, Amer. J. Math., **84** (1962), 485–496.

[3] E. Brieskorn, Rationale Singularitäten komplexer Flächen, Invent. Math., **4** (1968), 336–358.

[4] T. Fujita, On the topology of non-complete algebraic surfaces, J. Fac. Sci. Univ. Tokyo, **29** (1982), 503–566.

[5] R. V. Gurjar and M. Miyanishi, Affine surfaces with $\bar{\kappa} \leq 1$, Algebraic Geometry and Commutative Algebra, Kinokuniya, Tokyo, 1987, pp. 99–124.

[6] M. Miyanishi, Non-complete algebraic surfaces, Lecture Notes in Mathematics, No. **857**, Berlin-Heiderberg-New York, Springer, 1981.

[7] M. Miyanishi, Normal affine subalgebras of a polynomial ring, Algebraic and topological theories – to the memory of Dr. Takehiko Miyata, Kinokuniya, Tokyo, 1985, pp. 37–51.

[8] M. Miyanishi, Open algebraic surfaces, CRM Monograph Series **12**, Amer. Math. Soc., 2000.

[9] M. Miyanishi and T. Sugie, Homology planes with quotient singularities, J. Math. Kyoto Univ., **31** (1991), 755–788.

[10] J. A. Morrow, Minimal normal compactifications of $\mathbf{C}^2$, Rice Univ. Stud., **59** (1973), 97–112.

[11] T. Nishino and M. Suzuki, Sur les singularités essentielles et isolées des applications holomorphes â valeurs dans une surface complexe, Publ. Res. Inst. Math. Sci., **16** (1980), 461–497.

[12] C. P. Ramanujam, A topological characterization of the affine plane as an algebraic variety, Ann. of Math., **94** (1971), 69–88.

[13] M. Suzuki, Compactifications of $\mathbf{C} \times \mathbf{C}^*$ and $(\mathbf{C}^*)^2$, Tôhoku Math. J., **31** (1979), 453–468.

[14] T. Ueda, Compactifications of $\mathbf{C} \times \mathbf{C}^*$ and $(\mathbf{C}^*)^2$, Tôhoku Math. J., **31** (1979), 81–90.

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