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On the Semigroup Approach to a Class of Space-Dependent Porous Medium Systems

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The main goal of this paper is to treat the nonlinear system of porous medium equations,

$$\text{(PMS)}\qquad \begin{cases} \frac{\partial}{\partial t}u_1(x,t) = \Lambda\phi_1(x,u_1(x,t)) + f_1(x,u_2(x,t),u_1(x,t)) \\ \frac{\partial}{\partial t}u_2(x,t) = \Lambda\phi_2(x,u_2(x,t)) + f_2(x,u_1(x,t),u_2(x,t)) \\ u_1(\cdot,t)|_{\partial\Omega} = u_2(\cdot,t)|_{\partial\Omega} = 0 \ ,\ t \geq 0 \\ u_1(x,0) = u_1^0 \ ,\ \ u_2(x,0) = u_2^0 \quad \text{where } u_1^0, u_2^0 \in L^\infty(\Omega), \end{cases}$$

coupled in the reaction terms $f_1(x,u_2(x,t),u_1(x,t))$ and $f_2(x,u_1(x,t),u_2(x,t))$. Here Ω is a bounded open domain in $\mathbb{R}^n$, and its boundary $\partial\Omega$ is sufficiently smooth. We use the symbol Λ to represent a strongly elliptic operator such as the Laplace operator $\triangle$ on a given domain in $L^1(\Omega)$, defined in 2.9. In order to handle this system, we proceed in a number of stages.

Firstly, we consider the following single equation

$$\text{(PME)}\qquad \begin{cases} \frac{\partial}{\partial t}u(x,t) = \Lambda\phi(x,u(x,t)) + f(x,u(x,t)) \\ u(\cdot,t)|_{\partial\Omega} = 0 \\ u(x,0) = u_0(x) \ ,\ \ u_0 \in L^1(\Omega). \end{cases}$$

The functions ϕ and f are assumed to be continuous on $\overline{\Omega}\times\mathbb{R}$, and to satisfy certain natural assumptions which will be detailed in 2.6. Under those conditions we shall show that the theory of nonlinear semigroups can be applied to prove the existence of unique solutions in a generalized sense to (PME). This is done by first considering the semilinear equation $-\Lambda v + Gv = w$ for $w \in L^1$, G being the composition operator defined by $g(\cdot,v) = f(\cdot,\phi^{-1}(v))$, under appropriate conditions on g. Existence of solutions v to this simpler equation allows the *range condition* (RC) from nonlinear semigroup theory to be proven. Hence, by showing that the nonlinear operator $A = \Lambda + F$ is dissipative, the result is obtained.

Next, L^p- estimates and a comparison theorem for the problem (PME) are established. These are of importance both in a physical sense and in the final section, when we finally extend the results for the single equation to the case of a pair of equations given in (PMS). Employing an iteration scheme and proving its convergence using our estimations for (PME), we show that the conditions of the semigroup generation theorem are again satisfied, and hence unique solutions in the sense of distributions to (PMS) are constructed.

1 Background

This section deals with some results and notations that we shall use.

Let X be a real Banach space with norm $|\cdot|$. The term $F(x)$, where $x \in X$, shall be used to represent the *duality mapping* from X into the power set of the dual space X^* of X. When $x \in X$ and $x^* \in X^*$, we write $\langle x, x^* \rangle$ for the value of the functional x^* at the point x. An operator A with domain $D(A)$ in a Banach space X is said to be *dissipative* if, for $x_1, x_2 \in D(A)$ and $y_1 \in Ax_1, y_2 \in Ax_2$,

$$[y_1 - y_2, x_1 - x_2]_i \leq 0.$$

Here the *lower semi-inner product* $[\cdot,\cdot]_i$ is defined $[z_2, z_1]_i = \inf\{\langle z_2, z^* \rangle \mid z \in F(z_1)\}$, $z_1, z_2 \in X$. We also define the *upper semi-inner product* $[\cdot,\cdot]_s$ to be the supremum of the same set for $z_1, z_2 \in X$. An operator is said to be *accretive* if $[y_1 - y_2, x_1 - x_2]_s \geq 0$ for all $x_1, x_2 \in D(A)$ and $y_1 \in Ax_1, y_2 \in Ax_2$. It is known that an operator A is dissipative if and only if $-A$ is accretive.

1.1 Lemma. *Let ω be a real number. Given an operator A, the following are equivalent:*

(i) $[(y_1 - \omega x_1) - (y_2 - \omega x_2), x_1 - x_2]_i \leq 0$ *for* $x_i \in D(A), y_i \in Ax_i$, $i = 1, 2$.

(ii) $[y_1 - y_2, x_1 - x_2]_i \leq \omega \|x_1 - x_2\|^2$ *for* $x_i \in D(A), y_i \in Ax_i$, $i = 1, 2$.

(iii) $$\|(x_1 - \lambda y_1) - (x_2 - \lambda y_2)\| \geq (1 - \lambda\omega) \|x_1 - x_2\|$$ *for* $\lambda > 0$, $x_i \in D(A)$ *and* $y_i \in Ax_i$, $i = 1, 2$.

(iv) $$\big\|\big[(1 + \lambda\omega)x_1 - \lambda y_1\big] - \big[(1 + \lambda\omega)x_2 - \lambda y_2\big]\big\| \geq \|x_1 - x_2\|$$ *for* $\lambda > 0$, $x_i \in D(A)$, *and* $y_i \in Ax_i$, $i = 1, 2$.

*In other words the operator $A - \omega I$ is dissipative, and so A is said to be ω-*quasi dissipative.

We shall consider generalized solutions of abstract equations of the form

$$\left\{ \begin{array}{ll} u'(t) & \in Au(t), \\ u(0) & = v \in D, \end{array} \right. \tag{1.3}$$

in the Banach space X, where $u'(t)$ represents the derivative of the function $u(\cdot) : [0,\infty) \to X$, in some appropriate sense. The set D is understood to be the class of admissible initial data.

The generalized notion of solutions (*Integral solutions*) we shall deal with is defined as follows:

1.2 Definition. Let A be an operator on a Banach space X. For ω a real number, $u(t) : [0,\tau] \to \overline{D(A)}$ is an *integral solution of type* ω to equation (1.3) if

(i) $u(0) = v$ and $u(t)$ is continuous on $[0,\tau]$,

(ii) for every $s,t \in [0,\tau]$ with $s < t$ and every $x_0 \in D(A)$, $y_0 \in Ax_0$,

$$e^{-2\omega t}\|u(t)-x_0\|^2 - e^{-2\omega s}\|u(s)-x_0\|^2 \le \int_s^t e^{-2\omega r}[y_0, u(r)-x_0]_s dr.$$

Equivalent to (ii) above is the following slightly clearer condition:

(ii)′ For every $s,t \in [0,\tau]$ with $s < t$ and every $x_0 \in D(A)$ and $y_0 \in Ax_0$,

$$\|u(t)-x_0\|^2 - \|u(s)-x_0\|^2 \le 2\int_s^t [y_0, u(r)-x_0]_s dr + 2\omega\int_s^t \|u(t)-x_0\|\, dr.$$

Notice that if $u(t)$ is an integral solution of type ω_1, then $u(t)$ is an integral solution of type ω_2 to the same problem, whenever $\omega_1 \le \omega_2$.

The notion of integral solution relies heavily upon the quasidissipativity of the operators A under consideration. The uniform limit function $u(\cdot) \in \mathscr{C}([0,\tau];X)$ of step functions $u_h(\cdot)$ on $[0,\tau]$ defined

$$u_h(0) = v_h \in D \ ,\ u_h(t) = u_h^k \text{ for } t \in (t_k^{k-1}, t_h^k] \text{ and } k = 1,\cdots,N_h$$

gives an integral solution of type ω, under the assumption that A is ω quasidissipative and the sequences (u_h^k) satisfy a certain consistency condition.

Given a quasidissipative operator A of type ω in X, the class of integral solutions of type ω to the associated evolution equation (1.3) specifies an appropriate class of continuous functions on $[0,\infty)$ such that all the limits of solutions to the discrete schemes mentioned above are contained in the class and, as stated in Benilans uniqueness thoerem, each such function is uniquely determined in the class by the initial value v.

The following *generation theorem* for nonlinear semigroups shall be of use in later sections. For a more in-depth discussion of these concepts we refer to [6] and [7].

1.3 Theorem. *If $A - \omega I$ is a dissipative operator satisfying*

(RC) $$R(I - \lambda A) \supset D(A)$$

for all sufficiently small $\lambda > 0$, *then* A *generates a semigroup* $\{S(t)\}$ *on* $\overline{D(A)}$, *defined*

$$S(t)x = \lim_{\lambda \to 0}(I - \lambda A)^{-[t/\lambda]}x, \quad \textit{for } x \in \overline{D(A)} \tag{1.5}$$

such that $S(t)v : [0, \infty) \to \overline{D(A)}$ *is the unique integral solution of type* ω *to the equation*

$$\begin{cases} u'(t) &= Au(t) \\ u(0) &= v \in \overline{D(A)}. \end{cases} \tag{1.6}$$

Moreover, the following inequalities are satisfied

$$(1.7)\ \|S(t)x - S(t)y\| \quad \leq e^{\omega(t-s)}\,\|x - y\| \quad \textit{for } x, y \in \overline{D(A)} \textit{ and } t \geq 0$$

$$(1.8)\ \|S(t)x - S(s)x\| \quad \leq e^{2\omega_0(t+s)|}\,\|Ax\|\,|.|t - s| \quad \textit{for } x \in D(A) \textit{ and } 0 \leq s, t < \infty,$$

where $\omega_0 = \max\{0, \omega\}$. *If, in addition,* A *is closed and* $S(t)v$ *is strongly differentiable at some* t_0, *then* $S'(t_0)v \in AS(t_0)v$.

Define the operator Φ by $[\Phi v](\cdot) \equiv \phi(\cdot, v(\cdot))$, $v \in X$, for appropriate functions $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$, and similarly $Fv \equiv f(\cdot, v(\cdot))$.

1.4 Definition. We shall say that a function $u(t) : [0, \tau] \to L^1$ is an *integrated solution* to the problem (PME) on $[0, \tau]$ if, for all t, $0 \leq t \leq \tau$ we have:

$$u(t) = u_0 + \Lambda \int_0^t \phi(\cdot, u(s))ds + \int_0^t f(\cdot, u(s))ds \tag{1.9}$$

1.5 Lemma. *Let* A *be defined* $\Lambda\Phi + F$. *Any integral solution on* $[0, \tau]$ *to* (PME) *generated as in equation* (1.5) *in the statement of Theorem 1.3 is an integrated solution to* (PME).

Proof. Let $u(\cdot) = S(\cdot)u_0 : [0, \tau] \to \overline{D(A)}$ be a semigroup solution of the form in the statement of Theorem 1.3. Given $\lambda > 0$ let $u_\lambda(t)$ be defined

$$u_\lambda(t) = (I - \lambda A)^{-k}u_0 = u_\lambda^{(k)} \quad , \ t \in [0, \tau] \tag{1.10}$$

where $k = [t/\lambda]$. Then $u_\lambda(t)$ converges to $S(t)u_0$ uniformly with respect to t, and

$$u_\lambda(t) - u_0 \ = \sum_{i=1}^{[t/\lambda]} u_\lambda^{(i)} - (I - \lambda A)u_\lambda^{(i)} \ = \Lambda\lambda \sum_{i=1}^{[t/\lambda]} \Phi u_\lambda^{(i)} + \lambda \sum_{i=1}^{[t/\lambda]} F u_\lambda^{(i)},$$

for $t \in [0, \tau]$. The sums converge to integrals, and the closedness of Λ implies that,

$$\int_0^t \Phi S(s)u_0 ds \in D(\Lambda), \quad \text{and}$$

$$S(t)u_0 = u_0 + \Lambda \int_0^t \Phi S(s)u_0 ds + \int_0^t FS(s)u_0 ds \quad , \ t \in [0, \tau]. \tag{1.11}$$

□

1.6 Remark. The natural partial ordering of the space $L^p(\Omega)$ is defined

$$w_1 \leq w_2 \text{ iff } w_1(x) \leq w_2(x) \text{ for almost all } x \in \Omega.$$

This ordering has the propety that $w_1 \leq w_2$ implies that $\|w_1\|_p \leq \|w_2\|_p$, and in fact $L^p(\Omega)$ is a *Banach lattice* under this ordering (see [5]).

1.7 Definition. For a real number η, $[\eta]^+$ denotes η for $\eta \geq 0$ and 0 for $\eta < 0$. We then write $[f]^+$ to represent the nonnegative part of a function f, defined

$$[f]^+(x) = \begin{cases} f(x) & : f(x) \geq 0 \\ 0 & : f(x) < 0. \end{cases}$$

2 Semilinear Elliptic Equations

From now on, Ω denotes a given bounded open subset of $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$.

2.1 Assumptions. We assume that the function $f(x, \eta) : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following:

(i) $\eta_1 \leq \eta_2$ implies $f(x, \eta_1) \leq f(x, \eta_2)$ for $x \in \Omega$;

(ii) $f(x, 0) = 0$ for all $x \in \Omega$;

(iii) for any fixed $\eta \in \mathbb{R}$, the function $f(x, \eta)$ is measurable with respect to x, and $f(\cdot, \eta)$ belongs to $L^2(\Omega)$. Moreover, f is continuous with repect to η for fixed $x \in \Omega$.

2.2 Definition. We define operators Λ and F, and their domains, by

$$D(\Lambda) = \{w \in W_\circ^{1,1}(\Omega) \mid \Lambda w \in L^1(\Omega)\} \text{ and } \Lambda = \triangle,$$
$$D(F) = \{w \in L^1(\Omega) \mid f(\cdot, w(\cdot)) \in L^1(\Omega)\} \text{ and } [Fw](x) = f(x, w(x)), \quad x \in \Omega.$$

2.3 Lemma. *The following results hold:*

(i) *The linear operator Λ is densely defined, closed and m-dissipative in L^1.*

(ii) *For $1 \leq p \leq \infty$ we have $\|(I - \lambda\Lambda)^{-1}w\|_p \leq \|w\|_p$ for $w \in L^p(\Omega)$.*

(iii) *There exists an $\alpha > 0$ such that $\alpha \|w\|_1 \leq \|\Lambda w\|_1$ for $w \in D(\Lambda)$.*

Note that part (iii) relies on the boundedness of the domain. Although this important estimate is central to a number of parts of the proof of existence of solutions given below, it is not a trivial task to extend the results to the case of unbounded domains.

2.4 Proposition. *F is m-accretive in $L^1(\Omega)$ and $\overline{D(F)} = L^1(\Omega)$.*

Proof. Firstly, we show that $R(I + \lambda F) = L^1(\Omega)$. Let $w \in L^1(\Omega)$ and $\lambda > 0$. For any fixed x, we denote by $f_x(\eta)$ the number $f(x, \eta)$. Then $(1 + \lambda f_x)^{-1}$ is a contraction on $\mathbb{R}$, since $f_x(\cdot)$ is a strictly non-decreasing normalized function, by 2.1(ii). Let $u(x) = (1 + \lambda f_x)^{-1} w(x)$. Then, given $c \in \mathbb{R}$, we have

$$\{x \in \Omega \mid u(x) > c\} = \{x \in \Omega \mid w(x) > c + \lambda f(x, c)\}.$$

Since $w(x)$ and $f(x, c)$ are both measurable over Ω, the set on the right hand side is measurable, and it follows that $u(x)$ is a measurable function. We also know that $|u(x)| \le |w(x)|$ for almost every $x \in \Omega$, since $(1 + \lambda f_x)^{-1}$ is a contraction on $\mathbb{R}$ and f is normalized, and so $u(\cdot) \in L^1(\Omega)$. Thereby $u \in D(F)$, $(I + \lambda F)u = w$, and hence $R(I + \lambda F) = L^1(\Omega)$. The relations $(I + \lambda F)u_i = w_i$, $i = 1, 2$, imply

$$|u_1(x) - u_2(x)| \le |w_1(x) - w_2(x)| \quad a.e.\ x \in \Omega,$$

and thereby

$$\|u_1 - u_2\|_1 \le \|w_1 - w_2\|_1,$$

proving accretivity and hence m-accretivity of F. It remains to show that $\overline{D(F)} = L^1(\Omega)$. It is sufficient to prove that

$$C_0^\infty(\Omega) \subset D(F) = \{w \in L^1(\Omega) \mid f(\cdot, w(\cdot)) \in L^1(\Omega)\},$$

since $C_0^\infty(\Omega)$ is dense in L^1. So, given $w \in C_0^\infty(\Omega)$, there exists a sequence $\{w_i\}$ of finite-valued step functions converging uniformly to w such that $|w_i(x)| \le |w(x)|$ for all $x \in \Omega$. Then $f(x, w_i(x))$ is measurable and belongs to L^2. For, if we write $w_i = \sum_{j=1}^n a_{i,j} \chi_{E_{i,j}}$ where $a_{i,j} \in \mathbb{R}$ and $\cup_j E_{i,j} = \Omega$ then $f(x, w_i(x)) = \sum_{j=1}^n f(x, a_{i,j}) \chi_{E_{i,j}}$ and each $f(x, a_{i,j}) \chi_{E_{i,j}}$ is L^2. Hence $f(x, w_i(x)) \to f(x, w(x))$ for all $x \in \Omega$, and the functions $f(\cdot, w_i(\cdot))$ are all L^2 and bounded in L^2 by $f(x, \pm \|w\|_\infty)$. Thus the Lebesgue Dominated Convergence Theorem gives $f(x, w(x)) \in L^2(\Omega) \subset L^1(\Omega)$. □

2.5 Theorem. *Under the assumptions of 2.1, for any $w \in L^1(\Omega)$ there exists a unique $u \in D(\Lambda) \cap D(F)$ such that*

$$-\Lambda u + Fu = w \tag{2.1}$$

Let u_i be such that $-\Lambda u_i + Fu_i = w_i$, for $i = 1, 2$. Then

$$\|Fu_1 - Fu_2\|_1 \le \|w_1 - w_2\|_1 \text{ and} \tag{2.2}$$

$$\|\Lambda u_1 - \Lambda u_2\|_1 \le 2\|w_1 - w_2\|_1 \tag{2.3}$$

Proof. We show firstly that (2.2) holds, and thereby that (2.3) is also true. Assume that $(-\Lambda + F)u_i = w_i$, so that $-\Lambda(u_1 - u_2) + Fu_1 - Fu_2 = w_1 - w_2$. Multiplying both sides by $\mathrm{sgn}(u_1 - u_2)$ gives, by the fact that $\langle -\Lambda(u_1 - u_2), \mathrm{sgn}(u_1 - u_2)\rangle \geq 0$, that

$$\langle Fu_1 - Fu_2, \mathrm{sgn}(u_1 - u_2)\rangle \leq \langle w_1 - w_2, \mathrm{sgn}(u_1 - u_2)\rangle .$$

We note also that, by the monontonicity and normalization of f,

$$\begin{aligned} \|Fu_1 - Fu_2\|_1 &= \langle Fu_1 - Fu_2, \mathrm{sgn}(u_1 - u_2)\rangle \\ &\leq \langle w_1 - w_2, \mathrm{sgn}(u_1 - u_2)\rangle \leq \|w_1 - w_2\|_1 , \end{aligned}$$

proving (2.2). Applying (2.2) to $-\Lambda(u_1 - u_2) + Fu_1 - Fu_2 = w_1 - w_2$ implies (2.3).

In order to prove that $R(-\Lambda + F) = L^1$, we shall show firstly that $R(-\Lambda + F)$ is closed, and then that it is dense in L^1. To show closedness, first assume that $u_n \in D(\Lambda) \cap D(F)$ and that $-\Lambda u_n + Fu_n \to w$ in $L^1(\Omega)$. We let $w_n = -\Lambda u_n + Fu_n$. Then by (2.2), (2.3) above and by Lemma 2.3(iii),

$$\|Fu_n - Fu_m\|_1 \leq \|w_n - w_m\|_1$$

and

$$\alpha \|u_n - u_m\|_1 \leq \|\Lambda u_n - \Lambda u_m\|_1 \leq 2\,\|w_n - w_m\|_1$$

Therefore $\{u_n\}$, $\{\Lambda u_n\}$ and $\{Fu_n\}$ are all Cauchy sequences in $L^1(\Omega)$ and so, by the closedness of Λ and F, there exists $u \in D(\Lambda) \cap D(F)$ such that $u_n \to u$, $\Lambda u_n \to \Lambda u$ and $Fu_n \to Fu$. Thus $w = -\Lambda u + Fu$, which shows that $R(-\Lambda + F)$ is closed.

We prove denseness of $R(-\Lambda + F)$ in three steps. Firstly, we solve the equation $\varepsilon u - \Lambda u + F_\lambda u = w$ for $w \in L^1$, where F_λ is the Yosida approximation of F. Then the equation $\varepsilon u - \Lambda u + Fu = w$ for $w \in L^\infty$ is solved in the second step, and finally, we approximate elements of L^1 by elements of the form of w in step 2, and thus prove the result.

Step 1: We approximate f by f_λ for $\lambda > 0$, where

$$f_\lambda(x, \eta) = \lambda^{-1}\left(\eta - (1 + \lambda f_x)^{-1}\eta\right) \text{ for } \eta \in \mathbb{R}.$$

Next, we define the operator F_λ by

$$[F_\lambda w](x) = f_\lambda(x, w(x)) , \quad x \in \Omega \tag{2.4}$$

for every $w \in D(F_\lambda) = L^1(\Omega)$. Now, given η_1 and $\eta_2 \in \mathbb{R}$,

$$\begin{aligned} |f_\lambda(x,\eta_1) - f_\lambda(x,\eta_2)| &= \lambda^{-1}\left(\eta_1 - \eta_2 - (1 + \lambda f_x)^{-1}\eta_1 + (1 + \lambda f_x)^{-1}\eta_2\right) \\ &\leq 2\lambda^{-1}|\eta_1 - \eta_2|. \end{aligned}$$

In other words, $f_\lambda(x,\cdot)$ is Lipschitz continuous for any $x \in \Omega$ and the Lipschitz constant ($= 2\lambda^{-1}$) is independent of x. Thereby F_λ is Lipschitz continuous for $\lambda > 0$. Also, for $w \in L^1$ we have

$$[F_\lambda w](x) = \lambda^{-1}\left(w(x) - (1+\lambda f_x)^{-1} w(x)\right) = \lambda^{-1}\left(I - (I+\lambda F)^{-1}\right) w(x),$$

which means that F_λ is the Yosida approximation for the operator F. Therefore, given $\varepsilon > 0$, $\lambda > 0$ and $w \in L^1(\Omega)$ we solve

$$\varepsilon u - \Lambda u + F_\lambda u = w \tag{2.5}$$

which, by expanding F_λ as in the remark above, and rearranging the terms, may be written

$$u = \frac{1}{1+\lambda\varepsilon}\left(I - \frac{\lambda}{1+\lambda\varepsilon}\Lambda\right)^{-1}\left(\lambda w + (I+\lambda F)^{-1} u\right) \tag{2.6}$$

Labelling the operator on the right hand side T, we see that T is a strict contraction and so (2.5) has a unique fixed point. It will be of use to note here that if $p \in [1,\infty)$ and $w \in L^p$, and Λ_p, F_p and T_p are the associated operators acting on the space L^p, then T_p is a strict contraction on L^p. In particular, if we take $L^\infty(\Omega)$ with the norm $\|\cdot\|_\infty$ and $w \in L^\infty(\Omega)$, then T is a strict contraction from $M = \{v \in L^\infty(\Omega) \mid \|v\|_\infty \le \varepsilon^{-1}\|w\|_\infty\}$ into itself, and so $u \in D(\Lambda) \cap L^\infty(\Omega)$ and $\Lambda u \in L^\infty(\Omega)$ by (2.5).

Step 2: We find a solution to $\varepsilon u - \Lambda u + Fu = w$ for $w \in L^\infty(\Omega)$. Fix any $\varepsilon > 0$ and any $w \in L^\infty(\Omega)$ and denote by u_λ^ε the solution to (2.5). By (2.6) and the remarks mentioned above concerning the set M, we have

$$\|u_\lambda^\varepsilon\|_p \le \frac{1}{\varepsilon}\|w\|_p\ , \quad 1 \le p \le \infty. \tag{2.7}$$

We also note that

$$F_\lambda u_\lambda^\varepsilon(x) = f\left(x, (1+\lambda f_x)^{-1} u_\lambda^\varepsilon(x)\right)\ , \quad x \in \Omega \tag{2.8}$$

and that $|(1+\lambda f_x)^{-1} u_\lambda^\varepsilon(x)| \le \|u_\lambda^\varepsilon\|_\infty \le \frac{1}{\varepsilon}\|w\|_\infty \equiv c$, say. From this, it follows that

$$f(x,-c) \le F_\lambda u_\lambda^\varepsilon(x) \le f(x,c).$$

Since $f(\cdot, \pm c) \in L^2(\Omega)$, we see that $\{F_\lambda u_\lambda^\varepsilon\}_{\lambda>0}$ is bounded in L^2. In fact, as we shall now show, $\{u_\lambda^\varepsilon\}_{\lambda>0}$ and $\{F_\lambda u_\lambda^\varepsilon\}_{\lambda>0}$ are both Cauchy sequences in $L^2(\Omega)$ as $\lambda \to 0^+$.

For $\lambda, \mu > 0$, we have

$$\varepsilon(u_\lambda^\varepsilon - u_\mu^\varepsilon) - \Lambda(u_\lambda^\varepsilon - u_\mu^\varepsilon) + F_\lambda u_\lambda^\varepsilon - F_\mu u_\mu^\varepsilon = 0.$$

Hence, taking the inner product in L^2 of both sides with $(u_\lambda - u_\mu)$ gives

$$\varepsilon\left\|u_\lambda^\varepsilon - u_\mu^\varepsilon\right\|_2 + \left\langle F_\lambda u_\lambda^\varepsilon - F_\mu u_\mu^\varepsilon, u_\lambda^\varepsilon - u_\mu^\varepsilon\right\rangle \le 0 \tag{2.9}$$

since $\langle \Lambda(u_\lambda^\varepsilon - u_\mu^\varepsilon), u_\lambda^\varepsilon - u_\mu^\varepsilon \rangle \le 0$. We then substitute

$$u_\lambda^\varepsilon - u_\mu^\varepsilon = \{u_\lambda^\varepsilon - (I+\lambda F)^{-1}u_\lambda^\varepsilon\} + \{(I+\lambda F)^{-1}u_\lambda^\varepsilon - (I+\mu F)^{-1}u_\mu^\varepsilon\} + \{(I+\mu F)^{-1}u_\mu^\varepsilon - u_\mu^\varepsilon\}$$

into the inner product in (2.9). Equation (2.8) implies that $F_\lambda u_\lambda^\varepsilon = F\big((I+\lambda F)^{-1}u_\lambda^\varepsilon\big)$, and so the middle term of the resulting three, i.e.

$$\left\langle F\left((I+\lambda F)^{-1}u_\lambda^\varepsilon\right) - F\left((I+\mu F)^{-1}u_\mu^\varepsilon\right), (I+\lambda F)^{-1}u_\lambda^\varepsilon - (I+\mu F)^{-1}u_\mu^\varepsilon \right\rangle,$$

is non-negative, since F is accretive. This gives the inequality

$$\varepsilon \left\| u_\lambda^\varepsilon - u_\mu^\varepsilon \right\|_2 + \langle F_\lambda u_\lambda^\varepsilon - F_\mu u_\mu^\varepsilon, \lambda F_\lambda u_\lambda^\varepsilon - \mu F_\mu u_\mu^\varepsilon \rangle \le 0.$$

The boundedness of $\{F_\lambda u_\lambda^\varepsilon\}_{\lambda>0}$ implies that the second term converges to 0 as $\lambda,\mu \downarrow 0$ and thereby $\{u_\lambda^\varepsilon\}_{\lambda>0}$ is a Cauchy sequence in $L^2(\Omega)$ as $\lambda \downarrow 0$. Thus there exists some $u^\varepsilon \in L^2(\Omega)$ such that $\lim_{\lambda\to 0^+} u_\lambda^\varepsilon = u^\varepsilon$ in L^2. Since Ω is bounded, the convergence also holds in L^1 . Moreover, by (2.7) and the lower semicontinuity of the L^∞-norm, we know that $u^\varepsilon \in L^\infty(\Omega)$.

Notice that for some subsequence $\{\lambda_i\}$ with $\lambda_i \to 0$ as $i \to \infty$, $(1+\lambda_i f_x)^{-1}u_{\lambda_i}^\varepsilon(x)$ converges to $u^\varepsilon(x)$ for almost every $x \in \Omega$, and thus $F_{\lambda_i} u_\lambda^\varepsilon(x) = f(x, (1+\lambda_i f_x)^{-1}u_{\lambda_i}^\varepsilon(x))$ must converge to $f(x, u^\varepsilon(x))$ (recall that f is continuous for any fixed x). Hence by the Lebesque Dominated Convergence Theorem there exists some subsequence, which we again label $\{u_{\lambda_i}^\varepsilon\}$, such that $F_{\lambda_i} u_{\lambda_i}^\varepsilon$ converges to Fu^ε in L^2. Equation (2.5) thereby implies that $(\varepsilon I - \Lambda)u_{\lambda_i} \to w - Fu$ in L^2 and $u_{\lambda_i} \to u$ in L^1. Hence the closedness of $\varepsilon I - \Lambda$ implies that $u \in D(\Lambda)$ and $\varepsilon u - \Lambda u + Fu = w$.

Also, $f(\cdot) + \varepsilon$ satisies the conditions on f in 2.1, so we may apply part (iii) of Lemma 2.3 and (2.3) above to get uniqueness in the following way: Let $F_\varepsilon = \varepsilon I + F$. If $u_1, u_2 \in L^1$, and $-\Lambda u_1 + F_\varepsilon u_1 = w = -\Lambda u_2 + F_\varepsilon u_2$, then for some $\alpha > 0$ we have

$$\alpha \left\| u_1 - u_2 \right\|_1 \le \left\| \Lambda(u_1 - u_2) \right\|_1 \le 2 \left\| w - w \right\|_1 = 0$$

and hence $u_1 = u_2$.

STEP 3: We are now in a position to show the denseness of $R(-\Lambda + F)$ in $L^1(\Omega)$. Let w be an arbitrary element in $L^1(\Omega)$, then there exists a sequence $\{w_\varepsilon\} \subset L^\infty(\Omega)$ such that $w_\varepsilon \to w$ in L^1 as $\varepsilon \to 0$. Let u^ε be the (unique) solution to the equation in Step 2, so that $\varepsilon u^\varepsilon - \Lambda u^\varepsilon + Fu^\varepsilon = w_\varepsilon$. Again, applying part (iii) of Lemma 2.3 and (2.3) to $f + \varepsilon$ gives

$$\alpha \left\| u^\varepsilon \right\|_1 \le \left\| \Lambda u^\varepsilon \right\|_1 \le 2 \left\| w_\varepsilon \right\|_1$$

which shows that $\|u^\varepsilon\|_1$ is bounded and hence $\varepsilon u^\varepsilon \to 0$ in L^1 as $\varepsilon \to 0$. Thus $-\Lambda u^\varepsilon + Fu^\varepsilon = w_\varepsilon - \varepsilon u^\varepsilon \to w$ in L^1, i.e. $w \in \overline{R(-\Lambda + F)}$. Since $R(-\Lambda + F)$ is closed we have existence of the solution $u \in D(-\Lambda + F)$ to equation (2.1) and uniqueness follows immediately, using (2.3) and part (iii) of Lemma 2.3 in the same way as above. □

We now define the operator A, to match the expression on the right hand side of the equation (PME), and use the results obtained so far to prove that the hypothesis of 1.3 holds. We begin by placing some restrictions on ϕ and ψ, which, as will be seen later, allow us to make direct use of the results obtained above.

2.6 Assumptions.

(i) The functions ϕ, ψ belong to the class $\mathscr{C}(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$.

(ii) For fixed $x \in \overline{\Omega}$, $\phi(x, \eta)$ is *strictly increasing* with respect to η.

(iii) There exists an $M \in \mathbb{R}$ such that $\psi(x, \eta_1) - \psi(x, \eta_2) \leq M(\eta_1 - \eta_2)$ for η_1, $\eta_2 \in \mathbb{R}$ and $x \in \overline{\Omega}$. In other words, with respect to η, ψ can be decreasing at any rate but we have to be able to uniformly bound the rate at which it may increase.

(iv) $\lim_{\eta \to \pm\infty} \phi(x, \eta) = \pm\infty$ for $x \in \overline{\Omega}$

(v) $\phi(x, 0) = \psi(x, 0) = 0$ for $x \in \overline{\Omega}$

Before defining the new operators we need some properties of the function ϕ which we investigate in the following two lemmas.

2.7 Lemma. *Assume that* (i), (ii) *and* (iv) *from 2.6 hold. Then*

(1) $x_n \to x_0$, $\eta_n \uparrow +\infty$ *implies* $\phi(x_n, \eta_n) \to +\infty$

(2) $x_n \to x_0$, $\eta_n \downarrow -\infty$ *implies* $\phi(x_n, \eta_n) \to -\infty$

Proof. The proof of (2) is similar to (1), and so we prove only (1). For all $m \in \mathbb{N}$, choose $N(m) \geq m$ such that

$$|\phi(x_n, \eta_m) - \phi(x_0, \eta_m)| < 1 \quad \forall n \geq N(m) \geq m.$$

Then $\phi(x_0, \eta_m) - 1 < \phi(x_{N(m)}, \eta_m) < \phi(x_{N(m)}, \eta_{N(m)})$ and therefore $\lim_{m \to \infty} \phi(x_{N(m)}, \eta_{N(m)}) = +\infty$. This shows that for any $\{n_k\}$ with $n_k \uparrow +\infty$ we may extract a sub-sequence $\{n'_k\}$ such that $\lim_{k \to \infty} \phi(x_{n'_k}, \eta_{n'_k}) = +\infty$, proving that $\phi(x_n, \eta_n) \to +\infty$. □

We define the "inverse" funtion $\phi^* : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$\eta = \phi^*(x, \zeta) \iff \phi(x, \eta) = \zeta. \tag{2.10}$$

Then the following lemma holds:

2.8 Lemma. ϕ^* *is continuous on* $\overline{\Omega} \times \mathbb{R}$.

Proof. Let $(x_0, \zeta_0) \in \overline{\Omega} \times \mathbb{R}$ and $(x_n, \zeta_n) \to (x_0, \zeta_0)$. Let $\eta_n = \phi^*(x_n, \zeta_n)$, so that $\zeta_n = \phi(x_n, \eta_n)$. Then $\{\eta_n\}$ must be bounded, since if it were not we could extract a sequence satisfying either (1) or (2) in Lemma 2.7 which would imply that $\{\zeta_n\}$ is unbounded. Let $a = \sup_n\{\eta_n\}$. Then $\phi : \overline{\Omega} \times [-a, a] \to \mathbb{R}$ is uniformly continuous, so

(2.11) $$|\zeta_n - \phi(x_0, \eta_n)| = |\phi(x_n, \eta_n) - \phi(x_0, \eta_n)| \to 0 \text{ as } n \to \infty$$

Therefore $\phi(x_0, \eta_n) \to \zeta_0 = \phi(x_0, \eta_0)$ and $\eta_n \to \eta_0$, since $\phi(x_0, \eta)$ is strictly increasing. It follows that $\phi^*(x_n, \zeta_n) \to \phi^*(x_0, \zeta_0)$. □

2.9 Definition. We define two composition operators Φ, Ψ and a nonlinear diffusion operator A on $L^1(\Omega)$ by

(2.12) $$[\Phi w](x) = \phi(x, w(x)) \ ; \ D(\Phi) = \{w \in L^1 \mid \phi(\cdot, w(\cdot)) \in L^1\},$$

(2.13) $$[\Psi w](x) = \psi(x, w(x)) \ ; \ D(\Psi) = \{w \in L^1 \mid \psi(x, w(x)) \in L^1\},$$

(2.14) $$Aw = \Lambda\Phi w + \Psi w \quad ; \ D(A) = \{w \in L^1 \mid \Phi w \in W_0^{1,1}, \Lambda\Phi w \in L^1, \Psi w \in L^1\}.$$

2.10 Theorem. *Under assumptions 2.6, $D(A)$ is dense in $L^1(\Omega)$.*

Proof. In view of the density of $C_0(\Omega)$ in $L^1(\Omega)$, it is sufficient to prove that given $w \in C_0(\Omega)$ there exists a sequence of elements of $D(A)$ approximating w. So let $w \in C_0(\Omega)$ and let $z = \phi(\cdot, w(\cdot))$. The continuity and normality of ϕ ensures that $z \in C_0(\Omega)$ and so there exists a compact set $K \subset \Omega$ and a sequence $\{z_n\}$ in $C_0^\infty(\Omega)$ such that

(2.15) $$\|z_n\|_{C(\overline{\Omega})} \leq \|z\|_{C(\overline{\Omega})}, \quad z_n \to z \in C(\overline{\Omega}) \quad \text{and} \quad \operatorname{supp} z_n \cap \operatorname{supp} z \subset K.$$

Let $w_n = \phi^*(\cdot, z_n(\cdot))$. Then $w_n \in C(\overline{\Omega})$, $\Psi w_n \in L^1(\Omega)$ and $\Phi w_n \in C_0^\infty(\Omega)$ so $w_n \in D(A)$. It remains to show that w_n converges to w in $L^1(\Omega)$, but $\phi^* : K \times [-\|z_n\|_{C(\overline{\Omega})}, \|z_n\|_{C(\overline{\Omega})}] \to \mathbb{R}$ is uniformly continuous, and so $\phi^*(x, z_n(x)) \to \phi^*(x, z(x))$ uniformly over x as $n \to \infty$ and hence $w_n \to w$ in $L^1(\Omega)$. □

2.11 Theorem. *Under assumptions 2.6 we have*

(i) $A - M$ is dissipative in $L^1(\Omega)$

(ii) $R(I - \lambda A) = L^1(\Omega)$ for $\lambda \in (0, \gamma)$, where $\gamma = \begin{cases} 1/M & : M > 0, \\ \infty & : M = 0, \end{cases}$

where M is the constant from part (iii) of 2.6.

Proof. We prove the second statement first. Let $\lambda \in (0,\gamma)$ and $w \in L^1(\Omega)$. We seek an element $u \in D(A)$ such that $(I-\lambda A)u = w$. For this purpose we define

$$f(x,\eta) = \lambda^{-1}\phi^*(x,\eta) - \psi(x,\phi^*(x,\eta)) \text{ for } x \in \overline{\Omega}, \eta \in \mathbb{R}. \tag{2.16}$$

Clearly, f satisfies (ii) and (iii) of 2.1 by the continuity of ϕ and ψ, and the fact that $\phi(x,\eta)$ is strictly non-decreasing with respect to η. Note that by (iii) of 2.6,

$$\{\psi(x,\eta_1) - \psi(x,\eta_2)\}\,\mathrm{sgn}(\eta_1-\eta_2) \le M|\eta_1-\eta_2| \quad \text{for } \eta_1,\eta_2 \in \mathbb{R} \tag{2.17}$$

and that

$$\begin{aligned}&\lambda\{f(x,\eta_1) - f(x,\eta_2)\}\,\mathrm{sgn}\,(\eta_1-\eta_2)\\ &\quad = |\phi^*(x,\eta_1) - \phi^*(x,\eta_2)| - \lambda\{\psi(x,\phi^*(x,\eta_1)) - \psi(x,\phi^*(x,\eta_2))\}\,\mathrm{sgn}\,(\eta_1-\eta_2).\end{aligned}$$

Since $\mathrm{sgn}(\eta_1-\eta_2) = \mathrm{sgn}(\phi^*(x,\eta_1) - \phi^*(x,\eta_2))$, equation (2.17) gives

$$\lambda\{f(x,\eta_1) - f(x,\eta_2)\}\,\mathrm{sgn}\,(\eta_1-\eta_2) \ge (1-\lambda M)\,|\phi^*(x,\eta_1) - \phi^*(x,\eta_2)|. \tag{2.18}$$

In particular, when $\lambda \in (0,\gamma)$ we have $\lambda\{f(x,\eta_1) - f(x,\eta_2)\}\,\mathrm{sgn}\,(\eta_1-\eta_2) \ge 0$ for all $\eta_1,\eta_2 \in \mathbb{R}$, i.e. condition (i) from 2.1 is satisfied. Therefore we may apply Theorem 2.5 to obtain the existence of some $v \in D(-\Lambda) \cap D(F)$ such that

$$-\Lambda v + Bv = \lambda^{-1}w$$

We then define the function $u(\cdot)$ by $u(x) = \phi^*(x,v(x))$ for $x \in \Omega$. Then u is measurable and $\lambda|f(x,v(x))| \ge (1-\lambda M)|u(x)|$ for a.e. $x \in \overline{\Omega}$. Since $v \in D(F)$, we infer that $f(\cdot,v(\cdot)) \in L^1$. Moreover $u(\cdot) \in L^1$, so that $u \in D(A)$ and

$$\lambda^{-1}w = -\Lambda v + Bv = -\Lambda\Phi u + \lambda^{-1}u - \psi(x,u) = -Au + \lambda^{-1}u.$$

This means that $(I-\lambda A)u = w$ and that (ii) holds. We now show the dissipativity of $A-M$. Assume that $(I-\lambda A)u_i = w_i$, for $i=1,2$ and set $v_i = \Phi u_i$ so that $-\Lambda v_i + Bv_i = \lambda^{-1}w_i$, for $i=1,2$. We apply (2.2) from Theorem 2.5 to obtain

$$\|Bv_1 - Bv_2\|_1 \le \lambda^{-1}\,\|w_1 - w_2\|_1 \quad \lambda \in (0,\gamma).$$

Equation (2.18) now shows that

$$\lambda\,\|Bv_1 - Bv_2\|_1 \ge (1-\lambda M)\,\|u_1-u_2\|_1$$

and therefore $(1-\lambda M)\,\|u_1-u_2\|_1 \le \|w_1-w_2\|_1$, whereby the statement follows from Lemma 1.1. □

2.12 Remark. We note that the use of Theorem 2.5 implies the uniqueness of our solution $u \in D(I-\lambda A)$ to $(I-\lambda A)u = w$, however this also follows from the dissipatiivity of $A-M$.

The above theorem shows that $R(I-\lambda A) \supset D(A)$ for sufficiently small λ, and so the results for nonlinear semigroups, in particular 1.3, can now be applied to obtain the following result.

2.13 Theorem. *Under the assumptions of 2.6, we may construct a one-parameter family of operators $\{S(t) : t \geq 0\}$ generated through the exponential formula*

$$S(t)w = \lim_{\lambda \downarrow 0} \Big(I - \lambda A\Big)^{-[t/\lambda]} w \qquad \text{for } w \in L^1(\Omega),$$

and this family has the following properties:

(1) $S(0)w = w$ *and* $S(t+s)w = S(t)S(s)w$ *for* $t, s \geq 0$ *and* $w \in L^1$.

(2) For all $w \in L^1$, $S(t)w : [0, \infty) \to L^1$ *is continuous.*

(3) $\|S(t)w_1 - S(t)w_2\|_1 \leq e^{Mt} \|w_1 - w_2\|_1$ *for* $t \geq 0$, *and* $w_1, w_2 \in L^1$.

(4) $\|S(t)w - S(s)w\|_1 \leq e^{2\widehat{M}(t+s)} \|Aw\|_1 |t-s|$ *for* $s, t \geq 0$ *and* $w \in D(A)$,

where $\widehat{M} = \max\{0, M\}$.

For any u_0 *in* L^1 *the function* $S(t)x_0 : [0, \infty) \to \overline{D(A)} = L^1$ *is an integral solution of type* M *to the problem* (PME).

3 Comparison Theorem and L^p Estimates

In this section we investigate positivity and order properties of solutions to (PME). Note that nonlinearity means that even if a solution preserves positivity of initial data, it need not necessarily preserve order (see Remark 1.6). Positivity is particularly important when one considers what the function u may represent physically. Quantities such as concentration or density only have a physical meaning when they take non-negative values and therefore it is important that non-negative inital conditions do not yield results which are meaningless in this sense.

Estimates for the solutions to problems of the form $(I - \lambda A)u = w$ are also obtained, and these shall be of crucial importance in dealing with the equation (PMS).

3.1 Order preserving properties of $(I - \lambda A)^{-1}$ and $S(t)$

3.1 Lemma. *Under the assumptions of 2.1, if* u_1, u_2, w_1 *and* w_2 *satisfy*

$$-\Lambda u_1 + f(\cdot, u_1) = w_1 \quad \text{and} \quad -\Lambda u_2 + f(\cdot, u_2) = w_2,$$

then

$$\left\|[f(\cdot, u_1) - f(\cdot, u_2)]^+\right\|_1 \leq \left\|[w_1 - w_2]^+\right\|_1. \tag{3.1}$$

In particular, $w_1 \leq w_2$ *implies that* $f(\cdot, u_1) \leq f(\cdot, u_2)$, *as in Remark 1.6.*

Proof. Combining the two equations, we have

$$-\Lambda(u_1 - u_2) + f(\cdot, u_1) - f(\cdot, u_2) = w_1 - w_2.$$

Let $E = \{x \in \Omega \mid u_1(x) \geq u_2(x)\}$ and let $h(x) = \chi_E(x)$. Clearly $h(x) = [\text{sgn}(u_1(x) - u_2(x))]^+$. We shall show later that $\langle \Lambda(u_1 - u_2), h \rangle \leq 0$, giving

$$\langle f(\cdot, u_1) - f(\cdot, u_2), h \rangle = \int_E (w_1 - w_2) dx \leq \int_\Omega [w_1 - w_2]^+ dx.$$

The result follows, since

$$\langle f(\cdot, u_1) - f(\cdot, u_2), h \rangle = \int_E (f(\cdot, u_1) - f(\cdot, u_2)) dx = \int_\Omega [f(\cdot, u_1) - f(\cdot, u_2)]^+ dx.$$

Therefore it remains to show that when $u_1, u_2 \in W_0^{1,1}$ and $\Lambda u_1, \Lambda u_2 \in L^1$ we have $\langle \Lambda(u_1 - u_2), h \rangle \leq 0$. Define

$$r_n(t) = \begin{cases} 0 & : t \leq 0 \\ nt & : 0 < t < 1/n \\ 1 & : 1/n \leq t \end{cases}$$

Then $r_n'(t) \geq 0$, $r_n(t) \to [\text{sgn}(t)]^+$ in L^1 as $n \to \infty$ and $|r_n(t)| \leq 1$ so that

$$\langle \Lambda u_1, r_n(u_1) \rangle = -\langle \nabla u_1, r_n'(u_1) \nabla u_1 \rangle = -\int_\Omega |\nabla u_1|^2 . r_n'(u_1) dx \leq 0.$$

The Lebesgue Convergence Theorem thereby gives $\langle \Lambda(u_1 - u_2), h \rangle \leq 0$. □

3.2 Lemma. *Under the assumptions of 2.6, if $\lambda \in (0, \gamma)$ for γ as in 2.11, and $(I - \lambda A) u_i = w_i$ for $i = 1, 2$, then*

$$(1 - \lambda M) \left\| [u_1 - u_2]^+ \right\|_1 \leq \left\| [w_1 - w_2]^+ \right\|_1. \tag{3.2}$$

In particular, $w_1 \leq w_2$ implies $u_1 \leq u_2$ in $L^1(\Omega)$.

Proof. For $i = 1, 2$, we write the equations in the form

$$u_i - \lambda \Lambda \phi(\cdot, u_i) - \lambda \psi(\cdot, u_i) = w_i$$

and put $v_i = \phi(\cdot, u_i)$ and $f(x, \eta) = \phi^*(x, \eta) - \lambda \psi(x, \phi^*(x, \eta))$ for $(x, \eta) \in \overline{\Omega} \times \mathbb{R}$. Then $f(\cdot, v_i) = u_i - \lambda \psi(\cdot, u_i)$ and so

$$-\lambda \Lambda v_i + f(\cdot, v_i) = w_i$$

for $i = 1, 2$. Thus we may apply Lemma 3.1 to get

$$\left\| [f(\cdot, v_1) - f(\cdot, v_2)]^+ \right\|_1 \leq \left\| [w_1 - w_2]^+ \right\|_1$$

Since (2.18) implies

$$\left[f(x, v_1(x)) - f(x, v_2(x)) \right]^+ \geq (1 - \lambda M) \left[u_1(x) - u_2(x) \right]^+,$$

we obtain the desired estimate (3.2). □

3.3 Theorem. (Comparison Theorem) *The semigroup $\{S(t)\}$ generated in Theorem 2.13 has the order preserving property, in the sense that*

$$w, \widehat{w} \in L^1(\Omega),\ w \leq \widehat{w} \text{ implies } S(t)w \leq S(t)\widehat{w} \text{ for } t \geq 0. \tag{3.3}$$

In particular, $L^1(\Omega)^+$ is invariant under $S(t)$, namely $S(t)$ is positivity preserving.

Proof. Note that for arbitrary w in $L^1(\Omega)$ we may write

$$S(t)w = \lim_{\substack{\lambda\to 0\\ n\lambda\to t}} (I - \lambda A)^{-n} w.$$

Given $w, \widehat{w}$ in L^1, let $w_n = (I - \lambda A)^{-n}w$ and $\widehat{w}_n = (I - \lambda A)^{-n}\widehat{w}$, we have

$$(I - \lambda A)w_n = w_{n-1} \text{ and } (I - \lambda A)\widehat{w}_n = \widehat{w}_{n-1} \quad \text{for } n \geq 1,$$

where $w_0 = w$ and $\widehat{w}_0 = \widehat{w}$. Then, since $w_0 \leq \widehat{w}_0$, we see by Lemma 3.2 that $w_n \leq \widehat{w}_n$ for all $n \geq 0$. This holds for any $\lambda > 0$ and thus when we take sequences $\{w_{\lambda_n}\}$ and $\{\widehat{w}_{\lambda_n}\}$ generated as in the definition of $S(t)$, where $\lambda_n \downarrow 0$ and $n\lambda_n \to t$, the statement holds for every n, and thus holds for the limit. □

3.2 L^p-estimates and comparison of solutions on f, ψ and ϕ

3.4 Lemma. *Let functions $f_1(\cdot,\cdot)$ and $f_1(\cdot,\cdot)$ satisfy the assumptions of* (2.1). *Assume that $f_1(x,\eta) \leq f_2(x,\eta)$ for $(x,\eta) \in \Omega \times X$, and that $-\Lambda u_i + f_i(\cdot, u_i) = w_i$. Then we have*

(1) $\left\|[f_i(\cdot,u_2) - f_i(\cdot,u_1)]^+\right\|_1 \leq \left\|[w_2 - w_1]^+\right\|_1$ *for $i = 1, 2$.*

In particular,

(2) if $w_2 \leq w_1$, then $f_i(\cdot,u_2) \leq f_i(\cdot,u_1)$ for $i = 1, 2$,

(3) if either one of the f_i's is strictly increasing, $w_2 \leq w_1$ implies $u_2 \leq u_1$.

Proof. We may write the above equations in u_i, f_i and w_i as

$$\{\ -\Lambda u_1 + f_1(\cdot,u_1) = w_1 \text{ and } -\Lambda u_2 + f_1(\cdot,u_2) = w_2 + f_1(\cdot,u_2) - f_2(\cdot,u_2).$$

Using Lemma 3.1, we obtain

$$\left\|[f_1(\cdot,u_2) - f_1(\cdot,u_1)]^+\right\|_1 \leq \left\|[w_2 - w_1 + f_1(\cdot,u_2) - f_2(\cdot,u_2)]^+\right\|_1 \leq \left\|[w_2 - w_1]^+\right\|_1$$

since $f_1(\cdot,u_2) \leq f_2(\cdot,u_2)$. The result for f_2 follows in a similar way, and parts (2) and (3) follow from (1). □

3.5 Lemma. *Assume that ϕ_i and ψ_i satisfy the assumptions of 2.6 for $i = 1, 2$, and that*

$$\phi_1(x,\eta) \geq \phi_2(x,\eta) \text{ and } \psi_1(x,\eta) \geq \psi_2(x,\eta) \quad \text{for all } (x,\eta) \in \overline{\Omega} \times \mathbb{R}.$$

Let operators A_i be defined

$$A_i w = \Lambda\phi_i(\cdot, w) + \psi_i(\cdot, w) \text{ for all } w \in D(A_i).$$

Let $(I - \lambda A_i)u_i = w_i$, $v_i = \phi_i(\cdot, u_i)$ and $\lambda \in (0, \gamma)$, γ being defined as in 2.11. Then:

(1) $(I - \lambda M) \left\|[\phi_i^*(\cdot, v_2) - \phi_i^*(\cdot, v_1)]^+\right\|_1 \leq \left\|[w_2 - w_1]^+\right\|_1.$

(2) In particular, $w_2 \leq w_1$ implies that $\phi_2(\cdot, u_2) \leq \phi_1(\cdot, u_1)$.

Proof. Set $f_i(x, v) = \phi_i^*(x, v) - \lambda\psi_i(x, \phi_i^*(x, v))$, Then the equation for w_i becomes

$$-\lambda\Lambda v_i + f_i(\cdot, v_i) = w_i, \quad i = 1, 2.$$

Noting that $\phi_1(x, w) \geq \phi_2(x, w)$ implies $\phi_1^*(x, w) \leq \phi_2^*(x, w)$ for all $(x, w) \in \overline{\Omega} \times \mathbb{R}$ it is clear that $f_1(x, \eta) \leq f_2(x, \eta)$ for all $(x, \eta) \in \overline{\Omega} \times \mathbb{R}$. Therefore we may apply Lemma 3.4 to get

$$\left\|[f_i(\cdot, v_2) - f_i(\cdot, v_1)]^+\right\|_1 \leq \left\|[w_2 - w_1]^+\right\|_1.$$

Since $[f_i(\cdot, v_2) - f_i(\cdot, v_1)]^+ \geq (1 - \lambda M)[\phi_i^*(x, v_2) - \phi_i^*(x, v_1)]^+$ by (2.18), (1) follows. (2) is clear from (1). Note also that we require the restriction $\lambda \in (0, \gamma)$ in order that monotonicity of the f_i's is achieved. This allows us to apply Lemma 3.4. □

3.6 Lemma. *Assume that ϕ satisfies 2.6 and define ϕ_M and ϕ_m, respectively, by*

$$\phi_M(\eta) = \max_{x \in \overline{\Omega}} \phi(x, \eta) \quad \text{and} \quad \phi_m(\eta) = \min_{x \in \overline{\Omega}} \phi(x, \eta)$$

Then

(1) ϕ_M and ϕ_m are continuous and strictly increasing over $\mathbb{R}$.

(2) $\lim_{\eta \to \pm\infty} \phi_M(\eta) = \lim_{\eta \to \pm\infty} \phi_m(\eta) = \pm\infty.$

Proof. We show the statement for ϕ_M only, since the proof for ϕ_m is similar. The lower semicontinuity of ϕ_M is clear. Hence, in order to show (1), we prove that it is also upper semi-continuous. Assume then that $\eta_n \to \eta$. Let $\alpha = \overline{\lim}_{n\to\infty}\phi_M(\eta_n)$. Then we can find a subsequence $\{\eta_{n_k}\}$ such that $\alpha = \lim_{k\to\infty} \phi_M(\eta_{n_k})$. But for each n_k we have $\phi(\eta_{n_k}) = \phi(x_{n_k}, \eta_{n_k})$ for some $\{x_{n_k}\} \subset \overline{\Omega}$. By the compactness of $\overline{\Omega}$ we may choose a subsequence $\{x_{n'_k}\} \subset \{x_{n_k}\}$ converging to some $x \in \overline{\Omega}$. Then $\phi(x_{n'_k}, \eta_{n'_k}) \to \phi(x, \eta)$, by the continuity of ϕ, and therefore

$$\alpha = \lim_{k\to\infty} \phi_M(\eta_{n'_k}) = \lim_{k\to\infty} \phi(x_{n'_k}, \eta_{n'_k}) = \phi(x, \eta) \leq \phi_M(\eta).$$

Thus ϕ_M is upper semi-continuous and the continuity of ϕ follows. (1) and (2) are now clear from the properties of ϕ □

3.7 Theorem. *Assume that ϕ and ψ satisfy the hypotheses of 2.6, and that the ϕ_M and ϕ_m are as in Lemma 3.6. Define operators A_M, A_m and A by*

$$\begin{aligned} A_M w &= \Lambda\phi_M(w) + \psi(\cdot, w) \text{ for } w \in D(A_M)\ , \\ A_m w &= \Lambda\phi_m(w) + \psi(\cdot, w) \text{ for } w \in D(A_m)\ , \\ Aw &= \Lambda\phi(\cdot, w) + \psi(\cdot, w) \text{ for } w \in D(A). \end{aligned}$$

Then for u, u_i and $w \in L^1$ the relations $(I - \lambda A)u = w$, $(I - \lambda A_m)u_m = w$ and $(I - \lambda A_M)u_M = w$ together imply that

$$\phi_M^{-1}(\phi_m(u_m)) \leq u \leq \phi_m^{-1}(\phi_M(u_M)), \text{ for } \lambda \in (0, \gamma).$$

Proof. By Lemma 3.5 we know that $\phi_m(u_m) \leq \phi(\cdot, u) \leq \phi_M(u_M)$. Since $\phi_m(u) \leq \phi(x, u)$, we have $\phi_m(u) \leq \phi_M(u_M)$ and so $u = \phi_m^{-1}(\phi_m(u)) \leq \phi_m^{-1}(\phi_M(u_M))$.
Similarly, $\phi_M(u) \geq \phi(x, u) \geq \phi_m(u_m)$ implies that $\phi_M^{-1}(\phi_m(u_m)) \leq \phi_M^{-1}(\phi_M(u)) = u$.
□

3.8 Lemma. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be continuous and non-decreasing. Then*

$$-\left(\Lambda w, \phi(w)|\phi(w)|^{p-2}\right) \geq 0 \quad \text{for } w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

Proof. Since ϕ can be approximated by smooth functions, we assume that $\phi \in C^1(\mathbb{R})$ and that $\phi' \geq 0$. Note that $\phi(w)|\phi(w)|^{p-2} \in L^\infty(\Omega) \subset L^q(\Omega)$, where $p^{-1} + q^{-1} = 1$, and so

$$\begin{aligned} -\left\langle \Lambda w, \phi(w)|\phi(w)|^{p-2} \right\rangle &= (p-1)\left\langle \nabla w, \nabla w \phi'(w)|\phi(w)|^{p-2} \right\rangle \\ &= (p-1) \int_\Omega |\nabla w|^2 \phi'(w)|\phi(w)|^{p-2} dx \ \geq 0 \end{aligned}$$

□

3.9 Theorem. (L^p **Estimates**) *Assume that 2.6 holds, and that $\phi(x, \eta)$ is a funtion only of η. Define*

$$\overline{M} = \sup_{\substack{x \in \overline{\Omega} \\ \eta \neq 0}} \eta^{-1}\psi(x, \eta) \ \leq M \quad \text{and} \quad Aw = \Lambda\phi(w) + \psi(\cdot, w).$$

For $\lambda \in (0, \gamma)$ and $w \in L^\infty(\Omega)$ there exists a unique $u \in D(A)$ such that $(I - \lambda A)u = w$,

(i) $u \in C(\overline{\Omega})$ and $(1 - \lambda\overline{M})\,\|u\|_p \leq \|w\|_p$ for all p with $1 \leq p \leq \infty$,

(ii) $\phi(u) \in W^{2,p} \cap W_0^{1,p}$ for all p with $1 \leq p < \infty$, and $\Lambda\phi(u)$, $\psi(\cdot, u) \in L^\infty(\Omega)$.

Proof. The existence and uniqueness of $u \in D(A)$ is obtained by Theorem 2.11 (see Remark 2.12). Therefore we prove (i) and (ii). Let u be such that $u - \lambda\Lambda\phi(u) - \lambda\psi(\cdot, u) = w$. As before, we let $v = \phi(u)$ and $f(x,\eta) = \lambda^{-1}\phi^*(\eta) - \psi(x, \phi^*(\eta))$. Let $g = \lambda^{-1}w$, then $-\Lambda v + f(x, v) = g$.

For any ε, $\mu > 0$, the proof of Theorem 2.5 gives the existence of $v_\mu \in W^{2,p} \cap W_0^{1,p}$, $1 \le p < \infty$, such that

$$\varepsilon v_\mu - \Lambda v_\mu + B_\mu v_\mu = g \text{ and } \|v_\mu\|_p \le \varepsilon^{-1} \|g\|_p \text{ for } 1 \le p \le \infty,$$

where $f_\mu(x,\eta) = \mu^{-1}\{1 - (1 + \mu f_x)^{-1}\}\eta = f(x, (1 + \mu f_x)^{-1}\eta)$ and $[B_\mu z](x) = f_\mu(x, z(x))$, for all $z \in L^1(\Omega)$. Therefore we obtain

$$\|f_\mu(\cdot, v_\mu)\|_\infty \le \max_{x\in\Omega}\{f(x, \varepsilon^{-1}\|g\|_\infty), -f(x, -\varepsilon^{-1}\|g\|_\infty)\}.$$

Fixing any $\varepsilon > 0$, we see that $\{\varepsilon v_\mu\}_{\mu>0}$ and $\{f_\mu(\cdot, v_\mu)\}_{\mu>0}$ are both L^∞ bounded. From this it follows that $\{\Lambda v_\mu\}_{\mu>0}$ is also L^∞ bounded. Hence $\{v_\mu\}_{\mu>0}$ is $W^{2,p}$ bounded for all p with $1 \le p < \infty$ (see Tanabe [11]).

As shown in the proof of Theorem 2.5, we take an appropriate sequence $\{\mu_n\}$ with $\mu_n \downarrow 0$, to show that v_{μ_n} converges to the solution v of

$$\varepsilon v - \Lambda v + f(\cdot, v) = g \tag{3.4}$$

in $L^2(\Omega)$. Since Ω is bounded and $W^{2,p}$ is reflexive (and hence bounded sets are weakly sequentially compact), we infer that $v \in W^{2,p} \cap W_0^{1,p}$.

For each $\varepsilon > 0$, any solution to (3.4) satisfies

$$\varepsilon v^\varepsilon - \Lambda v^\varepsilon + \frac{1}{\lambda}\phi^*(v^\varepsilon) - \psi(\cdot, \phi^*(v^\varepsilon)) = g \tag{3.5}$$

$\|v^\varepsilon\|_p \le \frac{1}{\varepsilon}\|g\|_p$ for $1 \le p \le \infty$, and all terms belong to $L^\infty(\Omega)$.

Multiplying both sides of (3.5) by $\phi^*(v^\varepsilon)|\phi^*(v^\varepsilon)|^{p-2}$ and integrating the resulting identity, we have

$$\begin{aligned}\langle \varepsilon v^\varepsilon - \Lambda v^\varepsilon + \lambda^{-1}\phi^*(v^\varepsilon) - \psi(\cdot, \phi^*(v^\varepsilon)),\ & \phi^*(v^\varepsilon)|\phi^*(v^\varepsilon)|^{p-2}\rangle \\ &= \langle g, \phi^*(v^\varepsilon)|\phi^*(v^\varepsilon)|^{p-2}\rangle.\end{aligned}$$

Since $-\langle \Lambda v^\varepsilon, \phi^*(v^\varepsilon)|\phi^*(v^\varepsilon)|^{p-2}\rangle \ge 0$ by Lemma 3.8, and $\psi(x,\eta)\eta \le \overline{M}\eta^2$ for all $\eta \in \mathbb{R}$, we obtain

$$\frac{1}{\lambda}\|\phi^*(v^\varepsilon)\|_p^p - \overline{M}\|\phi^*(v^\varepsilon)\|_p^p \le \|g\|_p \cdot \|\phi^*(v^\varepsilon)\|_p^{p-1}.$$

Hence

$$(1 - \lambda\overline{M})\|\phi^*(v^\varepsilon)\|_p \le \lambda\|g\|_p = \|w\|_p \quad , \quad 1 \le p \le \infty. \tag{3.6}$$

Thus $\{\frac{1}{\lambda}\phi^*(v^\varepsilon) - \psi(\cdot,\phi^*(v^\varepsilon))\}_{\varepsilon>0}$ is L^∞ bounded (by the continuity of ψ) and so $\{\Lambda v^\varepsilon\}_{\varepsilon>0}$ is also L^∞ bounded. This shows that $\{v^\varepsilon\}_{\varepsilon>0}$ is $W^{2,p}$ bounded, for $1 \leq p < \infty$. The proofs of Theorems 2.5 and 2.11 show that $v^\varepsilon \to \phi(u)$, i.e. $\phi^*(v^\varepsilon) \to u$ in $L^1(\Omega)$. Hence (3.6) implies that

$$(1-\lambda\overline{M})\,\|u\|_p \leq \|w\|_p \quad , \quad 1 \leq p \leq \infty.$$

Since $\Lambda v^\varepsilon \to \Lambda\phi(u)$ in $L^1(\Omega)$ and $\{\Lambda v^\varepsilon\}_{\varepsilon>0}$ is L^∞ bounded, $\Lambda\phi(u) \in L^\infty$. The $W^{2,p}$ boundedness of $\{v^\varepsilon\}_{\varepsilon>0}$ implies that $\phi(u) \in W^{2,p}\cap W_0^{1,p}$ for $1 \leq p < \infty$. To complete the proof, we note that $W^{2,p} \hookrightarrow C(\overline{\Omega})$ for $p > n/2$ (see Brezis [2]). Therefore $u \in C(\overline{\Omega})$ and $\psi(\cdot,u) \in C(\overline{\Omega})$. □

3.10 Theorem. *Under the assumptions of 2.6, let $\phi(x,\eta)$ be a function of both x and η and let*

$$Aw = \Lambda\phi(\cdot,w) + \psi(\cdot,w), \; w \in D(A),$$

Then for any $\lambda \in (0,\gamma)$ and any $w \in L^\infty(\Omega)$, there exists a unique $u \in D(A)$ such that $(I-\lambda A)u = w$,

(i) $\|u\|_\infty \leq \max\{\phi_m^{-1}(\phi_M(c)), -\phi_M^{-1}(\phi_m(c))\} \quad , \quad c = (1-\lambda\overline{M})^{-1}\,\|w\|_\infty$

(ii) $\phi(\cdot,u) \in W^{2,p}\cap W_0^{1,p}$, $1 \leq p < \infty$ *for* $u \in C(\overline{\Omega})$, *and* $\Lambda\phi(\cdot,u), \psi(\cdot,u) \in L^\infty$.

Proof. Part (i) follows immediately from 3.7 and 3.9, so we need only show (ii). As in the proof of 3.9, for any $\varepsilon > 0$ there exist solutions $v_i^\varepsilon, v^\varepsilon \in W^{2,p}\cap W_0^{1,p}$ $1 \leq p < \infty$, to

$$(3.7) \qquad \begin{cases} \varepsilon v^\varepsilon - \Lambda v^\varepsilon + \lambda^{-1}\phi^*(\cdot,v^\varepsilon) - \psi(\cdot,\phi^*(v^\varepsilon)) = \lambda^{-1}w \\ \varepsilon v_i^\varepsilon - \Lambda v_i^\varepsilon + \lambda^{-1}\phi_i^{-1}(v_i^\varepsilon) - \psi(\cdot,\phi_i^{-1}(v^\varepsilon)) = \lambda^{-1}w \;, \quad (i = M,m), \end{cases}$$

such that $\phi_M^{-1}(\zeta) \leq \phi^*(x,\zeta) \leq \phi_m^{-1}(\zeta)$ for all $(x,\zeta) \in \Omega\times\mathbb{R}$. Thus Lemma 3.4 implies that $v_M^\varepsilon \geq v^\varepsilon \geq v_m^\varepsilon$. As in the proof of Theorem 3.9, in particular equation (3.6), the sequences $\{v_i^\varepsilon\}_{\varepsilon>0}$ are L^∞ bounded, and thus $\{v^\varepsilon\}_{\varepsilon>0}$ is also L^∞ bounded. It follows, then, that each term in (3.7) is also L^∞ bounded, in particular $\{\Lambda v^\varepsilon\}_{\varepsilon>0}$. Hence $\{v^\varepsilon\}_{\varepsilon>0}$ is $W^{2,p}$ bounded. Thereby, letting $\varepsilon \to 0^+$ gives the desired results. □

3.11 Theorem. **(L^p - invariance)** *Suppose that $\phi(x,\eta)$ is a function of η only. If we assume that 2.6 holds, then the semigroup generated in Theorem 2.13 has the following properties for $1 \leq p \leq \infty$:*

$$S(t): L^p(\Omega) \to L^p(\Omega), \quad \text{and} \quad \|S(t)w\|_p \leq e^{\overline{M}t}\,\|w\|_p, \text{ for } w \in L^p,$$

Proof. Part (i) from Theorem 3.9 implies that $J_\lambda \stackrel{\text{def}}{=} (I-\lambda A)^{-1}$ maps L^p into L^p, and that $\|J_\lambda w\|_p \leq (1-\lambda\overline{M})^{-1}\,\|w\|_p$. Hence $\|J_\lambda^n w\|_p \leq (1-\lambda\overline{M})^{-n}\,\|w\|_p$. Since $J_\lambda^n w \to S(t)w$ as $\lambda \downarrow 0$ and $n = [t/\lambda]$, we have $\|S(t)w\|_p \leq e^{\overline{M}t}\,\|w\|_p$. □

4 Coupled System of Porous Medium Equations

In this section we attempt to apply the results from the previous two sections to the weakly coupled porous medium system (PMS). We begin by placing restrictions on the functions ϕ_1, ϕ_2, ψ_1 and ψ_2 which resemble those in 2.6, along with some extra assumptions according to the nature of the coupling.

4.1 Assumptions.

(i) $\phi_i \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $\psi_i \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $i = 1, 2$,

(ii) $\phi_i(x, \eta)$ is strictly increasing with respect to η for $i = 1, 2$.

(iii) There exists $M_1 : (0, \infty) \to \mathbb{R}$ and $N_1 : (0, \infty) \to (0, \infty)$, such that

$$\psi_1(x, \eta_2, \eta_1) - \psi_1(x, \eta_2, \widehat{\eta}_1) \geq M_1(K)(\eta_1 - \widehat{\eta}_1) \ , \quad x \in \overline{\Omega}, \ |\eta_2| \leq K, \ \eta_1, \widehat{\eta}_1 \in \mathbb{R},$$

$$|\psi_1(x, \eta_2, \eta_1) - \psi_1(x, \widehat{\eta}_2, \eta_1)| \leq N_1(K) \, |\eta_2 - \widehat{\eta}_2| \ , \quad x \in \overline{\Omega}, \ |\eta_2|, |\widehat{\eta}_2| \leq K, \ |\eta_1| \leq K.$$

For the function ψ_2, we assume the existence of M_2 and N_2, and the corresponding conditions.

(iv) $\phi_i(x, 0) = \psi_i(x, \eta, 0) = 0$ for $x \in \overline{\Omega}$, $\eta \in \mathbb{R}$ and $i = 1, 2$.

(v) $\lim_{\eta \to \pm\infty} \phi_i(x, \eta) = \pm\infty$, for $x \in \overline{\Omega}$ and $i = 1, 2$.

Given $K \geq 0$, we define the three constants

$$\begin{aligned}
\overline{M}_1(K) &= \sup \left\{ \eta_1^{-1} \psi_1(x, \eta_2, \eta_1) \ \middle| \ x \in \overline{\Omega}, \ |\eta_2| \leq K, \ \eta_1 \neq 0 \right\}, \\
\overline{M}_2(K) &= \sup \left\{ \eta_2^{-1} \psi_2(x, \eta_1, \eta_2) \ \middle| \ x \in \overline{\Omega}, \ |\eta_1| \leq K, \ \eta_2 \neq 0 \right\}, \\
\omega(K) &= \max \left\{ M_1(K) + N_2(K) \ , \ M_2(K) + N_1(K) \right\}.
\end{aligned}$$

4.2 Definition. We define the space X to be $L^1(\Omega) \times L^1(\Omega)$ with norm $\|(w_1, w_2)\|_X = \|w_1\|_{L^1} + \|w_2\|_{L^1}$. The operator A and its domain $D(A)$ are then defined

$$A \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \Lambda\phi_1(\cdot, w_1) + \psi_1(\cdot, w_2, w_1) \\ \Lambda\phi_2(\cdot, w_2) + \psi_2(\cdot, w_1, w_2) \end{pmatrix} \ , \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in D(A)$$

$$D(A) = \left\{ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in X \ \middle| \ \phi_1(\cdot, w_1), \phi_2(\cdot, w_2) \in W_0^{1,1}, \ \Lambda\phi_1(\cdot, w_1), \Lambda\phi_2(\cdot, w_2) \in L^1(\Omega), \right.$$
$$\left. \psi_1(\cdot, w_2, w_1), \psi_2(\cdot, w_1, w_2) \in L^1(\Omega) \right\}$$

We also emply the convex subsets

$$D_K = \left\{ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in L^\infty \times L^\infty \;\middle|\; \|w_1\|_\infty, \|w_2\|_\infty \le K \right\}, \quad K > 0.$$

and the restriction A_K of A to D_K for each $K > 0$.

4.3 Theorem. *The operator $A_K - \omega(K)I$ is dissipative on D_K.*

Proof. Assume that $(I - \lambda A_K)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ and $(I - \lambda A_K)\begin{pmatrix} \widehat{u}_1 \\ \widehat{u}_2 \end{pmatrix} = \begin{pmatrix} \widehat{w}_1 \\ \widehat{w}_2 \end{pmatrix}$, so that

$$\text{(4.1)} \qquad \begin{cases} u_1 - \lambda\Lambda\phi_1\big(\cdot, u_1(\cdot)\big) - \lambda\psi_1\big(\cdot, u_2(\cdot), u_1(\cdot)\big) = w_1(\cdot) \\ u_2 - \lambda\Lambda\phi_2\big(\cdot, u_2(\cdot)\big) - \lambda\psi_2\big(\cdot, u_1(\cdot), u_2(\cdot)\big) = w_2(\cdot) \end{cases}$$

and

$$\text{(4.2)} \qquad \begin{cases} \widehat{u}_1 - \lambda\Lambda\phi_1\big(\cdot, \widehat{u}_1(\cdot)\big) - \lambda\psi_1\big(\cdot, \widehat{u}_2(\cdot), \widehat{u}_1(\cdot)\big) = \widehat{w}_1(\cdot) \\ \widehat{u}_2 - \lambda\Lambda\phi_2\big(\cdot, \widehat{u}_2(\cdot)\big) - \lambda\psi_2\big(\cdot, \widehat{u}_1(\cdot), \widehat{u}_2(\cdot)\big) = \widehat{w}_2(\cdot). \end{cases}$$

with $\|u_i\|_\infty, \|\widehat{u}_i\|_\infty \le K$. Using the fact that

$$\begin{aligned} \langle \psi_1(\cdot, u_2, u_1) - \psi_1(\cdot, \widehat{u}_2, \widehat{u}_1), \operatorname{sgn}(u_1 - \widehat{u}_1)\rangle &= \langle \psi_1(\cdot, u_2, u_1) - \psi_1(\cdot, u_2, \widehat{u}_1), \operatorname{sgn}(u_1 - \widehat{u}_1)\rangle \\ &\quad + \langle \psi_1(\cdot, u_2, \widehat{u}_1) - \psi_1(\cdot, \widehat{u}_2, \widehat{u}_1), \operatorname{sgn}(u_1 - \widehat{u}_1)\rangle \\ &\le M_1(K)\|u_1 - \widehat{u}_1\|_1 + N_1(K)\|u_2 - \widehat{u}_2\|_1, \end{aligned}$$

subtracting the top equation of (4.2) from the top equation in (4.1), multiplying both sides by $\operatorname{sgn}(u_1 - \widehat{u}_1)$ and integrating, we have

(4.3)

$$\begin{aligned} \|w_1 - \widehat{w}_1\|_1 &\ge \langle w_1 - \widehat{w}_1, \operatorname{sgn}(u_1 - \widehat{u}_1)\rangle \\ &= \|u_1 - \widehat{u}_1\|_1 + \big\langle -\lambda\Delta\left[\phi_1\big(\cdot, u_1(\cdot)\big) - \phi_1\big(\cdot, \widehat{u}_1(\cdot)\big)\right], \operatorname{sgn}(u_1 - \widehat{u}_1)\big\rangle \\ &\quad - \big\langle \lambda\big(\psi_1(\cdot, u_2, u_1) - \psi_1(\cdot, \widehat{u}_2, \widehat{u}_1)\big), \operatorname{sgn}(u_1 - \widehat{u}_1)\big\rangle \\ &\ge \big(1 - \lambda M_1(K)\big)\|u_1 - \widehat{u}_1\|_1 - \lambda N_1(K)\|u_2 - \widehat{u}_2\|_1. \end{aligned}$$

Similarly, we obtain

$$\text{(4.4)} \qquad \|w_2 - \widehat{w}_2\|_1 \ge \big(1 - \lambda M_2(K)\big)\|u_2 - \widehat{u}_2\|_1 - \lambda N_2(K)\|u_1 - \widehat{u}_1\|_1.$$

Combinign these estimates gives

$$\begin{aligned} \left\| \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - \begin{pmatrix} \widehat{w}_1 \\ \widehat{w}_2 \end{pmatrix} \right\|_X &\ge \left[1 - \lambda\big(M_1(K) + N_2(K)\big)\right]\|u_1 - \widehat{u}_1\|_1 \\ &\quad + \left[1 - \lambda\big(M_2(K) + N_1(K)\big)\right]\|u_2 - \widehat{u}_2\|_1 \\ &\ge \big(1 - \lambda\omega(K)\big)\left\| \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} \widehat{u}_1 \\ \widehat{u}_2 \end{pmatrix} \right\|_X, \end{aligned}$$

whereby the result follows from Lemma 1.1. □

4.4 Theorem. *$D(A)$ is dense in X.*

Proof. We approximate an arbitrary element $(w_1, w_2) \in C_0(\Omega) \times C_0(\Omega)$ by elements in $D(A)$. Let the functions z_1 and z_2 be defined by

$$z_1(x) = \phi_1\big(x, w_1(x)\big) \text{ and } z_2(x) = \phi_2\big(x, w_2(x)\big) \quad , \; x \in \Omega.$$

Both of these belong to $C_0(\Omega)$ by Assumption 4.1, and so there exists some compact set $K \subset \Omega$ and functions $z_{1,n}, z_{2,n} \in C_0^\infty(\Omega)$ such that

(i) $\|z_{i,n}\|_{C(\overline{\Omega})} \le \|z_i\|_{C(\overline{\Omega})}$ for $n \in \mathbb{N}$ and $i = 1, 2$.

(ii) $\operatorname{supp} z_{i,n} \cup \operatorname{supp} z_i \subset K$, for $i = 1, 2$.

(iii) $z_{i,n} \to z_i$ in $C(\overline{\Omega})$, for $i = 1, 2$.

Let $w_{i,n} = \phi_i^*(\cdot, z_{i,n}(\cdot))$ for $i = 1, 2$. Then $(w_{1,n}, w_{2,n}) \in D(A)$ and the function $\phi_i^* : K \times [-\|z_i\|_{C(\overline{\Omega})}, \|z_i\|_{C(\overline{\Omega})}] \to \mathbb{R}$ is uniformly continuous, so $w_{i,n} \to w_i$ in $C(\overline{\Omega})$, $i = 1, 2$. Hence $(w_{1,n}, w_{2,n}) \to (w_1, w_2)$ in X. □

4.5 Definition. For $i = 1, 2$ and $\eta \in \mathbb{R}$, we define $\phi_{i,M}(\eta)$ and $\phi_{i,m}(\eta)$ as in the previous section by

$$\phi_{i,M}(\eta) = \max_{x \in \overline{\Omega}} \phi_i(x, \eta) \quad , \quad \phi_{i,m}(\eta) = \min_{x \in \overline{\Omega}} \phi_i(x, \eta).$$

We also define

$$\Theta_1(K) = \max\left\{\phi_{1,m}^{-1}(\phi_{1,M}(2K)), -\phi_{1,M}^{-1}(\phi_{1,m}(2K))\right\},$$

and define $\Theta_2(K)$ for $\phi_{2,M}$ and $\phi_{2,m}$ similarly. We shall also employ the numbers

$$\begin{aligned} \Theta(K) &= \max\{\Theta_1(K), \Theta_2(K)\}, \\ \gamma(K) &= \min\left\{\frac{1}{2M_1(\Theta(K))}, \frac{1}{2M_2(\Theta(K))}, \frac{1}{2\sqrt{N_1(\Theta(K))N_2(\Theta(K))}}\right\}, \end{aligned}$$

where we replace $1/2M_i(\Theta(K))$ by the value $+\infty$., if $M_i(\Theta(K)) \le 0$. Note that

$$\text{(4.5)} \qquad \lambda \in (0, \gamma(K)) \Longrightarrow 1/2 < 1 - \lambda M_i(\Theta(K)) \text{ for } i = 1, 2.$$

4.6 Theorem. *For $K > 0$, $\lambda \in (0, \gamma(K))$ and $(w_1, w_2) \in D_K$, there exists a $(u_1, u_2) \in D(A)$ such that*

$$(I - \lambda A)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

and the following hold:

(i) $u_1, u_2 \in C(\overline{\Omega})$, $\|u_1\|_{C(\overline{\Omega})} \le \Theta_1(K)$, $\|u_2\|_{C(\overline{\Omega})} \le \Theta_2(K)$.

(ii) $\phi_1(\cdot, u_1), \phi_2(\cdot, u_2) \in W^{2,p} \cap W_0^{1,p}$ *for* $1 \le p < \infty$.

(iii) $\Lambda\phi_1(\cdot, u_1), \Lambda\phi_2(\cdot, u_2), \psi_1(\cdot, u_2, u_1), \psi_2(\cdot, u_1, u_2) \in L^\infty(\overline{\Omega})$.

Proof. We seek solutions to the coupled system

$$\begin{cases} u_1 - \lambda\Delta\phi_1(\cdot, u_1) - \lambda\psi_1(\cdot, u_2, u_1) = w_1 \\ u_2 - \lambda\Delta\phi_2(\cdot, u_2) - \lambda\psi_2(\cdot, u_1, u_2) = w_2 \end{cases}$$

For simplicity, we abbreviate $\Theta_1(K)$, $\Theta_2(K)$ and $\Theta(K)$ by Θ_1, Θ_2 and Θ, respectively. Let $u_{2,0} \in C(\overline{\Omega})$ be such that $\|u_{2,0}\|_{C(\overline{\Omega})} \le \Theta_2$ and consider the equation for u_1 and w_1 above, with $u_{2,0}$ in place of u_2. Theorem 3.10 yields a unique element $u_{1,0}$ such that $u_{1,0} - \lambda\Lambda\phi_1(\cdot, u_{1,0}) - \lambda\psi_1(\cdot, u_{2,0}, u_{1,0}) = w_1$ and

$$\|u_{1,0}\|_{C(\overline{\Omega})} \le \max\left\{\phi_{1,m}^{-1}(\phi_{1,M}(c)), -\phi_{1,M}^{-1}(\phi_{1,m}(c))\right\},$$

where, in view of (4.5),

$$\text{(4.6)} \qquad \begin{aligned} c \equiv \left(1 - \lambda\overline{M}_1(\Theta_2)\right)^{-1} \|w_1\|_\infty &\le \left(1 - \lambda\overline{M}_1(\Theta)\right)^{-1} \|w_1\|_\infty \\ &\le \left(1 - \lambda M_1(\Theta)\right)^{-1} \|w_1\|_\infty \le 2K. \end{aligned}$$

Therefore $\|u_{1,0}\|_{C(\overline{\Omega})} \le \Theta_1$. We then solve

$$u_2 - \lambda\Lambda\phi_2(\cdot, u_2) - \lambda\psi_2(\cdot, u_{1,0}, u_2) = w_2$$

to get $u_{2,1}$ satisfying $\|u_{2,1}\|_{C(\overline{\Omega})} \le \max\left\{\phi_{2,m}^{-1}(\phi_{2,M}(c')), -\phi_{2,M}^{-1}(\phi_{2,m}(c'))\right\}$, where

$$\begin{aligned} c' \equiv \left(1 - \lambda\overline{M}_2(\Theta_1)\right)^{-1} \|w_2\|_\infty &\le \left(1 - \lambda\overline{M}_2(\Theta)\right)^{-1} \|w_2\|_\infty \\ &\le \left(1 - \lambda M_2(\Theta)\right)^{-1} \|w_2\|_\infty \le 2K. \end{aligned}$$

Hence $\|u_{2,1}\|_{C(\overline{\Omega})} \le \Theta_2$. Substituting this back into the first equation and solving, we get a $u_{1,1}$, etc. Repeating this procedure we obtain sequences $\{u_{1,n}\}$ and $\{u_{2,n}\}$ satisfying

$$\text{(4.7)} \qquad u_{1,n} - \lambda\Lambda\phi_1(\cdot, u_{1,n}) - \lambda\psi_1(\cdot, u_{2,n}, u_{1,n}) = w_1,$$

$$\text{(4.8)} \qquad u_{2,n} - \lambda\Lambda\phi_2(\cdot, u_{2,n}) - \lambda\psi_2(\cdot, u_{1,n-1}, u_{2,n}) = w_2,$$

$$\|u_{1,n}\|_{C(\overline{\Omega})} \le \Theta_1, \quad \|u_{2,n}\|_{C(\overline{\Omega})} \le \Theta_2,$$

$$\phi_1(\cdot, u_{1,n}), \phi_2(\cdot, u_{2,n}) \in W^{2,p} \cap W_0^{1,p}, \quad 1 \le p < \infty,$$

The continuity of ψ_1 and ψ_2 implies that $\|\psi_j(\cdot, u_{2,n}, u_{1,n})\|_{C(\overline{\Omega})}$, $j = 1, 2$, are bounded over $n \in \mathbb{N}$, so that $\{\Delta\phi_j(\cdot, u_{1,n})\}_{n\in\mathbb{N}}$, $j = 1, 2$ are also L^∞-bounded. This shows that $\{\phi_1(\cdot, u_{1,n})\}_{n\in\mathbb{N}}$ and $\{\phi_2(\cdot, u_{2,n})\}_{n\in\mathbb{N}}$ are bounded in $W^{2,p}$ for all p, $1 \le p < \infty$.

From the proof of Theorem 4.3, in particular estimates (4.3) and (4.4), it is seen that equations (4.7) and (4.8) imply

$$(1-\lambda M_1(\Theta))\,\|u_{1,n}-u_{1,m}\|_1 \le \lambda N_1(\Theta)\,\|u_{2,n}-u_{2,m}\|_1$$
$$(1-\lambda M_2(\Theta))\,\|u_{2,n+1}-u_{2,m+1}\|_1 \le \lambda N_2(\Theta)\,\|u_{1,n}-u_{1,m}\|_1$$

and therefore

$$\|u_{2,n+1}-u_{2,m+1}\|_1 \le \frac{\lambda^2 N_1(\Theta)N_2(\Theta)}{(1-\lambda M_1(\Theta))(1-\lambda M_2(\Theta))}\,\|u_{2,n}-u_{2,m}\|_1 .$$

We recall that, by equation (4.5), we have $1/2 < 1-\lambda M_i(\Theta(K))$ for $\lambda \in (0,\gamma(K))$ and $i=1,2$, so that

$$\|u_{2,n+1}-u_{2,m+1}\|_1 \le 4\lambda^2 N_1(\Theta)N_2(\Theta)\,\|u_{2,n}-u_{2,m}\|_1 .$$

Note that the term $\alpha \equiv 4\lambda^2 N_1(\Theta)N_2(\Theta)$ is independent of n and m, and that by the definition of $\gamma(K)$ we have $\alpha < 1$. Hence $\|u_{2,n+1}-u_{2,n}\|_1 \le \alpha^n \|u_{2,1}-u_{2,0}\|_1$, and so it follows that given integers n, m with $n > m$,

$$\|u_{2,n}-u_{2,m}\| \le \sum_{i=m}^{n-1} \alpha^i \|u_{2,1}-u_{2,0}\| \le \alpha^m \left(\frac{1}{1-\alpha}\right) \|u_{2,1}-u_{2,0}\| \to 0, \text{ as } m \to 0.$$

This proves that $\{u_{2,n}\}$, and thereby $\{u_{1,n}\}$, is a Cauchy sequence in $L^1(\Omega)$. Therefore there exist $u_1, u_2 \in L^1$ such that $u_{1,n} \to u_1$ and $u_{2,n} \to u_2$ in L^1.

For $p > n/2$ the injection from $W^{2,p}(\Omega)$ into $C(\overline{\Omega})$ is compact (see Brezis, [2]), and so we may extract a $C(\overline{\Omega})$-convergent subsequence from any subsequence of $\{\phi_1(\cdot, u_{1,n})\}$. Since $\phi_1^* : \overline{\Omega} \times [-\Theta(K), \Theta(K)] \to \mathbb{R}$ is uniformly continuous, we can extract a $C(\overline{\Omega})$ convergent subsequence from any subsequence of $\{u_{1,n}\}$.

We already know that $u_{1,n} \to u_1$ in $L^1(\Omega)$, and hence $u_{1,n}$ must also converge to u_1 in $C(\overline{\Omega})$. Similarly, $u_{2,n} \to u_2$ in $C(\overline{\Omega})$. It follows then that $\{\phi_1(\cdot, u_{1,n})\}$, $\{\phi_2(\cdot, u_{2,n})\}$, $\{\psi_1(\cdot, u_{2,n}, u_{1,n})\}$ and $\{\psi_2(\cdot, u_{1,n}, u_{2,n})\}$ must also converge in $C(\overline{\Omega})$, and thus $\{\Lambda\phi_1(\cdot, u_{1,n})\}$ and $\{\Lambda\phi_2(\cdot, u_{2,n})\}$ are L^∞-convergent. Since the linear operator Λ is L^p-closed so $\phi_1(\cdot, u_1), \phi_2(\cdot, u_2) \in W^{2,p} \cap W_0^{1,p}$ for $1 \le p < \infty$, u_1 and u_2 belong to the domain $D(A)$ of A and

$$u_1 - \lambda\Delta\phi_1(\cdot, u_1) - \lambda\psi_1(\cdot, u_2, u_1) = w_1,$$
$$u_2 - \lambda\Delta\phi_2(\cdot, u_2) - \lambda\psi_2(\cdot, u_1, u_2) = w_2.$$

Since $\|u_{1,n}\|_{C(\overline{\Omega})} \le \Theta_1(K)$ and $\|u_{2,n}\|_{C(\overline{\Omega})} \le \Theta_2(K)$, part (i) holds. □

We note that, for sufficiently small $\lambda \in (0, 1/\omega(K))$, uniqueness holds by the dissipativity of $A_K - \omega(K)$.

Since we cannot be sure that the solution (u_1, u_2) above is again contained in the domain D_K, we cannot yet show that the range condition (RC) holds, and thereby we are unable to apply the semigroup generation theorem. Initially, we assume the independence of the ϕ_i's on x to obtain a slightly better estimate on the L^∞-norm of the solution.

4.7 Theorem. *Assume that $\overline{M}_1(K) \equiv \overline{M}_1$ and $\overline{M}_2(K) \equiv \overline{M}_2$, for all $K > 0$, and that $\phi_i(x,\eta) \equiv \phi_i(\eta)$ for $i = 1,2$. Then for any $K > 0$ and $\lambda > 0$ such that*

$$0 < \lambda < \gamma(K) \equiv 1/2 \max\left\{M_1(2K), M_2(2K), \sqrt{N_1(2K)N_2(2K)}\right\}$$

and any $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in D_K$, there exists a $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in D(A)$ such that $(I - \lambda A)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ and the following hold:

(i) $u_1, u_2 \in C(\overline{\Omega})$, $\|u_i\|_p \leq (1 - \lambda\overline{M}_i)^{-1} \|w_i\|_p$ *for* $1 \leq p \leq \infty$, $i = 1,2$.

(ii) $\phi_i(\cdot, u_i(\cdot)) \in W^{2,p} \cap W_0^{1,p}$ *for* $1 \leq p < \infty$, $i = 1,2$.

(iii) $\Lambda\phi_1(\cdot,u_1), \Lambda\phi_2(\cdot,u_2), \psi_1(\cdot,u_2,u_1), \psi_2(\cdot,u_1,u_2) \in L^\infty(\Omega)$.

Proof. Following the proof of Theorem 4.6, we use Theorem 3.9 at the appropriate point, namely equation (4.6), to obtain the estimate

$$\|u_{i,n}\|_p \leq (1 - \lambda\overline{M}_i)^{-1} \|w_i\|_p \quad \text{for } i = 1,2.$$

Leaving the rest of the proof almost identical yields the result. □

It can now be seen that the following extra assumption guarantees that (u_1, u_2) is again inside D_K, allowing Theorem 1.3 to be applied.

4.8 Corollary. *If $\psi_1(x,\eta_2,\eta_1)/\eta_1 \leq 0$ and $\psi_2(x,\eta_1,\eta_2)/\eta_2 \leq 0$ for all x, η_1, η_2 and $\phi_1(x,\eta) \equiv \phi_1(\eta)$, $\phi_2(x,\eta) \equiv \phi_2(\eta)$, then for all $K > 0$ and $\lambda > 0$ such that*

$$0 < \lambda < \gamma(K) \equiv \min\left\{\frac{1}{2M_1(2K)}, \frac{1}{2M_2(2K)}, \frac{1}{2\sqrt{N_1(2K)N_2(2K)}}\right\},$$

we have $R(I - \lambda A_K) \supset D(A_K)$. Thereby it follows that A_K generates a semigroup on $\overline{D(A_K)}^{L^1} = D_K$, providing integral solutions of type $\omega(K)$ to the problem (PMS).

Proof. Given $(w_1, w_2) \in D_K$, Theorem 4.7 implies that there exists some $(u_1, u_2) \in L^\infty \times L^\infty$ such that

$$(I - \lambda A)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Moreover, since $\overline{M}_1, \overline{M}_2 \leq 0$, Theorem 4.7 also implies that

$$\|u_i\|_{C(\overline{\Omega})} \leq (1 - \lambda\overline{M}_i)^{-1} \|w_i\|_\infty \leq \|w_i\|_\infty \quad \text{for } i = 1,2.$$

Hence $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in D_K$ and so $R(I - \lambda A_K) \supset D(A_K)$. □

4.9 Remark. If we let $\{S_K(t)\}$ be the semigroup generated by A_K, then $A_K \subset A_{K'}$ for $K \leq K'$, and so

$$S_K(t) \subset S_{K'}(t),$$

in the sense that the graph of $S_K(t)$ is a subset of the graph of $S_{K'}(t)$ for all $t \geq 0$. Therefore we may define a semigroup $\{S(t)\}$ on $L^\infty \times L^\infty$ by

$$S(t)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = S_K(t)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \text{ when } \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in D(S_K) \equiv D_K.$$

4.10 Theorem. *The semigroup generated in Corollary 4.8 has the following properties:*

(i) For all $t \geq 0$, $S(t)$ maps $L^\infty \times L^\infty$ into $L^\infty \times L^\infty$.

(ii) $S(0) = I$ and $S(t+s) = S(t)S(s)$.

(iii)
$$\left\| S(t)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - S(t)\begin{pmatrix} \widehat{w}_1 \\ \widehat{w}_2 \end{pmatrix} \right\|_X \leq e^{\omega(K)t} \left\| \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - \begin{pmatrix} \widehat{w}_1 \\ \widehat{w}_2 \end{pmatrix} \right\|_X$$
for $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \begin{pmatrix} \widehat{w}_1 \\ \widehat{w}_2 \end{pmatrix} \in D_K$, and $t > 0$.

(iv) For $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in D_K$, we have

$$\left\| S(t)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - S(s)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\|_X \leq e^{2\omega_0(K)(t+s)} \left\| A_K \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\|_X .|t-s| \text{ for } t, s \geq 0,$$

where $\omega_0(K) = \max\{0, \omega(K)\}$

(v) For all $\begin{pmatrix} u_1^0 \\ u_2^0 \end{pmatrix} \in L^\infty \times L^\infty$, let $\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \equiv S(t)\begin{pmatrix} u_1^0 \\ u_2^0 \end{pmatrix}$. Then

$$\|u_1(t)\|_p \leq e^{\overline{M}_1 t} \|u_1^0\|_p \quad , \quad \|u_2(t)\|_p \leq e^{\overline{M}_2 t} \|u_2^0\|_p \quad \text{for } 1 \leq p \leq \infty.$$

Note that we assume here that $\overline{M}_1$, and $\overline{M}_2 \geq 0$ throughout.

Proof. We show only (v), since all other results follow immediately from Theorem 1.3. Theorem 4.7 implies that when $(I - \lambda A)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, then

$$\|u_1\|_p \leq (1 - \lambda \overline{M}_1)^{-1} \|w_1\|_p \quad , \quad \|u_2\|_p \leq (1 - \lambda \overline{M}_2)^{-1} \|w_2\|_p$$

Let $\begin{pmatrix} u_1^0 \\ u_2^0 \end{pmatrix} \in L^\infty \times L^\infty$ and set $\begin{pmatrix} u_{1,k} \\ v_{2,k} \end{pmatrix} = (I - \lambda A)^{-k} \begin{pmatrix} u_1^0 \\ u_2^0 \end{pmatrix}$. Then $(I - \lambda A) \begin{pmatrix} u_{1,k} \\ u_{2,k} \end{pmatrix} = \begin{pmatrix} u_{1,k-1} \\ u_{2,k-1} \end{pmatrix}$, and so

$$\|u_{i,k}\|_p \leq (1 - \lambda \overline{M}_i)^{-1} \|u_{i,k-1}\|_p \leq (1 - \lambda \overline{M}_i)^{-k} \left\|u_i^0\right\|_p \text{ for } i = 1, 2.$$

Letting $k \to \infty$ and $\lambda \to 0$ in the appropriate way, we obtain the desired result. □

4.11 Assumptions. We impose the conditions of 4.1, replacing (iii) by

$$\text{(iii)}' \quad \sup_{\substack{x \in \overline{\Omega}, \eta_2 \in \mathbb{R} \\ \eta_1 \neq \widehat{\eta}_1}} \frac{\psi_1(x, \eta_2, \eta_1) - \psi_1(x, \eta_2, \widehat{\eta}_1)}{\eta_1 - \widehat{\eta}_1} = M_1 < \infty$$

$$\sup_{\substack{x \in \overline{\Omega}, \eta_1 \in \mathbb{R} \\ \eta_2 \neq \widehat{\eta}_2}} \left| \frac{\psi_1(x, \eta_2, \eta_1) - \psi_1(x, \widehat{\eta}_2, \eta_1)}{\eta_2 - \widehat{\eta}_2} \right| = N_1 < \infty$$

$$\sup_{\substack{x \in \overline{\Omega}, \eta_1 \in \mathbb{R} \\ \eta_2 \neq \widehat{\eta}_2}} \frac{\psi_2(x, \eta_1, \eta_2) - \psi_2(x, \eta_1, \widehat{\eta}_2)}{\eta_2 - \widehat{\eta}_2} = M_2 < \infty$$

$$\sup_{\substack{x \in \overline{\Omega}, \eta_2 \in \mathbb{R} \\ \eta_1 \neq \widehat{\eta}_1}} \left| \frac{\psi_2(x, \eta_1, \eta_2) - \psi_2(x, \widehat{\eta}_1, \eta_2)}{\eta_1 - \widehat{\eta}_1} \right| = N_2 < \infty$$

We define $\overline{M}_1(K) \equiv \overline{M}_1$ and $\overline{M}_2 \equiv \overline{M}_2$ as was done previously. Now we can consider the whole space $L^\infty \times L^\infty$ and show that the range condition (RC) holds.

4.12 Theorem. *Suppose that the assumptions of 4.11 above hold. Then for all $\lambda \in \mathbb{R}$ such that*

$$0 < \lambda < \gamma \equiv 1/2 \max\left\{M_1, M_2, \sqrt{N_1 N_2}\right\},$$

where, again if $M_i \leq 0$ set $1/M_i = \infty$, we have

$$R(I - \lambda A_\infty) \supset D(A_\infty),$$

where A_∞ denotes the restriction of A to $L^\infty \times L^\infty$.

Proof. For any given $w_1, w_2 \in L^\infty$, the result follows immediately from the proof of Theorem 4.6. □

4.13 Corollary. *The operator A_∞ generates a semigroup $\{S(t)\}$ in X which provides unique integral solutions of type ω (as before $\omega = \max\{M_1 + N_2, M_2 + N_1\}$) to* (PMS) *and satisfies*

(i) $S(t) : X \to X$ for all $T \geq 0$.

(ii) $S(0) = I$ *and* $S(t+s) = S(t)S(s)$.

(iii)
$$\left\|S(t)\begin{pmatrix}w_1\\w_2\end{pmatrix} - S(t)\begin{pmatrix}\widehat{w}_1\\\widehat{w}_2\end{pmatrix}\right\|_X \leq e^{(\omega t)}\left\|\begin{pmatrix}w_1\\w_2\end{pmatrix} - \begin{pmatrix}\widehat{w}_1\\\widehat{w}_2\end{pmatrix}\right\|_X,$$
for $\begin{pmatrix}w_1\\w_2\end{pmatrix}, \begin{pmatrix}\widehat{w}_1\\\widehat{w}_2\end{pmatrix} \in D_K, t > 0$

(iv) *For* $\begin{pmatrix}w_1\\w_2\end{pmatrix} \in D(A_\infty)$, $t, s \geq 0$
$$\left\|S(t)\begin{pmatrix}w_1\\w_2\end{pmatrix} - S(s)\begin{pmatrix}w_1\\w_2\end{pmatrix}\right\|_X \leq e^{(2\omega_0(t+s))}\left\|A_\infty\begin{pmatrix}w_1\\w_2\end{pmatrix}\right\|_X .|t-s|,$$
where $\omega_0 = \max\{0, \omega\}$.

Note that (v) of Theorem 4.10 cannot be included.

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