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Some Properties of the Generalized Aluthge Transform

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Abstract

Let T be a bounded linear operator on a complex Hilbert space with the polar decomposition $T = U|T|$. Let $T(t) = |T|^t U|T|^{1-t}$ for $0 < t < 1$, and $T(0) = U^*UU|T|$ and $T(1) = |T|U$. $T(t)$ is called the generalized Aluthge transform of T. In this note, we discuss some properties of the generalized Aluthge transform.

Key Words and Phrases: Generalized Aluthge transform, polar decomposition, closed range, numerical range

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1 Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space with inner product $(\cdot,\cdot)$ and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For an operator $T \in \mathcal{B}(\mathcal{H})$, let $T = U|T|$ be a polar decomposition of T, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is a partial isometry with the initial space the closure of the range of $|T|$ and the final space the closure of the range of T. We denote the range space of T by $R(T)$ and the null space of T by $\ker(T)$. Let $\sigma(T), \sigma_p(T)$ and $\sigma_a(T)$ denote the spectrum, the point spectrum and the approximate point spectrum of T respectively. The numerical range $W(T)$ of T is defined by $W(T) = \{(Tx, x), x \in \mathcal{H} \text{ and } \|x\| = 1\}$. Put $\rho_D(T)= \{\lambda \in \mathbb{C} : R(T-\lambda) \text{ is closed}\}$, where $\mathbb{C}$ denote the complex plane. We call $\rho_D(T)$ the closed range points of T.

Recently, the Aluthge transform $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ has been studied by many authors(cf. [1, 2, 6, 7]). Some elementary spectral and numerical range properties and related results of $\tilde{T}$ are obtained. In [2], Cho and Tanahashi studied the generalized Aluthge transform $T(t)$ $(0 \le t \le 1)$, where $T(t) = |T|^t U|T|^{1-t}$ for $0 < t < 1$, $T(0) = U^*UU|T|$ and $T(1) = |T|U$. If we let $|T|^0 = U^*U$, then we have $T(t) = |T|^t U|T|^{1-t}$ for all $0 \le t \le 1$. Thus we let $|T|^0 = U^*U$ in this note. It is clear that $\tilde{T} = T(\frac{1}{2})$.

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In this note, we discuss some further properties of the generalized Aluthge transform. We generalize some properties of $\tilde{T}$ (see [6, 7]) to the generalized Aluthge transform $T(t)$, for $0 \le t \le 1$. We prove that T and $T(t)(0 \le t \le 1)$ have the same nonzero closed range points, that is, $\rho_D(T)\backslash\{0\} = \rho_D(T(t))\backslash\{0\}$ for all $t \in [0,1]$. Moreover some other properties are also obtained.

2 Main results

Let $T \in \mathcal{B}(\mathcal{H})$ have the polar decomposition $T = U|T|$. Then we have $\ker T = \ker|T| = \ker U$ and $\overline{R(T^*)} = \overline{R(|T|)}$. In term of the orthogonal decomposition $\mathcal{H} = \ker T \oplus \overline{R(T^*)}$ of $\mathcal{H}$, T has the following matrix form

$$T = \begin{pmatrix} 0 & A \\ 0 & B \end{pmatrix} \tag{1}$$

for some bounded linear operators A from $\overline{R(T^*)}$ to $\ker T$ and B on $\overline{R(T^*)}$. Now it is known that

$$U = \begin{pmatrix} 0 & U_1 \\ 0 & U_2 \end{pmatrix} \quad \text{and} \quad T^*T = \begin{pmatrix} 0 & 0 \\ 0 & A^*A + A^*A \end{pmatrix}$$

for some operators U_1 and U_2. By a direct calculus, we have

$$T(t) = \begin{pmatrix} 0 & 0 \\ 0 & (A^*A+B^*B)^{\frac{t}{2}}U_2(A^*A+A^*A)^{\frac{1-t}{2}} \end{pmatrix} \tag{2}$$

for $0 < t < 1$,

$$T(0) = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad T(1) = \begin{pmatrix} 0 & 0 \\ 0 & (A^*A+A^*A)^{\frac{1}{2}}U_2 \end{pmatrix} \tag{3}$$

respectively. Put $T_1(t) = (A^*A+B^*B)^{\frac{t}{2}}U_2(A^*A+B^*B)^{\frac{1-t}{2}}$ for $0 < t < 1$, $T_1(0) = B$ and $T_1(1) = (A^*A + A^*A)^{\frac{1}{2}}U_2$ respectively, then we have $T(t) = 0 \oplus T_1(t)$ for $0 \le t \le 1$. It easily follows that $\ker T$ is a reducing subspace of $T(t)$ for all $t \in [0,1]$ from (2) and (3).

Lemma 1. Let $T \in \mathcal{B}(\mathcal{H})$ have the matrix form (1). Then for any $\lambda \in \mathbb{C}$, $\lambda \ne 0$, $R(T-\lambda) = \ker T \oplus R(B-\lambda)$. Moreover, $R(T-\lambda)$ is closed if and only if $R(B-\lambda)$ is closed.

Proof. Let $z = x \oplus y \in \mathcal{H}$. Then $(T-\lambda)z = (-\lambda x + Ay) \oplus (B-\lambda)y \in \ker T \oplus R(B-\lambda)$. Conversely, for any $\xi \oplus \eta \in \ker T \oplus R(B-\lambda)$, there is an $y \in \overline{R(T^*)}$ such that $\eta = (B-\lambda)y$. Putting $x = \lambda^{-1}(Ay - \xi)$, we have $(T-\lambda)(x \oplus y) = \xi \oplus \eta$.

It is trivial that $R(T-\lambda)$ is closed if and only if $R(B-\lambda)$ is closed. The proof is complete.

The following two lemmas will be useful in the latter.

Lemma 2. (see [2]) Let $T(t) = |T|^t U|T|^{1-t} (0 \leq t \leq 1)$ be the generalized Aluthge transform of T. Then $\sigma(T) = \sigma(T(t))$ and $\sigma_p(T) = \sigma_p(T(t))$ for $0 \leq t \leq 1$.

Lemma 3. (see [2]) Let $T(t) = |T|^t U|T|^{1-t} (0 \leq t \leq 1)$ be the generalized Aluthge transform of T. Then $\sigma_a(T) = \sigma_a(T(t))$ for $0 \leq t < 1$. In general, $\sigma_a(T) \neq \sigma_a(T(1))$.

Now we consider a relationship between $\rho_D(T)$ and $\rho_D(T(t))$. In [7, Theorem 1], Ruan and Yan shown that if $\ker(T) \subset \ker(T^*)$, then $\rho_D(T) = \rho_D(\tilde{T})$. In fact, we generally have

Theorem 4. Let $T(t) = |T|^t U|T|^{1-t} (0 \leq t \leq 1)$ be the generalized Aluthge transform of T. Then $\rho_D(T) \backslash \{0\} = \rho_D(T(t)) \backslash \{0\}$ for all $t \in [0,1]$.

Proof. For $0 \leq t \leq 1$, let $\lambda \in \rho_D(T(t))$ and $\lambda \neq 0$. By Lemma 1, it is sufficient to show that $R(B-\lambda)$ is closed. Suppose $\eta \in \overline{R(B-\lambda)}$. Then there exists a sequence $\{\xi_n\} \subset \overline{R(T^*)}$ such that $\lim_{n\to\infty} (B-\lambda)\xi_n = \eta$. Put $z_n = 0 \oplus \xi_n$ and $z = 0 \oplus \eta$. Note that

$$(T(t)-\lambda)|T|^t = (|T|^t U|T|^{1-t} - \lambda)|T|^t = |T|^t (T-\lambda).$$

It follows that $\lim_{n\to\infty} (T(t)-\lambda)|T|^t z_n = \lim_{n\to\infty} |T|^t (T-\lambda) z_n = |T|^t z$. Since $\lambda \in \rho_D(T(t))$, there exists an $y \in \mathcal{H}$ such that $|T|^t z = (T(t)-\lambda)y$. Hence

$$\lambda y = T(t)y - |T|^t z = |T|^t (U|T|^{1-t} y - z).$$

Note that $\lambda \neq 0$, putting $x = \frac{1}{\lambda}(U|T|^{1-t}y - z) = x_1 \oplus x_2$, then $y = |T|^t x$ and

$$|T|^t z = (T(t)-\lambda)y = (T(t)-\lambda)|T|^t x = |T|^t (T-\lambda)x.$$

Then $z - (T-\lambda)x \in \ker(|T|^t) = \ker T$. It follows that $\eta = (B-\lambda)x_2$, which implies that $R(B-\lambda)$ is closed. Thus $\lambda \in \rho_D(T)$.

Conversely let $\lambda \in \rho_D(T)$ and $\lambda \neq 0$. Suppose $z = z_1 \oplus z_2 \in \overline{R(T(t)-\lambda)}$. Then there exists a sequence $\{x_n\}$ in $\mathcal{H}$ such that $\lim_{n\to\infty} (T(t)-\lambda)x_n = z$. Now we have $(T-\lambda)U|T|^{1-t} = U|T|^{1-t}(T(t)-\lambda)$ for $0 \leq t \leq 1$. Then

$$\lim_{n\to\infty} (T-\lambda)U|T|^{1-t} x_n = \lim_{n\to\infty} U|T|^{1-t}(T(t)-\lambda)x_n = U|T|^{1-t} z.$$

It follows that there exists an $y \in \mathcal{H}$ such that $U|T|^{1-t}z = (T - \lambda)y$. Thus

$$\lambda y = Ty - U|T|^{1-t}z = U|T|^{1-t}(|T|^t y - z).$$

Put $x = \frac{1}{\lambda}(|T|^t y - z) = x_1 \oplus x_2$, then $y = U|T|^{1-t}x$ and

$$U|T|^{1-t}z = (T - \lambda)U|T|^{1-t}x = U|T|^{1-t}(T(t) - \lambda)x.$$

It follows that $z - (T(t) - \lambda)x \in \ker U|T|^{1-t} = \ker T$. Then $z_2 = (T_1(t) - \lambda)x_2$. Put $\xi = -\lambda^{-1}z_1 \oplus x_2$, then we have $(T(t) - \lambda)\xi = z$, that is $R(T(t) - \lambda)$ is closed. Hence $\lambda \in \rho_D(T(t))$. Thus $\rho_D(T)\backslash\{0\} = \rho_D(T(t))\backslash\{0\}$ for all $0 \leq t \leq 1$. The proof is complete.

Corollary 5. Let $T \in B(\mathcal{H})$. If $\ker T \subset \ker T^*$, then $\rho_D(T) = \rho_D(T(t))$ for $0 \leq t < 1$.

Proof. By Theorem 4, it is sufficient to consider the case that $\lambda = 0$. Since $\ker T \subset \ker T^*$, $\ker T$ is a reducing subspace for both T and $T(t)$, $0 \leq t < 1$. Then $A = 0$ and B is injective in the matrix form (1). We thus have $T(t) = 0 \oplus B(t)$, for all $0 \leq t < 1$, where $B(t)$ is the generalized Aluthge transform of B. Note that $R(T)$(resp. $R(T(t))$) is closed if and only $R(B)$ (resp. $R(B(t))$ is closed. Thus without loss of generality, we may assume that $\ker T = \{0\}$. Then U is an isometry and $T(0) = T$. Suppose $0 < t < 1$. If $0 \in \rho_D(T)$, then by assumption, $0 \notin \sigma_a(T)$. By Lemma 2 and Lemma 3, we have $0 \notin \sigma_a(T(t))$. Thus $R(T(t))$ is closed, that is, $0 \in \rho_D(T(t))$. The converse is similar. The proof is complete.

However, we do not have $\rho_D(T) = \rho_D(T(t))$ for $0 \leq t \leq 1$ in general.

Example 6. Let A be a positive compact operator on $\mathcal{H}$ with $\ker A = \{0\}$ and put

$$T = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}.$$

Then we easily have $T(t) = 0$ for all $0 \leq t \leq 1$. Thus $R(T(t))$ is closed for all $0 \leq t \leq 1$ while $R(T)$ is not closed.

Theorem 7. For any operator $T \in \mathcal{B}(\mathcal{H})$ and $\forall \lambda \in \mathbb{C}, \lambda \neq 0$, we have $\dim(\ker(T - \lambda)^*)= \dim(\ker(T(t) - \lambda)^*)$, $0 \leq t \leq 1$.

Proof. First we prove $\dim(\ker(T-\lambda)^*)\leq \dim(\ker(T(t)-\lambda)^*)$ for any $\lambda \in \mathbb{C}, \lambda \neq 0$. If $\ker(T - \lambda)^* = \{0\}$, the proof is complete. Otherwise, let $\ker(T - \lambda)^* \neq \{0\}$. Suppose that $x_i \in \ker(T-\lambda)^*$ $(i = 1, 2, ..., n)$ are any finite vectors which are linearly independent. Since $(T-\lambda)U|T|^{1-t} = U|T|^{1-t}(|T|^tU|T|^{1-t}-\lambda) = U|T|^{1-t}(T(t)-\lambda)$ for all $0 \le t \le 1$, we have

$$(U|T|^{1-t})^*(U|T|-\lambda)^*x_i = (|T|^tU|T|^{1-t}-\lambda)^*(U|T|^{1-t})^*x_i.$$

Also since $x_i \in \ker(T-\lambda)^*$, $(U|T|^{1-t})^*x_i \in \ker(T(t)-\lambda)^*$ for $n = 1, 2, \cdots n$. We claim that $(U|T|^{1-t})^*x_1$, $(U|T|^{1-t})^*x_2$, ...,$(U|T|^{1-t})^*x_n$ are linearly independent. In fact, if there exist complex numbers $a_1, a_2, ..., a_n$, where not all of them are zero, such that $\Sigma_{i=1}^n a_i(U|T|^{1-t})^*x_i = 0$, then, $(U|T|^{1-t})^*(\Sigma_{i=1}^n a_ix_i) = 0$. Note that

$$T^*(\Sigma_{i=1}^n a_ix_i) = (|T|^t)^*(U|T|^{1-t})^*(\Sigma_{i=1}^n a_ix_i) = 0.$$

It follows that $\overline{\lambda}(\Sigma_{i=1}^n a_ix_i) = 0$ since $x_i \in \ker(T-\lambda)^*$. Thus $\Sigma_{i=1}^n a_ix_i = 0$ since $\lambda \neq 0$. This contradicts to the fact that x_1, x_2,..., x_n are linearly independent. Therefore $(U|T|^{1-t})^*x_1$, $(U|T|^{1-t})^*x_2$, ...,$(U|T|^{1-t})^*x_n$ are linearly independent. Hence $\dim(\ker(T-\lambda)^*) \le \dim(\ker(T(t)-\lambda)^*)$.

We can prove the converse inequality by the similar method. The proof is complete.

We recall that an operator $T = U|T|$ is a quasinormal operator if $U|T| = |T|U$. In [6], Jung *et al* proved that an operator is quasinormal if and only if $\tilde{T} = T$. Now we have the following generalized result.

Theorem 8. Let $T \in B(\mathcal{H})$ have the polar decomposition $T = U|T|$. Then the following are equivalent.

(1) $T(t) = T$ for all $t \in (0,1)$;
(2) $T(t) = T$ for some $t \in (0,1)$;
(3) $|T|^tU = U|T|^t$ for some $t \in (0,1)$; and
(4) T is quasinormal.

Proof. The implication from (1) to (2) is trivial.

(2) $\Longrightarrow$ (3) If $T(t) = T$ for some $t \in (0,1)$, then

$$(|T|^tU - U|T|^t)|T|^{1-t} = 0,$$

which implies that $|T|^tU - U|T|^t$ vanishes on $\overline{R(|T|^{1-t})} = \overline{R(|T|)} = \overline{R(T^*)}$. However, $\ker|T| = \ker U = \ker|T|^t = \ker T$, then $|T|^tU - U|T|^t$ vanishes on $\ker T$. Hence $|T|^tU = U|T|^t$ on $\mathcal{H}$. Thus $|T|^tU = U|T|^t$ for some $t \in (0,1)$.

(3) $\Longrightarrow$ (4) Let $w(|T|)$ (resp. $w(|T|^t)$) be the (self-adjoint) weakly closed subalgebra generated by $|T|$(resp. $|T|^t$). Then $w(|T|) = w(|T|^t)$. By (3), we have U is in the commutant $(w(|T|^t))'$ of $|T|^t$. Note that $(w(|T|^t))' = (w(|T|))'$. Then $U|T| = |T|U$, that is, T is quasinormal.

(4) $\Longrightarrow$ (1) We note that $|T|^tU = U|T|^t$ holds for all $t \in (0,1)$. Then

$$T = U|T| = U|T|^t|T|^{1-t} = |T|^tU|T|^{1-t} = T(t).$$

The proof is complete.

At last, we give the following elementary relationship between numerical ranges of the generalized Aluthge transform of T.

Proposition 9. Let $T(t) = |T|^tU|T|^{1-t}(0 \le t \le 1)$ be the generalized Aluthge transform of T. Then

$$W(T(t)) \subset W(|T|^{2t})W(U|T|^{1-2t}) \text{ if } 0 \le t \le \frac{1}{2},$$

and $W(T(1)) \subset W(U^*U)W(T)$.

Proof. Let $x \in \mathcal{H}$, $\|x\| = 1$, and $0 \le t \le \frac{1}{2}$. If $|T|x = 0$, then $|T|^{1-t}x = |T|^tx = 0$ and $(T(t)x, x) = (|T|^tU|T|^{1-t}x, x) = (|T|^{2t}x, x) = 0$. It follows that $0 \in W(|T|^{2t})W(U|T|^{1-2t})$. If $|T|x \ne 0$, then we have

$$\begin{aligned}(T(t)x,x) &= (|T|^tU|T|^{1-t}x,x) = (U|T|^{1-2t}|T|^tx, |T|^tx) \\ &= (|T|^tx, |T|^tx)\left(\tfrac{U|T|^{1-2t}|T|^tx}{\||T|^tx\|}, \tfrac{|T|^tx}{\||T|^tx\|}\right) \in W(|T|^{2t})W(U|T|^{1-2t}).\end{aligned}$$

Thus,

$$W(T(t)) \subset W(|T|^{2t})W(U|T|^{1-2t}), 0 \le t \le \frac{1}{2}.$$

Similarly, if $Ux = 0$, then we have $(T(1)x, x) = 0 \in W(U^*U)W(T)$. If $Ux \ne 0$, then we have

$$\begin{aligned}(T(1)x,x) &= (|T|Ux,x) = (U^*U|T|Ux,x) \\ &= (U^*Ux,x)\left(\tfrac{U|T|Ux}{\|Ux\|}, \tfrac{Ux}{\|Ux\|}\right) \in W(U^*U)W(U|T|).\end{aligned}$$

Thus

$$W(T(1)) \subset W(U^*U)W(T).$$

The proof is complete.

Remark. From Proposition 9, we have the following results:
$W(T(0)) = W(U^*UU|T|) \subset W(U^*U)W(T)$,
$W(T(\frac{1}{2})) = W(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) \subset W(|T|)W(U)$ (see Proposition 1.8 in [6]) and
$W(T(1)) = W(|T|U) \subset W(U^*U)W(T)$.

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