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Commutators of Orthogonal Projections

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Abstract

In this note we prove that a bounded linear operator T on a complex separable Hilbert space $\mathcal{H}$ is a commutator of projections if and only if $T^* = -T$, $\|T\| \leq \frac{1}{2}$ and T is unitarily equivalent to T^*.

Key Words and Phrases: generic pairs of projections, orthogonal projections, spectrum.
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1 Introduction

Let $\mathcal{H}$ be a separable complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. A operator C in $\mathcal{B}(\mathcal{H})$ is said to be a commutator of operators if $C = AB - BA$ for some $A, B \in \mathcal{B}(\mathcal{H})$. In [6], some elementary properties for an operator to be a commutator were considered and related results have been studied by several authors(cf.[1, 2, 4]). Very recently, Drnovesk *et al* in [3] considered a characterization of commutators of idempotents in an algebra. In this note, we consider the commutator of orthogonal projections. We intensify the results in [3] for self adjoint idempotent in a $*-$ algebra. We prove that an operator T is a commutator of orthogonal projections if and only if $T^* = -T$, $\|T\| \leq \frac{1}{2}$ and T is unitarily equivalent to T^*.

We next recall some notations and terminologies . For $A \in \mathcal{B}(\mathcal{H})$, $R(A)$, $N(A)$, $\sigma(A)$, $r(A)$ and $\sigma_p(A)$ denote the range, the null space, the spectrum, the spectrum radius and the point spectrum of A, respectively. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be positive if $(Ax, x) \geq 0$ for all $x \in \mathcal{H}$ and A is an idempotent if $A^2 = A$. An orthogonal projection is a positive idempotent. A pair (P, Q) of projections means two orthogonal projections P and Q in $\mathcal{B}(\mathcal{H})$. $\mathbb{N}$ and $\mathbb{R}$ denote the positive integer and real number, respectively. For a closed subspace M of $\mathcal{H}$, $\dim M$ denotes the dimension of it. Let $\{A_i\}$ be a net in $\mathcal{B}(\mathcal{H})$, $A_i \to A$ (SOT) means $\{A_i\}$ convergent to an operator A in $\mathcal{B}(\mathcal{H})$ in strong operator topology.

2 Main results

At first, we recall the following well-known result.

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Lemma 1. Let $\mathcal{H}$ have a orthogonal sum decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and let $A \in B(\mathcal{H})$ be an operator with the following operator matrix form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{1}$$

Then A is positive if and only if A_{ii} is a positive operator on $\mathcal{H}_i$ for $i = 1, 2$, and $A_{21}^* = A_{12} = A_{11}^{\frac{1}{2}} D A_{22}^{\frac{1}{2}}$ for a contraction D from $\mathcal{H}_2$ into $\mathcal{H}_1$.

Definition 2.([5]) Let P and Q are two projections in $\mathcal{B}(\mathcal{H})$. If P and Q have no common eigenvalues, then (P, Q) is called a generic pair.

Note that (P, Q) is a generic pair if and only if

$$R(P) \cap R(Q) = R(P) \cap N(Q) = N(P) \cap R(Q) = N(P) \cap N(Q) = \{0\}.$$

It is clear that (P, Q) is a generic pair of projections if and only if so are $(I - P, Q)$, $(I - P, I - Q)$, and $(P, I - Q)$, where I denotes the identity on $\mathcal{H}$. For convenience, we do not distinguish the identities acting on different spaces and denote them by I. If P and Q are two projections on $\mathcal{H}$, then by Lemma 1, they have the following operator matrix forms

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} Q_{11} & Q_{11}^{\frac{1}{2}} D Q_{22}^{\frac{1}{2}} \\ Q_{22}^{\frac{1}{2}} D^* Q_{11}^{\frac{1}{2}} & Q_{22} \end{pmatrix} \tag{2}$$

corresponding to the space decomposition $\mathcal{H} = R(P) \oplus R(P)^{\perp}$.

Lemma 3. If (P, Q) is a generic pair such that P and Q have the operator matrix forms as (2), then

(1) Q_{11} and Q_{22} are positive operators on $R(P)$ and $N(P)$, respectively;

(2) 0 and 1 are not in $\sigma_p(Q_{ii})$ for $i = 1, 2$ and consequently Q_{11}, Q_{22}, $I - Q_{11}$ and $I - Q_{22}$ are injective.;

(3) D is a unitary operator from $N(P)$ onto $R(P)$ and is uniquely determined by (P, Q);

(4) $Q_{11} = D(I - Q_{22})D^*$ and $Q_{22} = D^*(I - Q_{11})D$;

(5) $\dim R(P) = \dim N(P)$.

Proof. (1) This follows from Lemma 1.

(2) For the space decomposition $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{R}(P)^{\perp}$, P and Q have operator matrix forms as (2). Since the pair (P, Q) of projections is generic, it is easy to show that both 0 and 1 are not in $\sigma_p(Q_{ii})$ for $i = 1, 2$. Consequently Q_{11}, Q_{22}, $1 - Q_{11}$ and $1 - Q_{22}$ are injective.

(3) Q is a projection, so

$$\begin{pmatrix} Q_{11} & Q_{11}^{\frac{1}{2}} D Q_{22}^{\frac{1}{2}} \\ Q_{22}^{\frac{1}{2}} D^* Q_{11}^{\frac{1}{2}} & Q_{22} \end{pmatrix} = \begin{pmatrix} Q_{11}^2 + Q_{11}^{\frac{1}{2}} D Q_{22} D^* Q_{11}^{\frac{1}{2}} & Q_{11}^{\frac{3}{2}} D Q_{22}^{\frac{1}{2}} + Q_{11}^{\frac{1}{2}} D Q_{22}^{\frac{3}{2}} \\ Q_{22}^{\frac{1}{2}} D^* Q_{11}^{\frac{3}{2}} + Q_{22}^{\frac{3}{2}} D^* Q_{11}^{\frac{1}{2}} & Q_{22}^2 + Q_{11}^{\frac{1}{2}} D^* Q_{11} D Q_{22}^{\frac{1}{2}} \end{pmatrix}.$$

Comparing the entries of matrices in two side of the above equation , we have

$$\begin{cases} Q_{11} = Q_{11}^2 + Q_{11}^{\frac{1}{2}} D Q_{22} D^* Q_{11}^{\frac{1}{2}}, \\ Q_{22} = Q_{22}^2 + Q_{11}^{\frac{1}{2}} D^* Q_{11} D Q_{22}^{\frac{1}{2}}, \\ Q_{11}^{\frac{1}{2}} D Q_{22}^{\frac{1}{2}} = Q_{11}^{\frac{3}{2}} D Q_{22}^{\frac{1}{2}} + Q_{11}^{\frac{1}{2}} D Q_{22}^{\frac{3}{2}}. \end{cases}$$

Considering that Q_{11} and Q_{22} are injective, hence

$$\begin{cases} I_1 = Q_{11} + D Q_{22} D^*, \\ I_2 = Q_{22} + D^* Q_{11} D, \\ D = Q_{11} D + D Q_{22}, \end{cases} \tag{3}$$

where I_1 and I_2 are identities on $\mathcal{R}(P)$ and $\mathcal{N}(P)$, respectively. From the first equation of (3), we obtain that $I_1 - Q_{11} = DQ_{22}D^*$. Note that $I_1 - Q_{11}$ is injective since that 1 is not in $\sigma_p(Q_{11})$, then both D and D^* are injective. Substituting the first equation of (3) into the third equation of (3), $D = (I_1 - DQ_{22}D^*)D + DQ_{22}$, so $DQ_{22}(D^*D - I_2) = 0$. Note that both D and Q_{22} are injective, then

$$D^*D = I_2. \tag{4}$$

Similarly,

$$DD^* = I_1. \tag{5}$$

Combining (4) with (5), D is a unitary operator from $\mathcal{N}(P)$ onto $\mathcal{R}(P)$.

(4) By the second equation of (3) and (4), $Q_{22} = D^*(I_1 - Q_{11})D$. Similarly, $Q_{11} = D(I - Q22)D^*$, then the conclusion holds.

(5) This follows from the fact that D is unitary from $\mathcal{N}(P)$ onto $\mathcal{R}(P)$.

The proof is completed.

Assume that P and Q are two orthogonal projections on $\mathcal{H}$. Put $\mathcal{H}_1 = R(P) \cap R(Q)$, $\mathcal{H}_2 = R(P) \cap N(Q)$, $\mathcal{H}_3 = N(P) \cap R(Q)$, $\mathcal{H}_4 = N(P) \cap N(Q)$. It is easy to see that $\mathcal{H}_i$ $(i = 1, 2, 3, 4)$ is a reducing subspace of P and Q, and $\mathcal{H}_i \perp \mathcal{H}_j$, $i \neq j$. Set $\mathcal{H}_5 = \mathcal{H} \ominus (\oplus_{i=1}^4 \mathcal{H}_i)$, then $\mathcal{H} = \oplus_{i=1}^5 \mathcal{H}_i$ and so P and Q have the following operator matrix form

$$\begin{pmatrix} I &&&& \\ & I &&& \\ && 0 && \\ &&& 0 & \\ &&&& P_0 \end{pmatrix} \text{ and } \begin{pmatrix} I &&&& \\ & 0 &&& \\ && I && \\ &&& 0 & \\ &&&& Q_0 \end{pmatrix} \tag{6}$$

corresponding to the space decomposition $\mathcal{H} = \oplus_{i=1}^{5}\mathcal{H}_i$, where the missed entries are all 0.

Clearly, (P_0, Q_0) is a pair of generic orthogonal projections on $\mathcal{H}_5$. So P_0 and Q_0 have the forms in (2).

The following lemma might be known, but we cannot find the reference. Here we give an elementary proof.

Lemma 4. If T is a commutator of a pair of orthogonal projections, then
(1) T is normal and $\|T\| \leq 1/2$.
(2) $\sigma(T) \subseteq \{ib : b \in \mathbb{R} \text{ and } |b| \leq 1/2\}$.
(3) T is unitarily equivalent to T^*.

Proof. (1)Assume that $T = PQ - QP$ where P and Q are orthogonal projections. Since $T^* = QP - PQ = -(PQ - QP) = -T$, we have T is normal. If P and Q have the forms in (6), then by Lemma 3(4), we have

$$\begin{aligned} \|PQ - QP\| &= \left\| \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & Q_{11}^{\frac{1}{2}} D Q_{22}^{\frac{1}{2}} \\ & & & & -Q_{22}^{\frac{1}{2}} D^* Q_{11}^{\frac{1}{2}} & 0 \end{pmatrix} \right\| \\ &= \|Q_{11}^{\frac{1}{2}} D Q_{22}^{\frac{1}{2}}\| = \|Q_{11}^{\frac{1}{2}} D Q_{22} D^* Q_{11}^{\frac{1}{2}}\|^{\frac{1}{2}} \\ &= \|Q_{11}^{\frac{1}{2}}(1 - Q_{11}) Q_{11}^{\frac{1}{2}}\|^{\frac{1}{2}} \quad \text{by } Q_{22} = D^*(1 - Q_{11})D \\ &= \|Q_{11}(1 - Q_{11})\|^{\frac{1}{2}}. \end{aligned}$$

Since Q_{11} is a positive operator, applying functional calculus , we have $\|Q_{11}(1 - Q_{11})\| \leq \sup\{|\lambda(1 - \lambda)| : \lambda \in \sigma(Q_{11})\} \leq \frac{1}{4}$. Hence $\|T\| \leq \frac{1}{2}$.

(2) Noting that $(iT)^* = -iT^* = iT$, then $\sigma(T) \subseteq \{ib : b \in \mathbb{R} \text{ and } |b| \leq 1/2\}$.

(3) Let $U = 2P - 1$. Then $U^* = U$ and $UU = 1$. Hence U is a self-adjoint unitary operator. Meanwhile $UT + TU = 0$, thus $T^* = -T = UTU$.

The proof is completed.

Theorem 5. T is a commutator of a pair of orthogonal projections if and only if $T^* = -T$, $\|T\| \leq \frac{1}{2}$ and T is unitarily equivalent to T^*.

Proof. Necessity. By the proof of Lemma 4, it is clear.

Sufficiency. By the assumption, we have $\sigma(T) \subseteq \{ib : b \in \mathbb{R} \text{ and } |b| \leq 1/2\}$. Let E be the spectral measure of T, and let $C_1 = \{ib : b \in \mathbb{R}^+ - \{0\}\}$, $C_2 = \{ib : b \in \mathbb{R}^- - \{0\}\}$. Then $\mathcal{H} = E(C_1)\mathcal{H} \oplus E(C_2)\mathcal{H} \oplus E(0)\mathcal{H}$. Put $A = \int_{C_1} \lambda dE_\lambda$, $B = \int_{C_2} \lambda dE_\lambda$. Then both A and B are injective operators. By assumption, there is a unitary operator U such that $T^* = UTU^*$. Then according to this spatial decomposition we have

$$T = \begin{pmatrix} A & & \\ & B & \\ & & 0 \end{pmatrix}, \quad T^* = \begin{pmatrix} -A & & \\ & -B & \\ & & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} A & & \\ & B & \\ & & 0 \end{pmatrix} = \begin{pmatrix} -A & & \\ & -B & \\ & & 0 \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix}. \quad (7)$$

Since both A and B are injective, it follows that all of U_{31}, U_{32}, U_{13} and U_{23} are 0. We next prove that both U_{11} and U_{22} are also 0.

If 0 is an isolated point of $\sigma(T)$, then both A and B are invertible operators. So by $U_{11}A = -AU_{11}, U_{22}B = -BU_{22}$, we have $U_{11} = U_{22} = 0$.

Otherwise, if 0 is not an isolated point of $\sigma(T)$. Put $C_{11}^n = \{ib : b \in [1/n, \infty)\}$, $C_{12}^n = \{ib : b \in (0, 1/n)\}$, $C_{21}^n = \{ib : b \in (-\infty, -1/n]\}$ and $C_{22}^n = \{ib : b \in (-1/n, 0)\}$. Let $A_n = \int_{C_{11}^n} \lambda dE_\lambda$, $B_n = \int_{C_{21}^n} \lambda dE_\lambda$ and $C_n = \int_{C_{12}^n \cup \{0\} \cup C_{22}^n} \lambda dE_\lambda$. Then $\mathcal{H} = E(C_{11}^n)\mathcal{H} \oplus E(C_{21}^n)\mathcal{H} \oplus E(C_{12}^n \cup \{0\} \cup C_{22}^n)\mathcal{H}$, and

$$\begin{pmatrix} U_{11}^n & U_{12}^n & U_{13}^n \\ U_{21}^n & U_{22}^n & U_{23}^n \\ U_{31}^n & U_{32}^n & U_{33}^n \end{pmatrix} \begin{pmatrix} A_n & & \\ & B_n & \\ & & C_n \end{pmatrix} = \begin{pmatrix} -A_n & & \\ & -B_n & \\ & & -C_n \end{pmatrix} \begin{pmatrix} U_{11}^n & U_{12}^n & U_{13}^n \\ U_{21}^n & U_{22}^n & U_{23}^n \\ U_{31}^n & U_{32}^n & U_{33}^n \end{pmatrix}.$$

Since both A_n and B_n are invertible, by the proceeding analysis, $U_{11}^n = U_{22}^n = 0$ for each $n \in \mathbb{N}$. Meanwhile, $E(C_{12}^n) \to 0$ (SOT) as $n \to \infty$ and $U_{11} = E(C_{11}^n)U_{11}E(C_{11}^n) + E(C_{11}^n)U_{11}E(C_{12}^n) + E(C_{12}^n)U_{11}$ $(n \in \mathbb{N})$. Since $U_{11}^n = 0$ for each $n \in \mathbb{N}$, $E(C_{11}^n)U_{11}E(C_{11}^n) = 0$. Then for each $x \in E(C_1)\mathcal{H}$,

$$\|U_{11}x\| \le \|E(C_{11}^n)U_{11}E(C_{12}^n)x\| + \|E(C_{12}^n)U_{11}x\| \to 0 \text{ as } (n \to \infty).$$

Hence $U_{11} = 0$. Similarly, we also have $U_{22} = 0$.

It now follows that U_{12} is unitary and that B is unitarily equivalent to A^*. Hence without loss of generality, we may assume that

$$T = \begin{pmatrix} A & & \\ & A^* & \\ & & 0 \end{pmatrix} = \begin{pmatrix} A & & \\ & -A & \\ & & 0 \end{pmatrix}.$$

Define $B = \int_{C_1} (\frac{1}{4} + \lambda^2)^{\frac{1}{2}} dE_\lambda$. Since $\sigma(T) \subseteq \{ib : |b| \le 1/2\}$, we have $\frac{1}{4} + \lambda^2 \ge 0$. Then $B^* = B$. Note that

$$\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} = \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}.$$

So it suffices to prove that $\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$ is the commutator of a pair of orthogonal projections. Define $P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} \frac{1}{2}I - B & A \\ -A & \frac{1}{2}I + B \end{pmatrix}$. It is clear that P and Q both are self-adjoint. By direct calculus, we have they are also idempotents and $PQ - QP = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$. It easily follows that T is also a commutator of projections. The proof is completed.

Lemma 6. (P, Q) is a pair of generic projections if and only if $PQ - QP$ is injective.

Proof. If (P, Q) is a pair of generic projections, then by the forms in (2) we have

$$PQ - QP = \begin{pmatrix} 0 & Q_{11}^{\frac{1}{2}} D Q_{22}^{\frac{1}{2}} \\ -Q_{22}^{\frac{1}{2}} D^* Q_{11}^{\frac{1}{2}} & 0 \end{pmatrix}.$$

Hence $N(PQ - QP) = \{0\}$, since $N(Q_{ii}^{\frac{1}{2}}) = 0$ $(i = 1, 2)$ by Lemma 3(2).

If $PQ - QP$ is injective, then $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 = \{0\}$. So $P = P_0$ and $Q = Q_0$ in the forms (6). Thus (P, Q) is a generic pair. The proof is completed.

Corollary 7. *T is commutator of a pair of generic projections if and only if $T^* = -T$, $\|T\| \leq \frac{1}{2}$, $N(T) = \{0\}$ and T is unitarily equivalent to T^*.*

Proof. It is clear by Theorem 4 and Lemma 5.

Remark. If $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, then $\|PQ - QP\| = \frac{1}{2}$. Hence the equation $\|T\| = \frac{1}{2}$ may hold.

Proposition 8. *Let P and Q be the orthogonal projections. Then the norm equation $\|PQ - QP\| = \frac{1}{2}$ holds if and only if $\frac{1}{2} \in \sigma(PQP)$.*

Proof. Sufficiency. Let $T = PQ - QP$. Then by the proof of Lemma 4, we have $\|(PQ - QP)\| = \|Q_{11}(1 - Q_{11})\|^{\frac{1}{2}}$. By functional calculus of positive operators and $1/2 \in \sigma(PQP)$, we have $\|Q_{11}(1 - Q_{11})\| = 1/4$. So $\|PQ - QP\| = 1/2$.

Necessity. By the proceeding analysis, if $\|PQ - QP\| = 1/2$, we have $\|Q_{11}(1 - Q_{11})\| = 1/4$. So there exists a $\lambda \in \sigma(Q_{11})$ such that $|\lambda(1 - \lambda)| = 1/4$. Then $\lambda = 1/2$. Hence $1/2 \in \sigma(PQP)$. The proof is completed.

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