Nihonkai Math. J.
Vol.15(2004), 75-91

THE SOLVABLE STRUCTURE OF THE C^*-ALGEBRAS OF CERTAIN SUCCESSIVE SEMI-DIRECT PRODUCTS

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ABSTRACT. As the main results we construct finite composition series of group C^*-algebras of certain successive semi-direct products of $\mathbb{C}^n$ by $\mathbb{R}$, $\mathbb{T}$ or $\mathbb{Z}$ such that their subquotients are tensor products involving commutative C^*-algebras, non-commutative tori and the C^*-algebra of all compact operators. As an application, we estimate the stable rank and connected stable rank of the group C^*-algebras of these connected or disconnected solvable Lie groups. Also, we introduce a class of C^*-algebras that are C^*-solvable in some sense.

0. INTRODUCTION

First recall that any simply connected, solvable Lie group G is isomorphic to a successive semi-direct product by $\mathbb{R}$ as follows:

$$G \cong (\cdots((\mathbb{R}^n \rtimes_{\alpha^1} \mathbb{R}) \rtimes_{\alpha^2} \mathbb{R})\cdots) \rtimes_{\alpha^m} \mathbb{R}$$

for $n \geq 1$ and $m \geq 0$ and α^j $(1 \leq j \leq m)$ actions by $\mathbb{R}$ (cf.[OV, Section 3 in Chapter 2]). It has been an interesting and important problem to study the (algebraic) structure of group C^*-algebras for all or certain G. Some remarkable results related with this problem were obtained by Green [Gr1] [Gr2] (for the imprimitivity theorem for crossed products and simple quotients of group C^*-algebras), Poguntke [Pg] (for simple quotients of group C^*-algebras of connected Lie groups) and Rosenberg [Rs] (for group C^*-algebras of certain solvable Lie groups with lower dimensions). On the other hand, we have explicitly studied the structure of group C^*-algebras in the case of semi-direct products of $\mathbb{C}^n$ by $\mathbb{R}$, $\mathbb{T}$ or $\mathbb{Z}$ (see [Sd5], [Sd10], [Sd6] respectively) and in the case of semi-direct products of $\mathbb{C}^n$ by $\mathbb{R}^n$ or $\mathbb{Z}^n$ with the diagonal actions (see [Sd9], [Sd8] respectively). In this paper we consider the structure of group C^*-algebras of some successive semi-direct products of $\mathbb{C}^n$ by $\mathbb{R}$, $\mathbb{T}$ or $\mathbb{Z}$ by using some results of [Sd5], [Sd10], [Sd6] inductively (cf. [Sd7], [Sd10] for group C^*-algebras of certain semi-direct products by the generalized discrete Heisenberg groups or Heisenberg Lie groups respectively). This attempt for the problem is still far from the general case, but should be a steady step and

2000 Mathematics Subject Classification. Primary 22D25. Secondary 46L05, 46L80
Key words: Group C*-algebras, Semi-direct products, Stable rank, Solvable

could be helpful for further study in the future. Indeed, this time we have reached a new and some reasonable notion of C^*-algebras being C^*-solvable generalizing the old notion of being solvable.

In details, this paper is organized as follows. In Section 1, we consider the structure (that is, finite composition series with subquotients that are tensor products involving commutative C^*-algebras, noncommutative tori and the C^*-algebra of all compact operators) of group C^*-algebras of successive semi-direct products of $\mathbb{C}^n$ by $\mathbb{R}$ under certain assumptions on actions which allow us to be able to analyze their group C^*-algebras. We also consider the case of solvable Lie groups splitting by nilradicals in a different view of point. In Sections 2 and 3, under the similar assumptions as in Section 1 we consider the case of successive semi-direct products of $\mathbb{C}^n$ by $\mathbb{T}$ and the case of successive semi-direct products of $\mathbb{C}^n$ by $\mathbb{Z}$, and the cases of linearlizable connected or disconnected solvable Lie groups. Those assumptions on actions are somewhat restrictive and technical. In fact, the group C^*-algebras of the Dixmier group and its variations such as the generalized Dixmier groups and certain semi-direct products by the generalized discrete Heisenberg groups (considered in [Sd10], [Sd7] respectively) are not contained in the class of the group C^*-algebras studied in those cases. However, the group C^*-algebras of successive semi-direct products in those cases are analyzable or solvable in some sense. In Section 4, as one application, (as usual) we consider the stable rank estimate for the analyzable or solvable group C^*-algebras of those connected or disconnected solvable Lie groups. As another one, we consider the case of amenable Lie groups such as semi-direct products of those solvable Lie groups by compact Lie groups. In Section 5, we introduce a class of C^*-algebras that are C^*-solvable in the sense that the class should contain all the C^*-algebras of solvable Lie groups, and discuss some basic properties of the class and our related conjectures for group C^*-algebras of Lie groups and their stable rank.

Notation: Let $C^*(G)$ denote the (full) group C^*-algebra of a Lie group G (cf.[Dx]). Let $\hat{G}_1$ be the space of all 1-dimensional representation of G, and $\hat{G}$ the space of all equivalence classes of irreducible unitary representations of G with the hull-kernel topology. Let $\mathfrak{A}^\wedge$ be the spectrum of a C^*-algebra $\mathfrak{A}$ consinting of all equivalence classes of irreducible representations of $\mathfrak{A}$. For a locally compact Hausdorff space X, we denote by $C_0(X)$ the C^*-algebra of all continuous complex-valued functions on X vanishing at infinity, and let $C(X) = C_0(X)$ when X is compact. Let $\mathfrak{A} \rtimes_\alpha G$ be the (full) crossed product of a C^*-algebra by G with α an action, that is, a homomorphism from G to the automorphism group of $\mathfrak{A}$ (cf.[Pd]). We often omit the symbol α in what follows. Let $\mathbb{K}$ be the C^*-algebra of all compact operators on a countably infinite dimensional Hilbert space.

1. Successive semi-direct products by $\mathbb{R}$

We first define the following (Lie) semi-direct products:

$$G_{n,m} = (\cdots((\mathbb{C}^n \rtimes_{\alpha^1} \mathbb{R}) \rtimes_{\alpha^2} \mathbb{R})\cdots) \rtimes_{\alpha^m} \mathbb{R}$$

for $n \geq 1$ and $m \geq 0$, where some restricted conditions on the actions α_j $(1 \leq j \leq m)$ are supposed in what follows. Let $C^*(G_{n,m})$ be the group C^*-algebra of $G_{n,m}$. Then we have the following isomorphism:

$$C^*(G_{n,m}) \cong (\cdots((C^*(\mathbb{C}^n) \rtimes_{\alpha^1} \mathbb{R}) \rtimes_{\alpha^2} \mathbb{R}) \cdots) \rtimes_{\alpha^m} \mathbb{R}$$

where the right hand side means a successive C^*-crossed product by $\mathbb{R}$. Then via Fourier transform we get $C^*(\mathbb{C}^n) \rtimes_{\alpha^1} \mathbb{R} \cong C_0(\mathbb{C}^n) \rtimes_{\hat{\alpha}^1} \mathbb{R}$, where the action $\hat{\alpha}^1_t$ is defined by the complex conjugate of α_t for $t \in \mathbb{R}$ via the duality of $\mathbb{C}^n$. If α^1 is trivial, then $C_0(\mathbb{C}^n) \rtimes_{\hat{\alpha}^1} \mathbb{R} \cong C_0(\mathbb{C}^n) \otimes C_0(\mathbb{R})$. By the similar reason we may assume that actions α^j are not trivial for all j in what follows.

We now review the structure of $C_0(\mathbb{C}^n) \rtimes_{\hat{\alpha}^1} \mathbb{R}$ briefly.

Theorem A [Sd5, Theorem 2.1]. *The crossed product $C_0(\mathbb{C}^n) \rtimes_{\hat{\alpha}^1} \mathbb{R}$ for the semi-direct product $G_{n,1}$ has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^K$ whose subquotients are given by*

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \begin{cases} C_0(\hat{G}_1) = C_0(\mathbb{C}^{n_0+u} \times \mathbb{R}), & j = K, \\ C_0(\mathbb{C}^{n_0+s_j} \times (\mathbb{C} \setminus \{0\})^{t_j} \times \mathbb{T}) \otimes \mathbb{K}, & \textit{or} \\ C_0(\mathbb{C}^{n_0+s_j} \times (\mathbb{C} \setminus \{0\})^{t_j} \times \mathbb{R}) \otimes \mathbb{K}, & \textit{or} \\ C_0(\mathbb{C}^{n_0+s_j} \times \mathbb{R}_+^{u_j}) \otimes \mathfrak{A}_{\Theta_j} \otimes \mathbb{K}, & \textit{for } 1 \leq j \leq K-1, \end{cases}$$

where $\mathbb{R}_+ = (0, \infty)$, and $\mathfrak{A}_{\Theta_j}$ is a noncommutative torus of the form $C(\mathbb{T}^{u_j-1}) \rtimes \mathbb{Z}$, and $0 \leq n_0 \leq n$ and $0 \leq s_j, t_j \leq n - n_0$ and $2 \leq u_j \leq n - n_0$ and $s_j + t_j + 1 \leq n - n_0$ and $s_j + u_j \leq n - n_0$.

Remark. For the proof, we use the Jordan decomposition of $\hat{\alpha}^1$ on $\mathbb{C}^n$ by taking a suitable base of $\mathbb{C}^n$ and replacing it as the canonical base to obtain the decomposition of $\mathbb{C}^n$ into $\hat{\alpha}^1$-invariant subspaces $\{Y_{1_j}\}$ with $\dim Y_{1_j} \geq \dim Y_{1_{j-1}}$ such that their corresponding subquotients $\mathfrak{I}_{K-j+1}/\mathfrak{I}_{K-j}$ are given in this theorem. In particular, $Y_{1_1} = \mathbb{C}^{n_0+u}$. We denote by X_{1_j} the spectrums of the commutative tensor factors of $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ $(1 \leq j \leq K-1)$. This convention indexing subquotients is used in the similar situations in what follows.

Now suppose that all the subquotients $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ (or their spectrums) are invariant under α^2 (or $\hat{\alpha}_2$). Then we may assume that $(C_0(\mathbb{C}^n) \rtimes_{\hat{\alpha}^1} \mathbb{R}) \rtimes_{\alpha^2} \mathbb{R}$ has a finite composition series whose subquotients are given by

$$(\mathfrak{I}_j/\mathfrak{I}_{j-1}) \rtimes_{\alpha^2} \mathbb{R} \cong \begin{cases} C_0(\hat{G}_1) \rtimes_{\hat{\alpha}^2} \mathbb{R} = C_0(\mathbb{C}^{n_0+u} \times \mathbb{R}) \rtimes_{\hat{\alpha}^2} \mathbb{R}, & j = K, \\ (C_0(\mathbb{C}^{n_0+s_j} \times (\mathbb{C} \setminus \{0\})^{t_j} \times \mathbb{T}) \rtimes_{\hat{\alpha}^2} \mathbb{R}) \otimes \mathbb{K}, \\ (C_0(\mathbb{C}^{n_0+s_j} \times (\mathbb{C} \setminus \{0\})^{t_j} \times \mathbb{R}) \rtimes_{\hat{\alpha}^2} \mathbb{R}) \otimes \mathbb{K}, \\ ([C_0(\mathbb{C}^{n_0+s_j} \times \mathbb{R}_+^{u_j}) \otimes \mathfrak{A}_{\Theta_j}] \rtimes_{\alpha^2} \mathbb{R}) \otimes \mathbb{K}, \end{cases}$$

for $1 \leq j \leq K-1$, where $\hat{\alpha}_2$ means the action on the spectrums of the commutative C^*-algebras involved in those crossed products. Note that

$$\begin{aligned} &([C_0(\mathbb{C}^{n_0+s_j} \times \mathbb{R}_+^{u_j}) \otimes \mathfrak{A}_{\Theta_j}] \rtimes_{\alpha^2} \mathbb{R}) \\ &\cong ([C_0(\mathbb{C}^{n_0+s_j} \times \mathbb{R}_+^{u_j}) \otimes C(\mathbb{T}^{k_j})] \rtimes_{\hat{\alpha}^2} \mathbb{R}) \rtimes_{\hat{\alpha}^1} \mathbb{Z}, \end{aligned}$$

where $\mathfrak{A}_{\Theta_j} = C(\mathbb{T}^{k_j}) \rtimes_{\hat{\alpha}^1} \mathbb{Z}$ and $k_j = u_j - 1$. Thus, it suffices to consider the following crossed products:

$$C_0(X_{1_j}) \rtimes_{\hat{\alpha}^2} \mathbb{R} = \begin{cases} C_0(\mathbb{C}^{n_2} \times \mathbb{R}) \rtimes_{\hat{\alpha}^2} \mathbb{R}, \\ C_0(\mathbb{C}^{n_2} \times (\mathbb{C} \setminus \{0\})^t \times \mathbb{T}) \rtimes_{\hat{\alpha}^2} \mathbb{R}, \\ C_0(\mathbb{C}^{n_2} \times (\mathbb{C} \setminus \{0\})^t \times \mathbb{R}) \rtimes_{\hat{\alpha}^2} \mathbb{R}, \\ C_0(\mathbb{C}^{n_2} \times \mathbb{R}_+^u \times \mathbb{T}^k) \rtimes_{\hat{\alpha}^2} \mathbb{R}. \end{cases}$$

The crossed products in the alternative are obtained by taking suitable quotients from

$$\begin{cases} C_0(\mathbb{C}^{n_2+1}) \rtimes_{\hat{\beta}^2} \mathbb{R}, \\ C_0(\mathbb{C}^{n_2+t+1}) \rtimes_{\hat{\beta}^2} \mathbb{R}, \\ C_0(\mathbb{C}^{n_2+t+1}) \rtimes_{\hat{\beta}^2} \mathbb{R}, \\ C_0(\mathbb{C}^{n_2+u+k}) \rtimes_{\hat{\beta}^2} \mathbb{R} \end{cases}$$

respectively. In fact, we can construct liftings from the spaces involved in the alternative of $C_0(X_{1_j}) \rtimes_{\hat{\alpha}^2} \mathbb{R}$ to the direct products $\mathbb{C}^{n_2+x}$ for $x = 1, t+1, u+k$ involved in the crossed products just above by assuming that the action $\hat{\beta}^2$ of $\mathbb{R}$ on the torus direction in $\{0_{n_2}\} \times \mathbb{C}$ of $\mathbb{C}^{n_2+1}$ with $\mathbb{C} \setminus \{0\} \approx \mathbb{T} \times \mathbb{R}_+$ (homeomorphic) is trivial and the same as $\hat{\alpha}^2$ elsewhere in the first case, where 0_x means the origin of $\mathbb{C}^x$, and the action $\hat{\beta}^2$ of $\mathbb{R}$ on the origin 0_t of $\mathbb{C}^t$ in $\mathbb{C}^{n_2+t+1}$ and on either the radius direction in $\{0_{n_2}\} \times \{0_t\} \times \mathbb{C}$ of $\mathbb{C}^{n_2+t+1}$ with $\mathbb{C} \setminus \{0\} \approx \mathbb{T} \times \mathbb{R}_+$ or the torus direction in $\{0_{n_2}\} \times \{0_t\} \times \mathbb{C}$ of $\mathbb{C}^{n_2+t+1}$ with $\mathbb{C} \setminus \{0\} \approx \mathbb{T} \times \mathbb{R}_+$ is trivial and the same as $\hat{\alpha}^2$ elsewhere in the second and third cases respectively, where we may identify $\mathbb{R}_+$ with $\mathbb{R}$, and the action $\hat{\beta}^2$ of $\mathbb{R}$ on the torus directions in $\{0_{n_2}\} \times \mathbb{C}^u \times \{0_k\}$ of $\mathbb{C}^{n_2+u+k}$ with $(\mathbb{C} \setminus \{0\})^u \approx (\mathbb{T} \times \mathbb{R}_+)^u$ and on the radius directions in $\{0_{n_2}\} \times \{0_u\} \times \mathbb{C}^k$ of $\mathbb{C}^{n_2+u+k}$ with $(\mathbb{C} \setminus \{0\})^k \approx (\mathbb{T} \times \mathbb{R}_+)^k$ is trivial and the same as $\hat{\alpha}^2$ elsewhere in the forth case. Therefore, we may assume from Theorem A that $C_0(X_{1_j}) \rtimes_{\hat{\alpha}^2} \mathbb{R}$ has a finite composition series whose subquotients are given by

$$\begin{cases} C_0(X_{2_j}), \\ C_0(X_{2_j}) \otimes \mathbb{K}, \\ C_0(X_{2_j}) \otimes (C(\mathbb{T}^{2_j}) \rtimes_{\hat{\alpha}^2} \mathbb{Z}) \otimes \mathbb{K} \quad j \geq 2 \end{cases}$$

where the spaces X_{2_j} are obtained from taking suitable quotients of the spectrums X_{1_j} of the commutative direct factors as in the subquotients in Theorem A. To sum up we obtain

Lemma 1.1. *If the structure of $C_0(\mathbb{C}^n) \rtimes_{\hat{\alpha}^1} \mathbb{R}$ as in Theorem A is invariant under α^2 (or $\hat{\alpha}^2$), then the crossed product $(C_0(\mathbb{C}^n) \rtimes_{\hat{\alpha}^1} \mathbb{R}) \rtimes_{\alpha^2} \mathbb{R}$ for the successive semi-direct product $G_{n,2}$ has a finite composition series whose subquotients are given* *by*

$$\begin{cases} C_0(X_{2_1}) = C_0((G_{n,2})^{\wedge}_1) = C_0(\mathbb{R}^{d_{n,2}}), & \textit{for some } d_{n,2} \geq 1, \\ C_0(X_{2_j}) \otimes \mathbb{K}, & \\ C_0(X_{2_j}) \otimes (C(\mathbb{T}^{2_j}) \rtimes_{\hat{\alpha}^p} \mathbb{Z}) \otimes \mathbb{K}, & p = 1, 2, \\ C_0(X_{2_j}) \otimes ((C(\mathbb{T}^{2_j}) \rtimes_{\hat{\alpha}^2} \mathbb{Z}) \rtimes_{\hat{\alpha}^1} \mathbb{Z}) \otimes \mathbb{K}, & \textit{for } j \geq 2, \end{cases}$$

and $2_j \geq 1$ *for* $\mathbb{T}^{2_j}$, *and* X_{2_j} *are certain locally compact Hausdorff spaces.*

By the same analysis as above, we obtain

Lemma 1.2. *If the structure of* $(C_0(\mathbb{C}^n) \rtimes_{\hat{\alpha}^1} \mathbb{R}) \rtimes_{\alpha^2} \mathbb{R}$ *as in Lemma* 1.1 *is invariant under* α^3 *(or* $\hat{\alpha}^3$*), then the crossed product* $((C_0(\mathbb{C}^n) \rtimes_{\hat{\alpha}^1} \mathbb{R}) \rtimes_{\alpha^2} \mathbb{R}) \rtimes_{\alpha^3} \mathbb{R}$ *for the successive semi-direct product* $G_{n,3}$ *has a finite composition series whose subquotients are given by*

$$\begin{cases} C_0(X_{3_1}) = C_0((G_{n,3})^{\wedge}_1) = C_0(\mathbb{R}^{d_{n,3}}), & \textit{for some } d_{n,3} \geq 1, \\ C_0(X_{3_j}) \otimes \mathbb{K}, & \\ C_0(X_{3_j}) \otimes (C(\mathbb{T}^{3_j}) \rtimes_{\hat{\alpha}^p} \mathbb{Z}) \otimes \mathbb{K}, & 1 \leq p \leq 3, \\ C_0(X_{3_j}) \otimes ((C(\mathbb{T}^{3_j}) \rtimes_{\hat{\alpha}^{p_2}} \mathbb{Z}) \rtimes_{\hat{\alpha}^{p_1}} \mathbb{Z}) \otimes \mathbb{K}, & 1 \leq p_1 < p_2 \leq 3, \\ C_0(X_{3_j}) \otimes (((C(\mathbb{T}^{3_j}) \rtimes_{\hat{\alpha}^3} \mathbb{Z}) \rtimes_{\hat{\alpha}^2} \mathbb{Z}) \rtimes_{\hat{\alpha}^1} \mathbb{Z}) \otimes \mathbb{K}, & \textit{for } j \geq 2, \end{cases}$$

and $3_j \geq 1$ *for* $\mathbb{T}^{3_j}$, *and* X_{3_j} *are certain locally compact Hausdorff spaces.*

Remark. Note that $d_{n,2} \geq d_{n,3} \geq 1$.

In general, we obtain

Theorem 1.3. *If the structures of* $C^*(G_{n,p})$ $(1 \leq p \leq m-1)$ *as in Lemmas* 1.1 *and* 1.2 *are invariant under* $\hat{\alpha}^{p+1}$ *respectively, then the group* C^**-algebra* $C^*(G_{n,m})$ *for the successive semi-direct product* $G_{n,m}$ *has a finite composition series whose subquotients are given by*

$$\begin{cases} C_0(X_{m_1}) = C_0((G_{n,m})^{\wedge}_1) = C_0(\mathbb{R}^{d_{n,m}}), & \textit{for } d_{n,m} \geq 1, \\ C_0(X_{m_j}) \otimes \mathbb{K}, & \\ C_0(X_{m_j}) \otimes (C(\mathbb{T}^{m_j}) \rtimes_{\hat{\alpha}^p} \mathbb{Z}) \otimes \mathbb{K}, & 1 \leq p \leq m, \\ C_0(X_{m_j}) \otimes ((C(\mathbb{T}^{m_j}) \rtimes_{\hat{\alpha}^{p_2}} \mathbb{Z}) \rtimes_{\hat{\alpha}^{p_1}} \mathbb{Z}) \otimes \mathbb{K}, & 1 \leq p_1 < p_2 \leq m, \\ C_0(X_{m_j}) \otimes (((C(\mathbb{T}^{m_j}) \rtimes_{\hat{\alpha}^{p_3}} \mathbb{Z}) \rtimes_{\hat{\alpha}^{p_2}} \mathbb{Z}) \rtimes_{\hat{\alpha}^{p_1}} \mathbb{Z}) \otimes \mathbb{K}, & 1 \leq p_1 < p_2 < p_3 \leq m, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, & \\ C_0(X_{m_j}) \otimes (\cdots((C(\mathbb{T}^{m_j}) \rtimes_{\hat{\alpha}^m} \mathbb{Z}) \rtimes_{\hat{\alpha}^{m-1}} \mathbb{Z}) \cdots \rtimes_{\hat{\alpha}^1} \mathbb{Z}) \otimes \mathbb{K}, & \textit{for } j \geq 2, \end{cases}$$

and $m_j \geq 1$ *for* $\mathbb{T}^{m_j}$, *and* X_{m_j} *are certain locally compact Hausdorff spaces.*

Remark. If a semi-direct product of the form:

$$G = (\cdots((\mathbb{R}^n \rtimes_{\alpha^1} \mathbb{R}) \rtimes_{\alpha^2} \mathbb{R}) \cdots) \rtimes_{\alpha^m} \mathbb{R}$$

is a quotient of $G_{n,m}$, then by taking a suitable quotient of $C^*(G_{n,m})$ we obtain the structure of $C^*(G)$ as in the theorem. Note that $\hat{G}_1$ is isomorphic to $\mathbb{R}^d$ for $d \geq 1$ since G is a simply connected solvable Lie group (cf.[ST2, Lemma 2.1]). The above successive crossed products by $\mathbb{Z}$ are noncommutative tori, and

$$(\cdots((C(\mathbb{T}^{m_j}) \rtimes_{\hat{\alpha}^m} \mathbb{Z}) \rtimes_{\hat{\alpha}^{m-1}} \mathbb{Z}) \cdots \rtimes_{\hat{\alpha}^1} \mathbb{Z}) \cong C(\mathbb{T}^{m_j}) \rtimes \mathbb{Z}^m.$$

The structure of these crossed products was studied by [Sd5] to decompose into subquotients which are AH-algebras, i.e, inductive limits of homogeneous C^*-algebras. The assumption on the structure invariances for $C^*(G_{n,p})$ $(1 \leq p \leq m-1)$ is crucial and restricted. In fact, it is known that the group C^*-algebras of the generalized Dixmier groups have subquotients as the C^*-algebras of continuous fields (non-split into tensor products) (cf. [Sd10, Theorems 2.1 and 2.2]). However, our C^*-algebras $C^*(G_{n,m})$ are analyzable or solvable in some sense. Also, it is easy to construct artificially such group C^*-algebras of successive semi-direct products and obtain their structures.

Solvable Lie groups splitting by nilradicals.

It is known that for any simply connected, solvable Lie group G, there exists the maximal simply connected nilpotent Lie group N (the nilradical of G) such that for some $n \in \mathbb{N}$, we have the following exact sequence of groups:

$$1 \to N \to G \to \mathbb{R}^n \to 1.$$

We assume that G splits into the semi-direct product $N \rtimes_\alpha \mathbb{R}^n$ (a solvable Lie group splitting by N). This is always true if $n = 1$. Unfortunately, G does not split in general for $n \geq 2$. The C^*-algebra $C^*(G)$ for G splitting by N is isomorphic to the crossed product $C^*(N) \rtimes_\alpha \mathbb{R}^n$. On the other hand, the structure of $C^*(N)$ for N a simply connected, nilpotent Lie group was obtained by [Sd1] as follows:

Theorem A_2 [Sd1, Theorem 3]. *Let N be a simply connected, nilpotent Lie group. Then there exists a finite composition series $\{\mathfrak{I}_j\}_{j=1}^k$ of $C^*(N)$ with $\mathfrak{I}_0 = 0$ and $\mathfrak{I}_k = C^*(N)$ such that*

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \begin{cases} C_0(\mathbb{R}^m) = C_0(\hat{N}_1) & \text{for } m \geq 1, \quad j = k, \\ \Gamma_0(\Omega_j, \{\mathbb{K}\}_{\pi\in\Omega_j}) & 1 \leq j \leq k-1, \end{cases}$$

where $\hat{N}_1$ is the space of all characters of N, and $\Gamma_0(\Omega_j, \{\mathbb{K}\}_{\pi\in\Omega_j})$ are the C^*-algebras of continuous fields vanishing at infinity over locally compact Hausdorff spaces Ω_j with the constant fiber $\mathbb{K}$, and the union of $\hat{N}_1$ and Ω_j $(1 \leq j \leq k-1)$ is $\hat{N} = C^*(N)^\wedge$.

Then $C_0(\hat{N}_1)$ is invariant under the action α of $\mathbb{R}$ since $\hat{N}_1$ is invariant under $\hat{\alpha}$, and $N/[N,N] \cong \mathbb{R}^m$ and $C^*(\mathbb{R}^m) \rtimes_\alpha \mathbb{R}^n \cong C_0(\mathbb{R}^m) \rtimes_{\hat{\alpha}} \mathbb{R}^n$ with $\hat{\alpha}$ the conjugate of α via the duality of $\mathbb{R}^n$. Note that

$$C_0(\mathbb{R}^m) \rtimes_{\hat{\alpha}} \mathbb{R}^n \cong (\cdots((C_0(\mathbb{R}^m) \rtimes_{\hat{\alpha}} \mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{R}) \cdots) \rtimes_{\hat{\alpha}} \mathbb{R}.$$

We also may assume that the subquotients $\Gamma_0(\Omega_j, \{\mathbb{K}\}_{\pi\in\Omega_j})$ split into the C^*-tensor products $C_0(\Omega_j) \otimes \mathbb{K}$. In fact, these continuous field C^*-algebras of this type are locally trivial (cf.[Dx, Section 10.8]), and use finite induction associated with analyzing the coadjoint orbit space as below. Moreover, we assume that those subquotients are also invariant under $\hat{\alpha}$. Then note that

$$(C_0(\Omega_j) \otimes \mathbb{K}) \rtimes_\alpha \mathbb{R}^n \cong (C_0(\Omega_j) \rtimes_{\hat{\alpha}} \mathbb{R}^n) \otimes \mathbb{K}.$$

Now note the following commutative diagram:

$$\begin{array}{ccc} N & \xrightarrow{\alpha_t} & N \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{N} & \xrightarrow{d\alpha_t} & \mathfrak{N} \end{array}$$

where $\mathfrak{N}$ is the Lie algebra of N and $d\alpha_t$ for $t \in \mathbb{R}^n$ is the differential of α_t at the origin, and exp means the exponential map. Also note that

$$\begin{array}{ccc} \mathfrak{N}^* & \xrightarrow{d\alpha_t^*} & \mathfrak{N}^* \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathfrak{N}^*/N & \xrightarrow{[\alpha_t]} & \mathfrak{N}^*/N \end{array}$$

where $\mathfrak{N}^*$ is the real dual space of $\mathfrak{N}$, the quotient space $\mathfrak{N}^*/N$ is the coadjoint orbit space of N under the coadjoint action Ad^* of N, $d\alpha_t^*$ is the transpose of $d\alpha_t$, and $[\alpha_t]$ is the action induced by $d\alpha_t^*$ and the quotient map q. By the Kirillov-Bernat correspondence, the orbit space $\mathfrak{N}^*/N$ is homeomorphic to the spectrum $\hat{N}$ of N (cf.[Sd1]), so that the action $[\alpha_t]$ on $\hat{N}$ is in fact given by $[\alpha_t][\pi] = [\pi \circ \alpha_{-t}]$ for $[\pi](= \pi) \in \hat{N}$.

Since Ω_j is a subspace of $\hat{N}$, it is also regarded as a subspace of $\mathfrak{N}^*/N$, so that the crossed product $C_0(\Omega_j) \rtimes_{\hat{\alpha}} \mathbb{R}^n$ is regarded as a subquotient of $C_0(\mathbb{R}^l) \rtimes_{\hat{\alpha}} \mathbb{R}^n$ for $l \le \dim N$ (this lifting might be nontrivial, however, if necessary we may analyze directly the subspace of $\mathfrak{N}^*$ corresponding to Ω_j via q^{-1}. In this case, the decomposition for $\mathfrak{N}^*$ given in [Sd1, Lemmas 2 and 4] would be useful). Since this crossed product $C_0(\mathbb{R}^l) \rtimes_{\hat{\alpha}} \mathbb{R}^n$ is a quotient of $C_0(\mathbb{C}^l) \rtimes_{\hat{\alpha}} \mathbb{R}^n$, the crossed product $C_0(\Omega_j) \rtimes_{\hat{\alpha}} \mathbb{R}^n$ is a subquotient of $C_0(\mathbb{C}^l) \rtimes_{\hat{\alpha}} \mathbb{R}^n$. Therefore, the inductive method given before Theorem 1.3 also works for this crossed product $C_0(\mathbb{C}^l) \rtimes_{\hat{\alpha}} \mathbb{R}^n$. Hence, summing up we obtain

Proposition 1.4. *Let G be a simply connected solvable Lie group splitting by its nilradical N so that $G = N \rtimes_\alpha \mathbb{R}^n$ for some $n \ge 0$. Suppose that the structure of $C^*(N)$ in Theorem A_2 is invariant under the action α of $\mathbb{R}^n$, and the structures of the crossed products $C_0(\hat{N}_1) \rtimes_{\hat{\alpha}} \mathbb{R}^s$ and $C_0(\Omega_j) \rtimes_{\hat{\alpha}} \mathbb{R}^s$ for $1 \le j \le k-1$ and $1 \le s \le n$ are invariant under the actions of direct factors $\mathbb{R}$ in $\mathbb{R}^n$ respectively, where the union of $\hat{N}_1$ and Ω_j $(1 \le j \le k-1)$ is the spectrum of $C^*(N)$. Then the C^*-algebra $C^*(G) = C^*(N \rtimes_\alpha \mathbb{R}^n)$ has the same structure with subquotients as given in Theorem 1.3.*

2. Successive semi-direct products by $\mathbb{T}$

We first recall the following result (cf. [Sd5, Theorem 2.1]):

Theorem B [Sd10, Proposition 3.10]. *Let $H = \mathbb{C}^n \rtimes_\alpha \mathbb{T}$ be a semi-direct product of $\mathbb{C}^n$ by $\mathbb{T}$. Then there exists a finite composition series $\{\mathfrak{I}_j\}_{j=1}^r$ of $C^*(H)$ such that*

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \begin{cases} C_0(\mathbb{C}^{n_0} \times \mathbb{Z}) = C_0(\hat{H}_1),\ n_0 \geq 0 & j = r, \\ C_0(\mathbb{C}^{n_0} \times \mathbb{R}^{k_j} \times \mathbb{T}^{k_j-1}) \otimes \mathbb{K} & 1 \leq j \leq r-1. \end{cases}$$

Now define connected solvable Lie groups $H_{n,m}$ by the following successive semi-direct products:

$$H_{n,m} = (\cdots((\mathbb{C}^n \rtimes_{\alpha^1} \mathbb{T}) \rtimes_{\alpha^2} \mathbb{T})\cdots) \rtimes_{\alpha^m} \mathbb{T}.$$

Then $C^*(H_{n,m}) \cong C_0(\mathbb{C}^n) \rtimes_{(\alpha^1,\cdots,\alpha^m)} \mathbb{T}^m$ since the automorphic action α^2 of $\mathbb{T}$ is trivial on $\mathbb{T}$ of $\mathbb{C}^n \rtimes_{\alpha^1} \mathbb{T}$, and inductively, the action α^j is trivial on $\mathbb{T}^{j-1}$ of $H_{n,j-1}$. By using Theorem B as Theorem A in the previous section, we obtain

Theorem 2.1. *If the structures of $C^*(H_{n,p})$ $(1 \leq p \leq m-1)$ are invariant under $\hat{\alpha}^{p+1}$ respectively, then the group C^*-algebra $C^*(H_{n,m})$ for the successive semi-direct product $H_{n,m}$ has a finite composition series whose subquotients are given by*

$$\begin{cases} C_0(X_{m_1}) = C_0((H_{n,m})_1^\wedge) = C_0(\mathbb{C}^{d_{n,m}} \times \mathbb{Z}^m), & \text{for } d_{n,m} \geq 0, \\ C_0(X_{m_j}) \otimes \mathbb{K} \end{cases}$$

for $j \geq 2$, and X_{m_j} are certain locally compact Hausdorff spaces.

Further define connected solvable Lie groups $H'_{n,m}$ by the following successive semi-direct products:

$$H'_{n,m} = (\cdots((\mathbb{C}^n \rtimes_{\alpha^1} H_1) \rtimes_{\alpha^2} H_2)\cdots) \rtimes_{\alpha^m} H_m$$

where $H_p \cong \mathbb{R}$ or $\mathbb{T}$ $(1 \leq p \leq m)$. Then we have

Corollary 2.2. *Theorem 1.3 holds when we replace both $G_{n,m}$ and $(G_{n,m})_1^\wedge$ with $H'_{n,m}$ and $(H'_{n,m})_1^\wedge$ respectively.*

Remark. If a successive semi-direct product of the following form:

$$H = (\cdots((\mathbb{R}^n \rtimes_{\alpha^1} H_1) \rtimes_{\alpha^2} H_2)\cdots) \rtimes_{\alpha^m} H_m$$

with $H_p \cong \mathbb{R}$ or $\mathbb{T}$ $(1 \leq p \leq m)$ is a quotient of $H'_{n,m}$, then we can obtain the structure of $C^*(H)$ by taking a quotient of $C^*(H'_{n,m})$.

Linearizable connected solvable Lie groups.

Recall that a connected solvable Lie group H is linearizable if it has a faithful finite-dimensional representation. In this case, H is isomorphic to a semi-direct product $G \rtimes \mathbb{T}^k$ for G a simply connected solvable Lie group and $k \geq 0$ (cf. [OV, Theorem 7.1 in page 66]).

Proposition 2.3. *Let H be a linearizable connected solvable Lie group so that $H \cong G \rtimes \mathbb{T}^k$ for G a simply connected solvable Lie group and $k \geq 0$. If $C^*(G)$ has the structure as given in Theorem 1.3 and this structure is invariant under the action of $\mathbb{T}^k$, then $C^*(H)$ has the structure with noncommutative subquotients as given in Theorem 1.3 and $C_0(\hat{H}_1) \cong C_0(\mathbb{R}^d \times \mathbb{Z}^k)$ for some $d \geq 0$.*

3. Successive semi-direct products by $\mathbb{Z}$

We first recall the following result:

Theorem C [Sd6, Theorem 1.2]. *Let $K = \mathbb{C}^n \rtimes_\alpha \mathbb{Z}$ be a semi-direct product of $\mathbb{C}^n$ by $\mathbb{Z}$. Then there exists a finite composition series $\{\mathfrak{I}_j\}_{j=1}^s$ of $C^*(K)$ such that*

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \begin{cases} C_0(\mathbb{C}^g \times \mathbb{T}) = C_0(\hat{H}_1),\ g \geq 0 & j = s, \\ \begin{cases} C_0(X_j/\mathbb{Z}) \otimes \mathbb{K} \quad \text{or} \\ C_0(\mathbb{R}^{2g_0+u_j}) \otimes (C(\mathbb{T}^{u_j}) \rtimes_{\Theta_j} \mathbb{Z}) \end{cases} & 1 \leq j < s \end{cases}$$

where X_j are certain locally compact Hausdorff spaces with $\dim X_{j-1} \geq \dim X_j$, $X_j/\mathbb{Z}$ are their quotients by the action $\hat{\alpha}$ of $\mathbb{Z}$, and $u_{j-1} \geq u_j$, and the actions Θ_j of $\mathbb{Z}$ are multi-rotations on $\mathbb{T}^{u_j}$.

Now define disconnected solvable Lie groups $K_{n,m}$ by the following successive semi-direct products:

$$K_{n,m} = (\cdots((\mathbb{C}^n \rtimes_{\alpha^1} \mathbb{Z}) \rtimes_{\alpha^2} \mathbb{Z}) \cdots) \rtimes_{\alpha^m} \mathbb{Z}.$$

Then we have

$$C^*(K_{n,m}) \cong (\cdots((C^*(\mathbb{C}^n) \rtimes_{\alpha^1} \mathbb{Z}) \rtimes_{\alpha^2} \mathbb{Z}) \cdots) \rtimes_{\alpha^m} \mathbb{Z}.$$

By using Theorem C as in the connected case above, we have

Theorem 3.1. *If the structures of $C^*(K_{n,p})$ $(1 \leq p \leq m-1)$ are invariant under $\hat{\alpha}^{p+1}$ respectively, then the group C^*-algebra $C^*(K_{n,m})$ for the successive semi-direct product $K_{n,m}$ has a finite composition series whose subquotients are given by*

$$\begin{cases} C_0(X_{m_1}) = C_0((K_{n,m})_1^\wedge) = C_0(\mathbb{C}^{d_{n,m}} \times \mathbb{T}^m), \quad \text{for } d_{n,m} \geq 0, \\ C_0(X_{m_j}) \otimes \mathbb{K}, \\ C_0(X_{m_j}) \otimes (C(\mathbb{T}^{m_j}) \rtimes_{\hat{\alpha}^p} \mathbb{Z}), \quad 1 \leq p \leq m, \\ C_0(X_{m_j}) \otimes ((C(\mathbb{T}^{m_j}) \rtimes_{\hat{\alpha}^{p_1}} \mathbb{Z}) \rtimes_{\hat{\alpha}^{p_2}} \mathbb{Z}), \quad 1 \leq p_2 < p_1 \leq m, \\ C_0(X_{m_j}) \otimes ((C(\mathbb{T}^{m_j}) \rtimes_{\hat{\alpha}^{p_1}} \mathbb{Z}) \rtimes_{\hat{\alpha}^{p_2}} \mathbb{Z}) \otimes \mathbb{K}, \quad 1 \leq p_2 < p_1 \leq m, \\ \cdots\cdots\cdots\cdots\cdots\cdots, \\ C_0(X_{m_j}) \otimes (\cdots((C(\mathbb{T}^{m_j}) \rtimes_{\hat{\alpha}^m} \mathbb{Z}) \rtimes_{\hat{\alpha}^{m-1}} \mathbb{Z}) \cdots \rtimes_{\hat{\alpha}^1} \mathbb{Z}) \quad \text{for } j \geq 2, \end{cases}$$

and $m_j \geq 1$ for $\mathbb{T}^{m_j}$, and X_{m_j} are certain locally compact Hausdorff spaces.

Remark. If a successive semi-direct product of the form:

$$K = (\cdots((\mathbb{R}^n \rtimes_{\alpha^1} \mathbb{Z}) \rtimes_{\alpha^2} \mathbb{Z})\cdots) \rtimes_{\alpha^m} \mathbb{Z}$$

is a quotient of $K_{n,m}$, then by taking a suitable quotient of $C^*(K_{n,m})$ we obtain the structure of $C^*(K)$ as in this theorem. Note that $(K_{n,m})^\wedge_1 = X_{m_1}$. Also note that the actions $\hat{\alpha}^j$ on implementary unitaries of $\hat{\alpha}^{j+1}$ are trivial or the reflection since an automorphic action of $\mathbb{Z}$ on $\mathbb{Z}$ is trivial or the reflection.

Further define disconnected solvable Lie groups $K'_{n,m}$ by the following successive semi-direct products:

$$K'_{n,m} = (\cdots((\mathbb{C}^n \rtimes_{\alpha^1} L_1) \rtimes_{\alpha^2} L_2)\cdots) \rtimes_{\alpha^m} L_m$$

where $L_p = \mathbb{R}$, $\mathbb{T}$ or $\mathbb{Z}$ for $1 \leq p \leq m$. Then we obtain

Corollary 3.2. *If the structures of $C^*(K'_{n,p})$ $(1 \leq p \leq m-1)$ are invariant under $\hat{\alpha}^{p+1}$ respectively, then the group C^*-algebra $C^*(K'_{n,m})$ for the successive semi-direct product $K'_{n,m}$ has a finite composition series whose subquotients are given as in Theorems* 1.3, 2.1 *and* 3.1.

Remark. If a successive semi-direct product of the form:

$$K = (\cdots((\mathbb{R}^n \rtimes_{\alpha^1} L_1) \rtimes_{\alpha^2} L_2)\cdots) \rtimes_{\alpha^m} L_m$$

is a quotient of $K'_{n,m}$, then we can deduce the structure of $C^*(K)$ from that of $C^*(K'_{n,m})$.

Linearizable disconnected solvable Lie groups.

We say that a disconnected solvable Lie group K is linearizable if it is isomorphic to a successive semi-direct product $G \rtimes \mathbb{T}^k \rtimes \mathbb{Z}^l$ for G a simply connected solvable Lie group and $k, l \geq 0$.

Proposition 3.3. *Let K be a linearizable disconnected solvable Lie group so that $K \cong G \rtimes \mathbb{T}^k \rtimes \mathbb{Z}^l$ for G a simply connected solvable Lie group and $k, l \geq 0$. If $C^*(G \rtimes \mathbb{T}^k)$ has the structure as given in Proposition* 2.3, *and this structure and the structures of the successive crossed products $C^*(G \rtimes \mathbb{T}^k) \rtimes \mathbb{Z}^j$ $(1 \leq j \leq l-1)$ are invariant under the actions of direct factors $\mathbb{Z}$ of $\mathbb{Z}^l$ respectively, then $C^*(K)$ has the structure with noncommutative subquotients as given in Theorem* 3.1 *and $C_0(\hat{K}_1) \cong C_0(\mathbb{R}^d \times \mathbb{Z}^k \times \mathbb{T}^l)$ for some $d \geq 0$.*

4. Applications

As for applications given below, we first recall some notation and definition as follows.

Let $\mathfrak{A}$ be a C^*-algebra. When $\mathfrak{A}$ is nonunital, we always consider its unitization $\mathfrak{A}^+$. The stable rank of $\mathfrak{A}$ is defined to be the least integer denoted by $\mathrm{sr}(\mathfrak{A})$ such that if $\mathrm{sr}(\mathfrak{A}) \leq n$, then the open space $L_n(\mathfrak{A})$ of all elements $(a_j) \in \mathfrak{A}^n$ with $\sum_{j=1}^n a_j^* a_j$ invertible in $\mathfrak{A}$ is dense in $\mathfrak{A}^n$. The connected stable rank of $\mathfrak{A}$ is defined to be the least integer denoted by $\mathrm{csr}(\mathfrak{A})$ such that if $\mathrm{csr}(\mathfrak{A}) \leq n$, then for any $m \geq n$, the space $L_m(\mathfrak{A})$ is connected, this is equivalent to that the connected component of $GL_m(\mathfrak{A})$ with the identity matrix acts transitively on $L_m(\mathfrak{A})$. Some of the basic properties of the stable rank and connected stable rank are included in the proof of Theorem 4.1 given below. See the original work [Rf1] for more details.

As a corollary of Theorem 1.3, by using some formulas of the stable rank and connected stable rank for C^*-algebras we obtain that

Theorem 4.1. *Let $G = G_{n,m}$ as in Theorem 1.3. Then*

$$\begin{cases} \begin{cases} \mathrm{sr}(C^*(G)) = 2 \vee \dim_{\mathbb{C}} \hat{G}_1 & \text{if } \dim \hat{G}_1 \text{ is even,} \\ 2 \vee \dim_{\mathbb{C}} \hat{G}_1 \leq \mathrm{sr}(C^*(G)) \leq \dim_{\mathbb{C}} \hat{G}_1 + 1, & \text{if } \dim \hat{G}_1 \text{ is odd,} \end{cases} \\ \mathrm{csr}(C^*(G)) \leq 2 \vee \mathrm{csr}(C_0(\hat{G}_1)) = [(\dim \hat{G}_1 + 1)/2] + 1 \end{cases}$$

where $\dim_{\mathbb{C}} = [\dim(\cdot)/2] + 1$, and $[x]$ means the maximum integer $\leq x$, and $\dim(\cdot)$ is the covering dimension, and $\vee$ is the maximum.

Remark. We may replace G in the statement with its quotient as in Remark of Theorem 1.3.

Proof. We apply the following formulas by [Rf1, Theorems 3.6, 4.3, 4.4, 4.11 and 6.4] and [Sh, Theorems 3.9 and 3.10] to the finite composition series obtained in Theorem 1.3:

$$\begin{aligned} \mathrm{sr}(\mathfrak{I}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{I}) \leq \mathrm{sr}(\mathfrak{A}) &\leq \mathrm{sr}(\mathfrak{I}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{I}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{I}), \\ \mathrm{csr}(\mathfrak{A}) &\leq \mathrm{csr}(\mathfrak{I}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{I}) \end{aligned}$$

for an exact sequence of C^*-algebras: $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$, and

$$\mathrm{sr}(\mathfrak{A} \otimes \mathbb{K}) \leq 2, \quad \mathrm{csr}(\mathfrak{A} \otimes \mathbb{K}) \leq 2$$

for any C^*-algebra $\mathfrak{A}$. By [Rf1, Proposition 1.7] and [Sh, p.381] (cf.[Ns]), we have

$$\begin{cases} \mathrm{sr}(C_0(X)) = \dim_{\mathbb{C}} X^+, & \mathrm{csr}(C_0(\mathbb{R})) = 2, \\ \mathrm{csr}(C_0(\mathbb{R}^2)) = 1, & \mathrm{csr}(C_0(\mathbb{R}^d)) = [(d+1)/2] + 1 \quad \text{for } d \geq 3, \end{cases}$$

where X^+ means the one-point compactification of a locally compact Hausdorff space X. Moreover, note that $\mathrm{sr}(C^*(G)) \geq 2$ by [ST2, Lemma 3.7]. $\square$

As a corollary of Theorem 1.3 and Corollary 2.2, we obtain

Theorem 4.2. *Let* $G = G_{n,m}$ *or* $H'_{n,m}$ *as in Theorem* 1.3 *and Corollary* 2.2. *Then*

$$\begin{cases} \dim_{\mathbb{C}} \hat{G}_1 \leq \mathrm{sr}(C^*(G)) \leq \dim_{\mathbb{C}} \hat{G}_1 + 1, \\ \mathrm{csr}(C^*(G)) \leq 2 \vee \mathrm{csr}(C_0(\hat{G}_1)) = [(\dim \hat{G}_1 + 1)/2] + 1. \end{cases}$$

Remark. We may replace G with its quotient as in Remarks of Theorem 1.3 and Corollary 2.2.

Theorem 4.3. *Let* $G = G_{n,m}$, $H'_{n,m}$ *or* $K'_{n,m}$ *as in Theorem* 1.3 *and Corollaries* 2.2 *and* 3.2. *Then*

$$\begin{cases} \dim_{\mathbb{C}} \hat{G}_1 \vee (\max_{j\in J} \max_{s\in S_j} \mathrm{sr}(\mathfrak{K}_{j,s}/\mathfrak{K}_{j,s-1})) \\ \leq \mathrm{sr}(C^*(G)) \leq (\dim_{\mathbb{C}} \hat{G}_1 + 1) \vee (\max_{j\in J} \max_{s\in S_j} \mathrm{csr}(\mathfrak{K}_{j,s}/\mathfrak{K}_{j,s-1})), \\ \mathrm{csr}(C^*(G)) \leq 2 \vee \mathrm{csr}(C_0(\hat{G}_1)) \vee (\max_{j\in J} \max_{s\in S_j} \mathrm{csr}(\mathfrak{K}_{j,s}/\mathfrak{K}_{j,s-1})) \end{cases}$$

where J *is an index set of the subquotients obtained in Theorem* 1.3 *and Corollaries* 2.2 *and* 3.2 *which have some homogeneous subquotients* $\mathfrak{K}_{j,s}/\mathfrak{K}_{j,s-1}$ *for* $j \in J$, *and* $\{S_j\}_{j\in J}$ *are index sets of such homogeneous subquotients* $\mathfrak{K}_{j,s}/\mathfrak{K}_{j,s-1}$ *for* $s \in S_j$.

Remark. We may replace G with its quotient as in Remarks of Theorem 1.3 and Corollaries 2.2 and 3.2. For example, the following subquotient in Theorem 3.1:

$$C_0(X_{m_j}) \otimes (\cdots((C(\mathbb{T}^{m_j}) \rtimes_{\hat{\alpha}^m} \mathbb{Z}) \rtimes_{\hat{\alpha}^{m-1}} \mathbb{Z}) \cdots \rtimes_{\hat{\alpha}^1} \mathbb{Z})$$

has homogeneous subquotients when the actions $\hat{\alpha}^p$ $(1 \leq p \leq m)$ are rational rotations on $\mathbb{T}^{m_j}$, and the stable rank and connected stable rank of such homogeneous subquotients are computable by combining some formulas used in the proof of Theorem 4.1 with [Rf1, Theorem 6.1] and [Rf2, Theorem 4.7] (cf. [Sd7], [Sd8, Theorem 2.3]). The above theorems 4.1 to 4.3 partially answer Rieffel's question [Rf1, Question 4.14] on describing the stable rank of group C^*-algebras in terms of groups. See [Sd2], [Sd3], $\cdots$, [Sd9], [ST1], [ST2] for some partial results to this question.

Moreover, we consider the case of (non-solvable) amenable Lie groups as follows:

Theorem 4.4. *Let* M *be a semi-direct product* $G \rtimes_\beta L$ *of a compact connected Lie group* L *by a connected or disconnected successive semi-direct product* G *as given in Theorem* 1.3 *and Corollaries* 2.2 *and* 3.2. *Suppose that the structure of* $C^*(G)$ *given there is invariant under the action* $\hat{\beta}$ *of* L. *Then* $C^*(M)$ *has a finite composition series such that its subquotients are given by*

$$\begin{cases} C_0(\hat{G}_1) \rtimes_{\hat{\beta}} L, \\ (C_0(X_{m_j}) \rtimes_{\hat{\beta}} L) \otimes \mathbb{K}, \\ (([C_0(X_{m_j}) \otimes C(\mathbb{T}^{m_j})] \rtimes_{\hat{\beta}} L) \rtimes_{\hat{\alpha}^p} \mathbb{Z}) \otimes \mathbb{K}, \quad 1 \leq p \leq m, \\ ([C_0(X_{m_j}) \otimes C(\mathbb{T}^{m_j})] \rtimes_{\hat{\beta}} L) \rtimes_{\hat{\alpha}^p} \mathbb{Z}, \quad 1 \leq p \leq m, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ ((\cdots([C_0(X_{m_j}) \otimes C(\mathbb{T}^{m_j})] \rtimes_{\hat{\beta}} L) \rtimes_{\hat{\alpha}^m} \mathbb{Z} \cdots) \rtimes_{\hat{\alpha}^1} \mathbb{Z}) \otimes \mathbb{K}, \\ (\cdots([C_0(X_{m_j}) \otimes C(\mathbb{T}^{m_j})] \rtimes_{\hat{\beta}} L) \rtimes_{\hat{\alpha}^m} \mathbb{Z} \cdots) \rtimes_{\hat{\alpha}^1} \mathbb{Z}. \end{cases}$$

Remark. The structures of the crossed products $C_0(\hat{G}_1) \rtimes_{\hat{\beta}} L$ and $([C_0(X_{m_j}) \otimes C(\mathbb{T}^{m_j})] \rtimes_{\hat{\beta}} L$ are analyzable, and their stable rank and connected stable rank are computable (cf. [Sd3]). Also, we may take L as $SO(n)$ for an example. On the other hand, if L is replaced by a non-compact connected semi-simple Lie group such as $SL_n(\mathbb{R})$, then M is non-amenable (cf. [Sd2]). But, the same theorem holds under the same assumptions in the case of the full group C^*-algebra $C^*(M)$ (this would be true for the reduced group C^*-algebra of M).

In particular, by using [Sd3, Proposition 3.9] we have

Proposition 4.5. *With the notation as above, if G is a simply connected solvable Lie group, then*

$$\mathrm{sr}(C_0(\hat{G}_1) \rtimes_{\hat{\beta}} L)) = (2 \wedge \dim_{\mathbb{C}}(\hat{G}_1/L)) \vee \dim_{\mathbb{C}}(\hat{G}_1^L)$$

where $\wedge$ means the minimum, and $\hat{G}_1/L$ is the orbit space of $\hat{G}_1$ by L, and $\hat{G}_1^L$ is the fixed point subspace of $\hat{G}_1$.

Remark. With the same technique as the case of G being simply connected, the proof of [Sd3, Proposition 3.9] also works for the case where G is taken as in Corollaries 2.2 and 3.2 and for the case of $([C_0(X_{m_j}) \otimes C(\mathbb{T}^{m_j})] \rtimes_{\hat{\beta}} L$ above.

Remark. Also, we may define that an amenable Lie group M is linearizable if it is isomorphic to a successive semi-direct product $G \rtimes L \rtimes \mathbb{Z}^k$ for G a simply connected solvable Lie group and L a compact (connected) Lie group and $k \geq 0$. In this case, we can obtain that $C^*(M)$ has the structure as given in Theorem 4.4.

5. C^*-SOLVABLE C^*-ALGEBRAS

We first recall the following definition (cf. [Sc, p. 449]):

Definition 5.1. Let $\mathfrak{A}$ be a C^*-algebra. We say that $\mathfrak{A}$ is solvable if there exists a composition series $\{\mathfrak{I}_j\}_{j=0}^{\infty}$ of closed ideals of $\mathfrak{A}$ with $\mathfrak{I}_0 = 0$ such that the union $\cup_{j=0}^{\infty}\mathfrak{I}_j$ is dense in $\mathfrak{A}$ and each subquotient $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ has the following isomorphism:

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong C_0(X_j) \otimes \mathbb{K}(H_j)$$

for X_j a locally compact Hausdorff space and $\mathbb{K}(H_j)$ the C^*-algebra of all compact operators on a Hilbert space H_j.

Remark. Note that the Hilbert space H_j may be finite or infinite dimensional. In particular, $\mathbb{K}(\mathbb{C}) = \mathbb{C}$. Since any commutative C^*-algebra is isomorphic to $C_0(X)$ for X a locally compact Hausdorff space and $C_0(X) \otimes \mathbb{K}(H)$ has the same representation theory as $C_0(X)$, that is, $(C_0(X) \otimes \mathbb{K}(H))^\wedge = C_0(X)^\wedge$, this definition seems to be suitable. If the above composition series is finite, then this definition for being solvable is more fitting in a sense. However, this class of solvable C^*-algebras does not contain the class of the C^*-algebras of solvable Lie groups.

From this remark, we give the following new definition:

Definition 5.2. Let $\mathfrak{A}$ be a C^*-algebra. We say that $\mathfrak{A}$ is C^*-solvable if there exists a finite composition series $\{\mathfrak{I}_j\}_{j=0}^n$ of closed ideals of $\mathfrak{A}$ with $\mathfrak{I}_0 = 0$ and $\mathfrak{I}_n = \mathfrak{A}$ such that each subquotient $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ is isomorphic to one of the following C^*-algebras:

$$\begin{cases} C_0(X_j) \\ C_0(X_j) \otimes \mathbb{K}(H_j) \\ C_0(X_j) \otimes \mathbb{K}(H_j) \otimes \mathfrak{A}_{\Theta_j} \\ C_0(X_j) \otimes \mathbb{K}(H_j) \otimes \Gamma_0(Y_j, \{C_0(Z_{y_j}) \otimes \mathbb{K}(H_{y_j}) \otimes \mathfrak{A}_{\Theta_{y_j}}\}_{y_j \in Y_j}) \end{cases}$$

where $\mathfrak{A}_{\Theta_j}$ are noncommutative tori generated by unitaries U_k $(1 \leq k \leq n_j)$ subject to the relations $U_k U_l = e^{2\pi i \theta_{kl}} U_l U_k$ for $\Theta_j = (\theta_{kl})_{k,l=1}^{n_j}$ and $\theta_{kl} \in \mathbb{R}$, and $\Gamma_0(Y_j, \{C_0(Z_{y_j}) \otimes \mathbb{K}(H_{y_j}) \otimes \mathfrak{A}_{\Theta_{y_j}}\}_{y_j \in Y_j})$ mean the C^*-algebras of continuous fields vanishing at infinity on locally compact Hausdorff spaces Y_j with fibers $C_0(Z_{y_j}) \otimes \mathbb{K}(H_{y_j}) \otimes \mathfrak{A}_{\Theta_{y_j}}$ for Z_{y_j} locally compact Hausdorff spaces and H_{y_j} finite or infinite dimensional Hilbert spaces and $\mathfrak{A}_{\Theta_{y_j}}$ noncommutative tori, and these tensor factors of the fibers may vary over $y_j \in Y_j$.

Remark. By definition, solvable C^*-algebras with finite composition series in the sense of Definition 5.1 are C^*-solvable. As typical examples, the C^*-algebra of the Mautner group has a subquotient of the form $C_0(X) \otimes \mathbb{K} \otimes \mathfrak{A}_\Theta$ (see [Sd5, Theorem 2.1]), and the C^*-algebra of the Dixmier group has a subquotient of the form:

$$C_0(X) \otimes \mathbb{K} \otimes \Gamma_0(Y, \{C_0(Z) \otimes \mathbb{K} \otimes \mathfrak{A}_{\Theta_y}\}_{y \in Y})$$

(see [Sd10, Theorems 2.1 and 2.2]). Since the C^*-algebras of solvable Lie groups should be solvable, this definition seems to be more suitable than Definition 5.1. Note that possible simple subquotients of C^*-solvable C^*-algebras are given by either $\mathbb{C}$, $M_n(\mathbb{C})$, $\mathbb{K}$, $\mathbb{K} \otimes \mathfrak{A}_\theta$, $\mathfrak{A}_\theta$ or $\mathfrak{A}_\theta \otimes M_n(\mathbb{C})$, where $\mathfrak{A}_\theta$ is a simple noncommutative torus. At this moment we believe in that this definition 5.2 is right, but it might be necessary to add more members to subquotients in the definition. See more discussion below.

Then our conjecture is the following:

Conjecture I. *Let G be a connected solvable Lie group. Then the group C^*-algebra $C^*(G)$ of G is C^*-solvable in the sense of Definition* 5.2.

Remark. We have already obtained some positive evidences to this conjecture. See [Sd5, Theorem 2.1] for the case of semi-direct products of $\mathbb{C}^n$ by $\mathbb{R}$ including the Mautner group, and [Sd10, Theorems 2.1 and 2.2] for the case of generalized Dixmier groups including the Dixmier group, and [Sd9, Theorem 1.4] for the case of generalized Mautner groups of the form $\mathbb{C}^n \rtimes_\alpha \mathbb{R}^m$ with α a multi-rotation. But the general case remains open. Also, it is known by [Pg, Theorem 2] that simple subquotients of $C^*(G)$ for G a connected solvable Lie group are given by either $\mathbb{C}$, $\mathbb{K}$ or $\mathbb{K} \otimes \mathfrak{A}_\theta$, where $\mathfrak{A}_\theta$ is a simple noncommutative torus.

As a variation of the above conjecture, we may state the following:

Conjecture II. *Let G be a disconnected solvable Lie group. Then the C^*-algebra $C^*(G)$ of G is C^*-solvable.*

Remark. See [Sd6, Theorems 1.2 and 2.1], [Sd7, Theorem 1.1], [Sd8, Theorems 1.7 and 2.1] for some results supporting this Conjecture II. We may define that a disconnected solvable Lie group is a (disconnected) Lie group with a finite normal series such that its subquotients are commutative Lie groups such as the product groups: $\mathbb{R}^s \times \mathbb{T}^k \times \mathbb{Z}^l \times F$ for $s, k, l \geq 0$ and F a finite commutative group.

Proposition 5.3. *Let G be a connected solvable Lie group. If the C^*-algebra $C^*(G)$ of G is C^*-solvable, then*

$$\begin{aligned}\mathrm{sr}(C_0(\hat{G}_1)) \leq \mathrm{sr}(C^*(G)) &\leq 2 \vee \mathrm{sr}(C_0(\hat{G}_1)) \vee \mathrm{csr}(C_0(\hat{G}_1)),\\ \mathrm{csr}(C^*(G)) &\leq 2 \vee \mathrm{sr}(C_0(\hat{G}_1)) \vee \mathrm{csr}(C_0(\hat{G}_1))\end{aligned}$$

where $\hat{G}_1$ means the space of all characters of G.

Proof. See the proof of Theorem 4.1. Note that any irreducible representation of $C^*(G)$ is either one or infinite dimensional. Also, by [Gr2, Theorem 3.3] or [Pg, Theorem 2] any simple subquotient of $C^*(G)$ is stable or one-dimensional. □

Remark. The above estimates are not true in general in the case of disconnected solvable Lie groups. For this, see Theorem 4.3.

Our conjecture for the stable rank of group C^*-algebras of connected solvable Lie groups is the following:

Conjecture III. *Let G be a connected solvable Lie group. Then*

$$\begin{aligned}\mathrm{sr}(C_0(\hat{G}_1)) \leq \mathrm{sr}(C^*(G)) &\leq 2 \vee \mathrm{sr}(C_0(\hat{G}_1)) \vee \mathrm{csr}(C_0(\hat{G}_1)),\\ \mathrm{csr}(C^*(G)) &\leq 2 \vee \mathrm{sr}(C_0(\hat{G}_1)) \vee \mathrm{csr}(C_0(\hat{G}_1)).\end{aligned}$$

Remark. This conjecture follows from the Conjecture I. Even if the Conjecture I is false, this Conjecture III could be true. See [Sd3, Theorem 3.10], [Sd4, Theorem 3.1], [Sd5, Corollaries 2.3 and 3.3], [Sd9, Corollary 2.2], [Sd10, Corollaries 3.8 and 4.3], [ST1, Theorem 3.6] and [ST2, Theorem 3.9] for some affirmative results on this conjecture.

In a general setting, we have

Proposition 5.4. *Let $\mathfrak{A}$ be a C^*-solvable C^*-algebra. If any simple subquotient of $\mathfrak{A}$ is stable or one-dimensional, then*

$$\begin{aligned}\mathrm{sr}(C_0(\hat{\mathfrak{A}}_1)) \leq \mathrm{sr}(\mathfrak{A}) &\leq 2 \vee \mathrm{sr}(C_0(\hat{\mathfrak{A}}_1)) \vee \mathrm{csr}(C_0(\hat{\mathfrak{A}}_1)),\\ \mathrm{csr}(\mathfrak{A}) &\leq 2 \vee \mathrm{sr}(C_0(\hat{\mathfrak{A}}_1)) \vee \mathrm{csr}(C_0(\hat{\mathfrak{A}}_1))\end{aligned}$$

where $\hat{\mathfrak{A}}_1$ means the space of all characters of $\mathfrak{A}$, and a C^*-algebra $\mathfrak{B}$ is stable if $\mathfrak{B} \otimes \mathbb{K} \cong \mathfrak{B}$.

Remark. This is the case in Definition 5.2 such that $\mathbb{K}(H_j) = \mathbb{K}$ and $\mathbb{K}(H_{y_j}) = \mathbb{K}$, that is, they are the the C^*-algebra of compact operators on a countably infinite dimensional Hilbert space. Note that

$$\Gamma_0(Y_j, \{C_0(Z_{y_j}) \otimes \mathbb{K} \otimes \mathfrak{A}_{\Theta_{y_j}}\}_{y_j \in Y_j}) \cong \Gamma_0(Y_j, \{C_0(Z_{y_j}) \otimes \mathfrak{A}_{\Theta_{y_j}}\}_{y_j \in Y_j}) \otimes \mathbb{K}$$

since for any $n \in \mathbb{N}$,

$$\begin{aligned} \mathfrak{B} &\equiv \Gamma_0(Y_j, \{C_0(Z_{y_j}) \otimes M_n(\mathbb{C}) \otimes \mathfrak{A}_{\Theta_{y_j}}\}_{y_j \in Y_j}) \\ &\cong \Gamma_0(Y_j, \{C_0(Z_{y_j}) \otimes \mathfrak{A}_{\Theta_{y_j}}\}_{y_j \in Y_j}) \otimes M_n(\mathbb{C}) \equiv \mathfrak{D} \otimes M_n(\mathbb{C}), \end{aligned}$$

which is deduced from that the canonical open continuous map from the primitive ideal space of $\mathfrak{D}$ to Y_j induces an open continuous map from the primitive ideal space of $\mathfrak{B}$ to Y_j (cf.[Le]).

Finally, at this moment we have

Proposition 5.5. *The class of C^*-solvable C^*-algebras is closed under the following operations*:

(1) *Stable isomorphism*
(2) *Taking closed ideals, quotients and extensions*
(3) *Taking tensor products*

Proof. It is obvious. □

Remark. In Definition 5.2, if we allow to take countably infinite composition series of closed ideals with their union dense in $\mathfrak{A}$, then the class is closed under taking inductive extensions. However, it is unclear whether the class is stable under the operations: (4) taking inductive limits (this would be hopeless) and (5) taking crossed products by amenable groups, in particular, whether all the group C^*-algebras of amenable groups belong to the class. It is known that nuclear C^*-algebras are closed under the operations (1) to (5) (cf. [Bl]).

Acknowledgement. The author is gratful to the referee for reading the manuscript carefully and some comments for revision.

References

[Bl] B. Blackadar, *K-Theory for Operator algebras*, Second Edition, Cambridge, 1998.
[Dx] J. Dixmier, *C^*-algebras*, North-Holland, 1962.
[Gr1] P. Green, *C^*-algebras of transformation groups with smooth orbit space*, Pacific. J. Math. **72** (1977), 71–97.
[Gr2] ———, *The structure of imprimitivity algebras*, J. Funct. Anal. **36** (1980), 88–104.
[Le] R. Lee, *On the C^*-algebras of operator fields*, Indiana Univ. Math. J. **25** (1976), 303–314.

[Ns] V. Nistor, *Stable range for tensor products of extensions of $\mathcal{K}$ by $C(X)$*, J. Operator Theory **16** (1986), 387–396.

[OV] A.L. Onishchik and E.B. Vinberg (Eds.), *Lie Groups and Lie Algebras III*, EMS 41, Springer-Verlag, 1994.

[Pg] D. Poguntke, *Simple quotients of group C^*-algebras for two step nilpotent groups and connected Lie groups*, Ann. Scient. Éc. Norm. Sup. **16** (1983), 151–172.

[Rf1] M.A. Rieffel, *Dimension and stable rank in the K-theory of C^*-algebras*, Proc. London Math. Soc. **46** (1983), 301–333.

[Rf2] ———, *The homotopy groups of the unitary groups of non-commutative tori*, J. Operator Theory **17** (1987), 237–254.

[Rs] J. Rosenberg, *The C^*-algebras of some real and p-adic solvable groups*, Pacific. J. Math. **65** (1976), 175–192.

[Sc] C. Schochet, *Topological methods for C^*-algebras II: Geometric resolutions and the Künneth formula*, Pacific J. Math. **98** (1982), 443–458.

[Sh] A.J-L. Sheu, *A cancellation theorem for projective modules over the group C^*-algebras of certain nilpotent Lie groups*, Canad. J. Math. **39** (1987), 365–427.

[Sd1] T. Sudo, *Structure of the C^*-algebras of nilpotent Lie groups*, Tokyo J. Math. **19** (1996), 211–220.

[Sd2] T. Sudo, *Stable rank of the reduced C^*-algebras of non-amenable Lie groups of type I*, Proc. Amer. Math. Soc. **125** (1997), 3647–3654.

[Sd3] T. Sudo, *Stable rank of the C^*-algebras of amenable Lie groups of type I*, Math. Scand. **84** (1999), 231–242.

[Sd4] ———, *Dimension theory of group C^*-algebras of connected Lie groups of type I*, J. Math. Soc. Japan **52** (2000), 583–590.

[Sd5] T. Sudo, *Structure of group C^*-algebras of Lie semi-direct products $\mathbb{C}^n \rtimes \mathbb{R}$*, J. Operator Theory **46** (2001), 25–38.

[Sd6] ———, *Structure of group C^*-algebras of semi-direct products of $\mathbb{C}^n$ by $\mathbb{Z}$*, Nihonkai Math. J. **12** (2001), 135–143.

[Sd7] T. Sudo, *Structure of group C^*-algebras of the generalized disconnected Dixmier groups*, Sci. Math. Japon. **54** (2001), 449–454, :e4, 861-866.

[Sd8] ———, *Structure of group C^*-algebras of the generalized disconnected Mautner groups*, Linear Alg. Appl. **341** (2002), 317–326.

[Sd9] ———, *Structure of group C^*-algebras of the generalized Mautner groups*, J. Math. Kyoto Univ. **42-2** (2002), 393–402.

[Sd10] ———, *The structure of group C^*-algebras of the generalized Dixmier groups*, Publ. RIMS Kyoto Univ. **39-2** (2003), 205-225.

[ST1] T. Sudo and H. Takai, *Stable rank of the C^*-algebras of nilpotent Lie groups*, Internat. J. Math. **6** (1995), 439–446.

[ST2] ———, *Stable rank of the C^*-algebras of solvable Lie groups of type I*, J. Operator Theory **38** (1997), 67–86.

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Recieved December 19, 2003 Revised April 16, 2004