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Isomorphism classes of quasiperiodic tilings by the projection method

Kazushi KOMATSU and Kuniko SAKAMOTO

ABSTRACT. Let $\mathcal{T}(W_0)$ be the space of quasiperiodic tilings by the projection method in terms of $\mathbf{R}^d = E \oplus E^{\perp}$ with a lattice L and the orthogonal projection $\pi : \mathbf{R}^d \to E$. We will consider the case that $L = \mathbf{Z}^d$ or (E, L) which corresponds to an exceptional folding of Coxeter groups. We determine when two tilings in $\mathcal{T}(W_0)$ belong to the same isomorphism class if $\pi|L$ is injective. As its application we have uncountably many isomorphism classes of quasiperiodic tilings by the projection method.

1. Introduction

First, we will prepare several basic definitions. A tiling T of the space $\mathbf{R}^p$ is a countable family of closed sets called tiles: $T = \{T_1, T_2, \dots\}$ such that $\cup_{i=1}^{\infty} T_i = \mathbf{R}^p$ and Int $T_i \cap$ Int $T_j = \phi$ if $i \neq j$. An isomorphism of tilings is bijection between families of tiles that is induced by isometry of the space $\mathbf{R}^p$. An aperiodic tiling is one that admits no translation isomorphisms to itself. A tiling satisfies the local isomorphism property if for each bounded patch of the tiling there exists a positive real number r such that a translation of its patch appears in any ball of radius r. An quasiperiodic tiling is defined to be an aperiodic tiling with the local isomorphism property.

In 1981 de Bruijn [2], [3] introduced the projection method to construct quasiperiodic tilings such as Penrose tilings. The projection method was extended to the higher dimensional hypercubic lattices [5] and to more general lattices [6]. To construct tilings by the projection method, the hypercubic lattices are most frequently used. Furthermore some famous tilings are obtained from root lattices (cf. [1]). We recall the definitions of tilings by the projection method (cf. [5],[6],[9],[12]). Let L be a lattice in $\mathbf{R}^d$. Let E be a p-dimensional subspace of $\mathbf{R}^d$, and $E^{\perp}$ its orthogonal complement with respect to the standard inner product. Let $\pi : \mathbf{R}^d \to E$ be the orthogonal projection onto E, and $\pi^{\perp} : \mathbf{R}^d \to E^{\perp}$ the orthogonal projection onto $E^{\perp}$. Let $V(0)$ be the Voronoi cell in 0 of L. We put $W = \pi^{\perp}(V(0))$, which is called a window for the projection. W_0 is defined as the subset of W which consists of points s in W such that $\partial W \cap (s + \pi^{\perp}(L))$ is empty. For any $x \in \mathbf{R}^d$ such that $\pi(x) \in W_0$ we define $\Lambda(x)$ by $\Lambda(x) = \pi((W \times E) \cap (x + L))$. Let $\mathcal{V}(x)$ denote the Voronoi tiling induced by $\Lambda(x)$, which consists of the Voronoi cells of $\Lambda(x)$. For a vertex v in $\mathcal{V}(x)$ we define $S(v)$ by $S(v) = \bigcup\{P \in \mathcal{V}(x) | v \in P\}$. The tiling $T(x)$ given by the projection method is defined as the collection of tiles Conv $(S(v) \cap \Lambda(x))$, where Conv (B) denotes the convex hull of a set B. Note that $\Lambda(x)$ is the set of the vertices of $T(x)$. $\mathcal{T}(W_0)$ is defined to be the space $\{T(x) | x \in \mathbf{R}^d$ such that $\pi(x) \in W_0\}$ with a topology defined by a tiling metric (see for example [20]).

Let H be a folding of a Coxeter group G. If H is a non-crystallographic group, the folding is called exceptional. We have the settings of the root lattice L and the subspace E of projection method corresponding to an exceptional folding of a Coxeter group when G is A_4, B_4, F_4, D_6 or E_8-type and each folding H is $I_2(m)$ $(m = 5, 8, 12)$, H_3 or H_4-type (see [15],[16],[17]). It is known that H acts on W as isometries (for example see [8],[9]).

In this paper we consider the case that that $L = \mathbf{Z}^d$ or the above case that (E, L) corresponds to an exceptional folding.

We define an equivalence relation $s \sim t$ on W_0 by the following: $s \sim t$ $(s, t \in W_0)$ if there exists $s_0, s_1, \ldots, s_k \in W_0$ such that $s_0 = s, s_k = t$ and for any i $(i = 0, 1, \ldots k)$ $s_{i-1} - s_i \in \pi^{\perp}(L)$ or there exists isometry $g : W \to W$ such that $g(s_{i-1}) = s_i$ ($g \in H$ in the case of projection method corresponding to an exceptional folding).

For any $x \in \mathbf{R}^d$ such that $\pi(x) \in W_0$, we see that $T(x)$ is the translation of $T(\pi^{\perp}(x))$ by $\pi(x)$. Then each isomorphism classes in $\mathcal{T}(W_0)$ can be represented by $T(s)$ for some $s \in W_0$.

One of the purpose of this paper is to show the following theorem:

THEOREM. *For $s, t \in W_0$, let $T(s), T(t)$ be quasiperiodic tilings by the projection method in terms of $\mathbf{R}^d = E \oplus E^{\perp}$ with a lattice L and an orthogonal projection $\pi : \mathbf{R}^d \to E$. Assume that $L = \mathbf{Z}^d$ and $\pi|L$ is injective, or that (E, L) corresponds to an exceptional folding. Then $T(s)$ is isomorphic to $T(t)$ if and only if $s \sim t$.*

In the case that (E, L) corresponds to an exceptional folding, It is known that $\pi|L$ is injective ([15],[16],[17]). Two tilings are said to belong to the same local isomorphism class if every bounded patch that appears in one of them also appears in the other. Note that all tilings belong to a single local isomorphism class if π is injective. We define a map $\rho : W_0 \to \mathcal{T}(E)$ by $\rho(s) = T(s)$. Note that ρ is continuous, and that ρ induces a homeomorphism from $W_0/\sim$ to the space of isomorphism classes of $\mathcal{T}(W_0)$ by the Theorem if π is injective.

In [4] Danzer and Dolbilin show that there are uncountably many equivalence classes up to translation of quasiperiodic tilings obtained from a finite system of prototiles and local matching rules. In [18] Oger show the same result by applying finite Model theory.

As an application of the Theorem we give the simple proof of a similar result in the case of quasiperiodic tilings obtained by the projection method.

COROLLARY. *Assume that $L = \mathbf{Z}^d$ or that (E, L) corresponds to an exceptional folding. There are uncountably many isomorphism classes of quasiperiodic tilings by the projection method, in terms of $\mathbf{R}^d = E \oplus E^\perp$ with a lattice L, which are contained in a single local isomorphism class.*

When $L = \mathbf{Z}^d$, we have another variation of the projection method (see [11],[13],[14],[19]) to construct quasiperiodic tilings as Penrose tilings by rhomb tiles. In this variation we can also prove the similar theorems and Corollary by slightly modifying the proof in the section 2 and 3.

2. Proof of Theorem

In order to prove the Theorem it suffices to show the following proposition:

PROPOSITION. *For $s, t \in W_0$, let $T(s), T(t)$ be quasiperiodic tilings obatined by the projection method in terms of $\mathbf{R}^d = E \oplus E^\perp$ with a lattice L and the orthogonal projection $\pi : \mathbf{R}^d \to E$. Then,*
(1)$s - t \in \pi^\perp(L)$ if and only if $T(s)$ is a translation of $T(t)$.
(2) In the case that $L = \mathbf{Z}^d$, if there exists isometry $g : W \to W$ such that $g(s) = t$, then $T(s)$ is isomorphic to $T(t)$.
(3) In the case that $L = \mathbf{Z}^d$, if $T(s)$ is isomorphic to $T(t)$, then $s \sim t$.
(4) In terms of (E, L) corresponding to an exceptional folding, if there exists $g \in H$ such that $g(s) = t$, $T(s)$ is isomorphic to $T(t)$.

(5) In terms of (E, L) corresponding to an exceptional folding, if $T(s)$ is isomorphic to $T(t)$ by the underlying isometry ϕ and $\phi(0) = 0$, then ϕ belongs to H.

PROOF OF PROPOSITION

(1) A tiling obtained by projection method always satisfies the local isomorphism property. A tiling obtained by the projection method is aperiodic if and only if $\pi^{\perp}|L$ is injective by the aperiodic criterion([7] in the case that $L = \mathbf{Z}^d$, [12] L is an integral lattice). We define $f_s : W \cap (s + \pi^{\perp}(L)) \to \Lambda(s)$ by $f_s = (\pi|s + L) \circ (\pi^{\perp}|W \cap (s + \pi^{\perp}(L)))^{-1}$. Because $\pi^{\perp}|L$ and $\pi^{\perp}|L$ are injective, f_s is bijective.

If $T(s) = T(t)$, then $\Lambda(s) = \Lambda(t)$. We have a bijection $f_t^{-1} \circ f_s : W \cap (s + \pi^{\perp}(L)) \to W \cap (t + \pi^{\perp}(L))$. By the definition of f_t and f_s, we see that $f_t^{-1} \circ f_s$ is a translation map by a vector $t - s$. Due to [21] $\pi^{\perp}(L)$ is dense if and only if $\pi|L$ is injective. Then, we get that $W \cap (s + \pi^{\perp}(L)), W \cap (t + \pi^{\perp}(L))$ are dense in W. The assumption that $s \neq t$ implies the contradiction, and we get that $s = t$. Hence we obtain that $T(s) = T(t)$ if and only if $s = t$.

If $T(s)$ is a translation of $T(t)$, then $\Lambda(s) = v + \Lambda(t)$ for some $v \in \pi(L)$. We take $u \in L$ such that $v = \pi(u)$. Then we see that $(E \times W) \cap (s + L)$ $= v + (E \times W) \cap (t + L) = (E \times W) \cap (t + v + L) = (E \times W) \cap (t - \pi^{\perp}(u) + L)$. By the definition of the projection method in §1, we get that $T(s) = T(t - \pi^{\perp}(u))$. By the mentioned above $s = t - \pi^{\perp}(u)$, and we obtain that $s - t \in \pi^{\perp}(L)$.

If $s - t \in \pi^{\perp}(L)$, then we see that $(E \times W) \cap (s + L) = (E \times W) \cap (t + \pi^{\perp}(u) + L) = \pi(u) + (E \times W) \cap (t + L)$ for $u \in L$ such that $s - t = \pi^{\perp}(u)$. By the definition of the projection method in §1, we see that $\Lambda(s) = \pi(u) + \Lambda(t)$, and obtain that $T(s)$ is a translation of $T(t)$. The proof of Proposition (1) is completed.

(2) Assume that there exists an isometry $g : W \to W$ such that $g(s) = t$. Since $L = \mathbf{Z}^d$ is self-dual, the Voronoi cell $V(0)$ in 0 of L coincides with a translation of $A = \{\sum_{i=1}^{d} r_i e_i | 0 \leq r_i \leq 1\}$, where $\{e_i | i = 1, 2, \cdots, d\}$ is the standard basis of $L = \mathbf{Z}^d$. Since $W = \pi^{\perp}(V(0))$ and g is an isometry, $g(W \cap (s + \pi^{\perp}(L))) \subset W \cap (t + \pi^{\perp}(L))$. We define a bijection $\psi : \Lambda(s) \to \Lambda(t)$ by $\psi = f_t \circ g|(W \cap (s + \pi^{\perp}(L))) \circ f_s^{-1}$.

We will show that $\psi : \Lambda(s) \to \Lambda(t)$ is an isometry. We define $F : (E \times W) \cap (s + L) \to (E \times W) \cap (t + L)$ by $F = (\pi^{\perp}|(E \times W) \cap (t + L))^{-1} \circ g|(W \cap (s + \pi^{\perp}(L))) \circ (\pi^{\perp}|(E \times W) \cap (s + L))$. Then we see that $F(s + \sum_{i=1}^{d} \alpha_i e_i) = t + \sum_{1=1}^{d} \alpha_i g(e_i)$, $g(e_i) \in \{e_i \mid i = 1, 2, \cdots, d\}$ and $g(e_i) \neq g(e_j)$ if $i \neq j$. When lattice vectors in $\mathbf{Z}^d$ have the same combinations of coefficients for a basis $\{e_i\}$, the lengths of those lattice vectors are the same. Hence we get that F is an isometry. Because $F = \psi \times g$ and g is an isometry, ψ is also an isometry. Note that ψ extend to the isometry from E to E. Hence, $T(s)$ is isomorphic to $T(t)$. The proof of Proposition (2) is completed.

(3) If $T(s)$ is isomorphic to $T(t)$ by an isomorphism induced by the isometry $\phi : E \to E$, then we have a bijection $\phi : \Lambda(s) \to \Lambda(t)$. We take a vertex u of $T(s)$ and a vertex v of $T(t)$ such that $\phi(u) = v$. We define a bijection $h' : W \cap (s' + \pi^{\perp}(L)) \to W \cap (t' + \pi^{\perp}(L))$ by $h' = f_t^{-1} \circ \phi \circ f_s$. We put $s' = f_s^{-1}(u)$ and $t' = f_t^{-1}(v)$, and see that $h'(s') = t'$.

We will show that h' is an isometry. We define $G : (E \times W) \cap (s' + L) \to (E \times W) \cap (t' + L)$ by $G = (\pi|(E \times W) \cap (t' + L))^{-1} \circ \phi \circ (\pi|(E \times W) \cap (s' + L))$. Then we see that $G(s + \sum_{i=1}^{d} \beta_i e_i) = t + \sum_{i=1}^{d} \beta_i \phi(e_i)$, $\phi(e_i) \in \{e_i | i = 1, 2, \cdots, d\}$ and $\phi(e_i) \neq \phi(e_j)$ if $i \neq j$. When lattice vectors in $\mathbf{Z}^d$ have the same combinations of coefficients for a basis $\{e_i\}$, the lengths of those lattice vectors are the same. Hence we get that G is an isometry. Because $G = \phi \times h$ and ϕ is an isometry, h' is also an isometry. Due to [21] $\pi^{\perp}(L)$ is dense if and only if $\pi|L$ is injective. Then, we get that $W \cap (s + \pi^{\perp}(L))$ is dense in W, and that h' can extend to the isometry $h : W \to W$ such that $h(s') = t'$. Because $T(s)$ is translation of $T(s')$ and $T(t)$ is translation of $T(t')$, we see that $s - s', t - t' \in \pi^{\perp}(L)$ by Proposition (1) which has been proven above. Hence we get that $s \sim t$. The proof of Proposition (3) is completed.

(4) H acts on $(E \times W) \cap L$, E and W as isometries (see [8], [9] for example). By the similar argument to the proof of Proposition (2), we can prove that $T(s)$ is isomorphic to $T(t)$ if there exists $g \in H$ such that $g(s) = t$.

(5) Due to ([10],[15],[17],[18]) we recall the following results that are necessary for our proofs:

Let Σ be a root system of G that satisties the crystallographic condition. Then there exists a decomposition $\Sigma = \Sigma_\ell \coprod \Sigma_s$ and an inflation map $T : \mathbf{R}^d \to \mathbf{R}^d$ such that the following two conditions:

$\pi(\Sigma_s)$ is a root system of H,
$T(\Sigma_s) = \Sigma_\ell$,
$\pi(T(x)) = \alpha\pi(x)$ for $\forall x \in R^d$,
where $\alpha = \sqrt{3}$ if F_4-type, $\alpha = \sqrt{2}$ if B_4-type, $\alpha = \frac{1+\sqrt{5}}{2}$ if A_4, D_6, E_8-type.

If $T(s)$ is isomorphic to $T(t)$ by a underlying isometry $\phi : E \to E$ and $\phi(0) = 0$, then we have an isometry $\phi|\Lambda(s) : \Lambda(s) \to \Lambda(t)$. Hence the results quoted above imply that $\phi \in H$.

3. Proof of Corollary

W_0 is obtained by removing countably many $(d-p-1)$-dimensional polytopes from W. So, W_0 is an uncountable set. Since the isometry group of a $(d-p)$-dimensional polytope W is finite, each equivalence class for $\sim$ is a countable set. Hence $W_0/\sim$ is an uncountable set. By Theorem we obtain Corollary in the case that $L = \mathbf{Z}^d$ and $\pi|L$ is injective or that (E, L) corresponds to an exceptional folding. The rest of proof of Corollary is the case that $L = \mathbf{Z}^d$ and $\pi|L$ is not injective. Due to [21] there exist subspaces V_1 and V_2 which satisfy the following:

$E^{\perp} = V_1 \oplus V_2$,
$\pi^{\perp}((V_1 \oplus E) \cap L) = V_1 \cap \pi^{\perp}(L)$ is a discrete lattice in the subspace V_1,
$\pi^{\perp}((V_2 \oplus E) \cap L) = V_2 \cap \pi^{\perp}(L)$ is dense in the nontrivial subspace V_2.

Since $\pi|L$ is not injective, $\pi^{\perp}(L)$ is not dense. So, the subspace V_1 is nontrivial. We write the closure of $W \cap (s + \pi^{\perp}(L))$ by $C(s)$. Then we see that $C(s)$ is a union of q-dimensional polytopes, where $q = \dim V_2$. Note that for any $t \in C(s) \cap W_0$, every tiling $T(t)$ belongs to a single local isomorphism class (see [7],[11]). We can take $s_0 \in W_0$ which satisfies that $C(s_0) \cap W_0$ is an uncountable set. We consider the set $[s_0]$ which consists of $t \in C(s_0) \cap W_0$ such that $T(t)$ is isomorphic to $T(s_0)$. Since $\pi|(E \times W) \cap (s + L))$ is injective due to ([14],[19]), we can prove that $[s_0]$ is a countable set by the similar argument to the proof of Proposition (3). Hence we have uncountably many isomorphism classes of quasiperiodic tilings by the projection method. q.e.d.

References

[1] M. Baake, D. Joseph, P. Kramer and M. Schlottmann, Root lattices and quasicrystals, *J. Phys. A:Math. Gen.* **23**(1990), 1037–1041.

[2] N. G. de Bruijn, Algebraic theory of Penrose's non-periodic tilings of the plane. I, *Kon. Nederl. Akad. Wetesch. Proc Ser.A.***84** (=*Indag.Math.***43**)(1981),39–52.

[3] N. G. de Bruijn, Algebraic theory of Penrose's non-periodic tilings of the plane. II, *Kon. Nederl. Akad. Wetesch. Proc Ser.A.***84** (=*Indag.Math.***43**)(1981),53–66.

[4] L. Danzer and N. Dolbilin, Delone graphs:Some species and local rules, in *The Mathematics of Long-Range Aperiodic Order*, R.V.Moody(ed.), Kluwer Academic Publishers, 1997, 85–114.

[5] M. Duneau and A. Katz, Quasiperiodic patterns, *Phys. Rev. Lett.***54**(1985), 2688–2691.

[6] F. Gähler and J. Rhyner, Equivalence of the generalized grid and projection methods for the construction of quasiperiodic tilings, *J. Phys. A: Math. Gen.***19**(1986),267–277.

[7] C. Hillman, Sturmian dynamical systems, PhD. thesis, University of Washington, Seattle, 1998.

[8] S. Ishiguro, On symmetries of tilings obtained by folding root systems (in Japanese), Master thesis, Kochi Univ. (2001)

[9] S. Ishiguro and K. Komatsu, Symmetries of cut-and-project tilings corresponding to exceptional folding *Mem. Fac. Sci. Kochi Univ. (Math.)***23**(2002),93–97.

[10] N. Kawanaka, On non-crystallographic finite Coxeter groups (in Japanese), *The 12th algebraic combinatorial theory symposium*, Osaka Univ, (1995)

[11] K. Komatsu, On orbit closure decompositions of tiling spaces by the generalized projection method, *Hiroshima Math. J.* **30**(2000), 537–541

[12] K. Komatsu, Periods of cut-and-project tiling spaces obtained from root lattices, *Hiroshima Math. J.* **31**(2001), 435–438

[13] K. Komatsu, Digital approximations to affine spaces, *Mem. Fac. Sci. Kochi Univ. (Math.)* **24** (2003),61–65

[14] K. Komatsu and K. Nakahira, Some remarks on tilings by the projection method, *Mem. Fac. Sci. Kochi Univ. (Math.)***18**(1997),117–122.

[15] R. V. Moody and J. Patera, Quasicystals and icosians, *J. Phys. A : Math. Gen.* 26(1993)

[16] J. Morita and K. Sakamoto, Shell structure of dodecagonal quasicrystals associated with root system F_4 and cyclotomic field $\mathbf{Q}(\zeta_{12})$, *Comm. Algebra,* (1)**28**, 2000, pp.255-263.

[17] J. Morita and K. Sakamoto, Octagonal quasicrystals and a formula for shelling, *J. Phys.A. :Math. Gen.*, **31**,1998, pp.9321-9325.

[18] F. Oger, Algebraic and model-theoretic properties of tilings, to appear.

[19] C. Oguey, M. Duneau and A. Katz, A geometrical approach of quasiperiodic tilings, *Comm. Math. Phys.* **118**(1988), 99-118.

[20] E. Arthur Robinson, The dynamical theory of tilings and quasicrystallography, in *Ergodic theory of* $\mathbf{Z}^d$*-actions*, London Math. Soc. Lecture Note Ser. 228, Cambridge University Press, 1996, 451–473.

[21] Senechal, M. Quasicrystals and Geometry, Cambridge University Press, 1995.

K. Komatsu
Kochi University
Department of Mathematics
Faculty of Science
Kochi 780-8520 *Japan*
komatsu@math.kochi-u.ac.jp

K. Sakamoto
Food Engineering Division
National Food Research Institute
2-1-12 Kannondai Tsukuba Ibaraki
305-8642 *Japan*
kunikos@affrc.go.jp

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