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SCHOLZ ADMISSIBLE MODULI OF FINITE GALOIS EXTENSIONS OF ALGEBRAIC NUMBER FIELDS

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ABSTRACT. Let K be a finite Galois extension over an algebraic number field k with Galois group G. We call a modulus $\mathfrak{M}$ of K Scholz admissible when the Schur multiplier of G is isomorphic to the number knot of K/k modulo $\mathfrak{M}$. This paper develops a systematic treatment for Scholz admissibility. We first reduce the problem to the local case, in particular, to the strongly ramified case, and study this case in detail. A main object of local Scholz admissibility is $H^{-1}(G, U_K^{(s)})$ in the strongly ramified case. In the case where K/k is totally strongly ramified of prime power degree p^n, we prove that the natural homomorphism : $H^{-1}(G, U_K^{(r+s)}) \to H^{-1}(G, U_K^{(s)})$ is trivial for $s \geq 1$, where r denotes the last ramification number. This result describes a basic situation for vanishing of $H^{-1}(G, U_K^{(s)})$. Using this result for a Galois tower $K \supset L \supset k$ with a totally strongly ramified cyclic extension L/k we prove a relationship between Scholz admissible moduli of K/L and K/k. This gives a way to estimate for Scholz conductor of K/k from the ramification in K/k. As an application of this result we give an alternative proof of a result of Fröhlich.

1. INTRODUCTION

Let k be an algebraic number field of finite degree. Let K be a finite Galois extension over k with Galois group $G = Gal(K/k)$. Let $\mathfrak{M}$ be a Galois modulus of K/k, i.e., a finite product of primes of K which satisfies $\mathfrak{M}^\sigma = \mathfrak{M}$ for any $\sigma \in G$. It is known that the number knot of K/k modulo $\mathfrak{M}$ is an epimorphic image of Schur multiplier $H^{-3}(G, \mathbb{Z})$, i.e., there exists a natural epimorphism

$$H^{-3}(G, \mathbb{Z}) \to k \cap N_{K/k}(J_{K\mathfrak{M}})/N_{K/k}(K_{(\mathfrak{M})}) \tag{1.1}$$

(cf. Proposition 2.7). A. Scholz [9] developed the knot theory in relation to the Hasse norm principle, and established fundamental properties of this type without using cohomology. About forty years after, W. Jehne [6] gave a reformation and a generalization of Scholz's knot theory with new proofs using cohomology. Following Jehne, P. Heider [4] studied the knot theory modulo $\mathfrak{M}$. In his paper [4], Heider called $\mathfrak{M}$ a Scholz conductor when (1.1) is an isomorphism and studied some properties of Scholz conductors. In particular, he proved that a sufficiently large $\mathfrak{M}$ is a Scholz conductor of K/k. Before Heider's work, Shirai [8] determined such moduli of K/k in the case where K/k is tamely ramified. In the general case, however, it is not easy to determine such moduli explicitly.

In this paper we call $\mathfrak{M}$ Scholz admissible when (1.1) is an isomorphism with slightly modified terminology. The purpose of this paper is to develop a systematic treatment of Scholz admissibility.

Although Scholz admissibility is defined in the global case as above, the essential part is in the local case. So we first define Scholz admissibility in the local case in section 2.1. Next, in section 2.2, we give a reformulated definition in the global case and discuss the relationship between the local case and the global case. In particular we show that $\mathfrak{M}$ is global strong Scholz admissible if and only if $\mathfrak{P}$-part of $\mathfrak{M}$ is local Scholz admissible at $\mathfrak{P}$ for every prime $\mathfrak{P}$ of K (Proposition 2.9).

In section 3 we study the local case in detail. A main object of local Scholz admissibility is $H^{-1}(G, U_K^{(s)})$, in particular, in the case where K/k is strongly ramified. When K/k is totally strongly ramified of prime power degree p^n, we prove the natural homomorphism : $H^{-1}(G, U_K^{(r_n+s)}) \to H^{-1}(G, U_K^{(s)})$ is trivial for $s \geq 1$, where r_n denotes the last ramification number (Theorem 3.5). This result describes a basic situation for vanishing of $H^{-1}(G, U_K^{(s)})$. Using this result for a finite Galois tower $K \supset L \supset k$ with a totally strongly ramified cyclic extension L/k, we prove a relationship between Scholz admissible moduli of K/L and K/k (Theorem 3.12). This theorem gives a way of getting an estimate for Scholz conductor of K/k from the ramification in K/k. As an application of this result we give an alternative proof of [2, Theorem 3].

2. Definitions and Preliminaries

Throughout this paper an algebraic number field means a finite extension of the rational number field $\mathbb{Q}$.

In this section we give several definitions related to Scholz admissibility, and establish their fundamental properties.

2.1. **Local fields.** Let $\mathfrak{p}$ be a prime of an algebraic number field. Let k denote the completion of the field with respect to $\mathfrak{p}$. In this paper we assume that $\mathfrak{p}$ is finite since the results are rather trivial in the case where $\mathfrak{p}$ is infinite. In fact in the case where $\mathfrak{p}$ is infinite, although the notion of Scholz admissibility is defined similarly to the finite case, the Scholz admissibility coincides with Galois admissibility (cf. Proposition 2.2). Let K/k be a Galois extension of finite degree, and let $\mathfrak{P}$ be the prime of K over $\mathfrak{p}$. For non negative integers i and j we define functions $v_{K/k}$ and $u_{K/k}$ using the Hasse's function $\phi_{K/k} = \phi$ of $\mathfrak{P}$ with respect to K/k by

$$v_{K/k}(i) = v(i) = \phi(i-1) + 1, \tag{2.1}$$

$$u_{K/k}(j) = u(j) = \min\{i \in \mathbb{Z} \mid j \leq \phi(i)\}, \tag{2.2}$$

respectively. For the properties of the Hasse's function $\phi_{K/k}$ we refer to Iyanaga [5] or Neukirch [7]. From the definitions we see

$$\phi(u(j) - 1) < j \leq \phi(u(j)), \tag{2.3}$$

$$v(u(j)) \leq j < v(u(j) + 1), \tag{2.4}$$

and

(2.5) $$u(v(i)) = u(\phi(i)) = i.$$

Moreover by the definition we obtain transitivity of u and v easily as follows.

Lemma 2.1. *Let L be a Galois subextension of K/k. Then*
(1) $v_{K/k} = v_{K/L} \circ v_{L/k}$
(2) $u_{K/k} = u_{L/k} \circ u_{K/L}$.

Let $\mathfrak{m} = \mathfrak{p}^i$ be a modulus of k , i.e., a finite product of $\mathfrak{p}$. Then we define the lifting modulus $\mathfrak{v}_{K/k}(\mathfrak{m})$ of $\mathfrak{m}$ from k to K by

(2.6) $$\mathfrak{v}_{K/k}(\mathfrak{m}) = \mathfrak{v}(\mathfrak{m}) = \mathfrak{P}^{v(i)}.$$

Conversely, for a modulus $\mathfrak{M} = \mathfrak{P}^j$ of K we define the restricted modulus $\mathfrak{u}_{K/k}(\mathfrak{M})$ of $\mathfrak{M}$ from K to k by

(2.7) $$\mathfrak{u}_{K/k}(\mathfrak{M}) = \mathfrak{u}(\mathfrak{M}) = \mathfrak{p}^{u(j)}.$$

Since v and u are transitive, the liftings and the restrictions are also transitive.

For an integer $i \geq -1$, let $V_{K/k}(i)$ denote the i-th ramification group with respect to K/k, i.e.,

(2.8) $$V_{K/k}(i) = \{\sigma \in \mathrm{Gal}(K/k) \mid \sigma(\alpha) \equiv \alpha \pmod{\mathfrak{P}^{i+1}} \text{ for all } \alpha \in \mathfrak{O}_K\},$$

where $\mathfrak{O}_K$ denotes the valuation ring of K. Let $U_k = U_k^{(0)}$ denote the group of units in k, and for a positive integer i, put

(2.9) $$U_k^{(i)} = \{x \in U_k \mid x \equiv 1 \pmod{\mathfrak{p}^i}\}.$$

For a non negative integer j, $U_K^{(j)}$ is also defined similarly with respect to $\mathfrak{P}$.

Let L be a Galois subextension of K/k, and put $H = Gal(K/L)$. Then the Herbrand's theorem states

(2.10) $$V_{L/k}(u_{K/L}(j)) = V_{K/k}(j)H/H.$$

Moreover since $u(j)$ is transitive, we know from class field theory

(2.11) $$N_{K/k}(U_K^{(j)}) \subset U_k^{(u(j))},$$

where $u(j) = u_{K/k}(j)$ (e.g. Iyanaga [5, p.340]).

Using the ramification groups we define the Galois admissibility of a modulus of K/k as follows.

Definition 2.2. *Let $\mathfrak{M} = \mathfrak{P}^j$ be a modulus of K/k. If $V_{K/k}(j) = \{1\}$, then we say that $\mathfrak{M}$ is Galois admissible over k.*

Moreover a modulus $\mathfrak{m}$ of k is said to be Galois admissible with respect to K/k if $\mathfrak{v}_{K/k}(\mathfrak{m})$ is Galois admissible.

The Galois conductor $\mathfrak{f}_{K/k}$ of K/k is defined to be the least Galois admissible modulus of k with respect to K/k.

The definition of Galois conductor is due to Shirai [8].

In the case where K/k is an abelian extension, by the conductor theorem (e.g. Iyanaga [5, p.348]) we know that the ordinary conductor $\mathfrak{f}_{K/k}$ of K/k is of form $\mathfrak{p}^{r+1}$ where r is the integer such that $\phi(r)(= v(r+1)-1)$ is the last ramification number of K/k. Therefore in this case the Galois conductor of K/k coincides with the ordinary conductor of K/k.

Moreover from the definition of Galois conductors we immediately see the following.

(2.12) K/k is unramified if and only if $\mathfrak{f}_{K/k} = 1$.

(2.13) K/k is tamely ramified if and only if $\mathfrak{f}_{K/k} = \mathfrak{p}$.

For the relationship between the Galois conductor and the norm group of $U_K^{(j)}$, we obtain the following.

Lemma 2.3. *Let $\mathfrak{P}^j$ be a modulus of K, and assume $\mathfrak{P}^j$ is Galois admissible over k. Then*

$$N_{K/k}(U_K^{(j)}) = U_k^{(u(j))},$$

where $u(j)$ denotes $u_{K/k}(j)$.

Proof. If K/k is abelian, then the assertion is well-known (e.g. Iyanaga [5, p. 340]).

Let L/k be a Galois subextension of K/k. Then $\mathfrak{P}^j$ is clearly Galois admissible over L. Moreover we know from Herbrand's theorem (2.7) that $\mathfrak{P}'^{u'(j)}$ is also Galois admissible over k, where $\mathfrak{P}'$ denotes the prime of L and $u'(j) = u_{K/L}(j)$. Hence the assertion follows from the transitivity of $u_{K/k}$ since K/k is a solvable extension. □

The above lemma generalizes Shirai [8, Lemma 10], which asserts the result in the case where $j = v_{K/k}(i)$.

Let $G = \mathrm{Gal}(K/k)$. The main object in this paper is the Tate cohomology group $H^{-1}(G, U_K^{(j)})$. Using this the (local) Scholz admissibility is defined as follows.

Definition 2.4. *Let $\mathfrak{M} = \mathfrak{P}^j$ be a Galois admissible modulus of K/k. If the natural homomorphism*

$$1^{\#} : H^{-1}(G, U_K^{(j)}) \to H^{-1}(G, K^*)$$

is trivial, then we say that $\mathfrak{M}$ is (local) Scholz admissible over k. Moreover a modulus $\mathfrak{m}$ of k is said to be (local) Scholz admissible with respect to K/k if $\mathfrak{v}_{K/k}(\mathfrak{m})$ is (local) Scholz admissible.

The (local) Scholz conductor of K/k is defined to be the least (local) Scholz admissible modulus of k with respect to K/k.

If K/k is a cyclic extension, then $H^{-1}(G, K^*) = 1$ by Hilbert's theorem 90. Moreover, if K/k is unramified, then $H^{-1}(G, U_K^{(j)}) = 1$ for $j \geq 0$ (c.f. Lemma 3.2 below). Thus by the definition of Scholz admissibility we have the following.

(2.14) If K/k is cyclic, then a Galois admissible modulus is Scholz admissible.

(2.15) If K/k is unramified, then $\mathfrak{P}^j$ is Scholz admissible for $j \geq 0$.

In particular we have the following.

Proposition 2.5. *In the following cases* (1) *and* (2) *the Scholz conductor of* K/k *coincides with the Galois conductor of* K/k.

(1) K/k *is a cyclic extension.*
(2) K/k *is an unramified extension.*

2.2. **Global fields.** In this subsection we deal with global fields. Let K be a finite Galois extension over an algebraic number field k.

Let $\mathfrak{M} = \prod_{\mathfrak{P}} \mathfrak{P}^{e(\mathfrak{P})}$ be a Galois modulus of K/k, i.e., a finite product of primes $\mathfrak{P}$ of K which satisfies $\mathfrak{M}^\sigma = \mathfrak{M}$ for any $\sigma \in Gal(K/k)$. Let $\mathfrak{M}_{\mathfrak{P}} = \mathfrak{P}^{e(\mathfrak{P})}$ denote the $\mathfrak{P}$-component of $\mathfrak{M}$. For a prime $\mathfrak{P}$ of K and $\mathfrak{p} = \mathfrak{P} \cap k$, let $K_{\mathfrak{P}}$ and $k_{\mathfrak{p}}$ denote the completion of K by $\mathfrak{P}$ and the completion of k by $\mathfrak{p}$, respectively. Let $\mathfrak{u}_{\mathfrak{P}}(\mathfrak{M}_{\mathfrak{P}})$ denote the restricted modulus of $\mathfrak{M}_{\mathfrak{P}}$ from $K_{\mathfrak{P}}$ to $k_{\mathfrak{p}}$ defined locally in (2.6). Then the global restricted modulus of $\mathfrak{M}$ from K to k is defined by

$$\mathfrak{u}_{K/k}(\mathfrak{M}) = \mathfrak{u}(\mathfrak{M}) = \prod_{\mathfrak{P}|\mathfrak{M}} \mathfrak{u}_{\mathfrak{P}}(\mathfrak{M}_{\mathfrak{P}}),$$

where the product is taken over non conjugate prime divisors $\mathfrak{P}$ of $\mathfrak{M}$.

Similarly, let $\mathfrak{m} = \prod_{\mathfrak{p}} \mathfrak{p}^{e(\mathfrak{p})}$ be a modulus of k, and let $\mathfrak{m}_{\mathfrak{p}} = \mathfrak{p}^{e(\mathfrak{p})}$ denote the $\mathfrak{p}$-component of $\mathfrak{m}$. Let $\mathfrak{P}$ be a prime divisor of $\mathfrak{p}$ in K and let $\mathfrak{v}_{\mathfrak{P}}(\mathfrak{m}_{\mathfrak{p}})$ denoted the lifting modulus of $\mathfrak{m}_{\mathfrak{p}}$ from $k_{\mathfrak{p}}$ to $K_{\mathfrak{P}}$ defined locally in (2.5). Then the global lifting modulus of $\mathfrak{m}$ from k to K is defined by

$$\mathfrak{v}_{K/k}(\mathfrak{m}) = \mathfrak{v}(\mathfrak{m}) = \prod_{\mathfrak{P}|\mathfrak{m}} \mathfrak{v}_{\mathfrak{P}}(\mathfrak{m}_{\mathfrak{p}}),$$

where the product is taken over all of prime divisors $\mathfrak{P}$ of $\mathfrak{m}$.

The Galois admissibility in Global fields is defined by using local Galois admissibility given in the previous subsection.

Definition 2.6. *Let* $\mathfrak{M}$ *be a Galois modulus of* K/k. *If for every prime* $\mathfrak{P}$ *of* K *the* $\mathfrak{P}$*-component of* $\mathfrak{M}$ *is Galois admissible in* $K_{\mathfrak{P}}/k_{\mathfrak{p}}$, *then we say that* $\mathfrak{M}$ *is Galois admissible over* k. *Moreover a modulus* $\mathfrak{m}$ *of* k *is said to be Galois admissible with respect to* K/k *if* $\mathfrak{v}_{K/k}(\mathfrak{m})$ *is Galois admissible.*

The Galois conductor $\mathfrak{f}_{K/k}$ *of* K/k *is defined to be the least Galois admissible modulus of* k *with respect to* K/k.

Let J_K denote the idele group of K. For a Galois modulus $\mathfrak{M} = \prod_{\mathfrak{P}} \mathfrak{P}^{e(\mathfrak{P})}$ of K, we put

$$J_{K\mathfrak{M}} = \prod_{e(\mathfrak{P})>0} U_{K_{\mathfrak{P}}}^{(e(\mathfrak{P}))} \cdot \prod_{e(\mathfrak{P})=0}' K_{\mathfrak{P}}^*,$$

where $\prod'$ denotes the restricted product. Let $K_{(\mathfrak{M})}$ denote the ray number group of K modulo $\mathfrak{M}$, i.e.,

$$K_{(\mathfrak{M})} = \{x \in K^* \mid x \equiv 1 \pmod{\mathfrak{M}}\}.$$

Let $C_K = J_K/K^*$ denote the idele class group. Since $J_K \subset K^* J_{K\mathfrak{M}}$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_{(\mathfrak{M})} & \xrightarrow{i} & J_{K\mathfrak{M}} & \xrightarrow{j} & C_K & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & K^* & \xrightarrow{i} & J_K & \xrightarrow{j} & C_K & \longrightarrow & 1. \end{array}$$

Then taking Tate cohomology groups we have

$$\begin{array}{ccccccccc} \longrightarrow H^{-1}(G, J_{K\mathfrak{M}}) & \xrightarrow{j^\#} & H^{-1}(G, C_K) & \xrightarrow{\delta^\#} & H^0(G, K_{(\mathfrak{M})}) & \xrightarrow{i^\#} & H^0(G, J_{K\mathfrak{M}}) & \longrightarrow \\ \downarrow & & \| & & \downarrow & & \downarrow & \\ \longrightarrow H^{-1}(G, J_K) & \xrightarrow{j^\#} & H^{-1}(G, C_K) & \xrightarrow{\delta^\#} & H^0(G, K^*) & \xrightarrow{i^\#} & H^0(G, J_K) & \longrightarrow, \end{array}$$

where $G = Gal(K/k)$. Using this sequence we obtain an epimorphism from $H^{-3}(G, \mathbb{Z}) \cong H^{-1}(G, C_K)$ to the number knot modulo $\mathfrak{M}$. Indeed we have the following.

This result is essentially contained in [4] and proved by a standard manner using cohomology, so we omit the proof.

Proposition 2.7. *Let K/k be a Galois extension. Let $\mathfrak{M}$ be a Galois modulus of K/k. Then we have the following.*

$$(k \cap N_{K/k}(J_{K\mathfrak{M}}))/N_{K/k}(K_{(\mathfrak{M})}) \cong H^{-1}(G, C_K)/j^\#(H^{-1}(G, J_{K\mathfrak{M}}))$$

Now global Scholz admissibility is defined as follows.

Definition 2.8. *Let $\mathfrak{M}$ be a Galois admissible modulus of K/k. If the natural homomorphism*

$$j^\# : H^{-1}(G, J_{K\mathfrak{M}}) \to H^{-1}(G, C_K)$$

is trivial, then we say that $\mathfrak{M}$ is (global) Scholz admissible over k.

Moreover, a global Galois admissible modulus $\mathfrak{M}$ is called (global) strong Scholz admissible over k if the natural homomorphism

$$H^{-1}(G, J_{K\mathfrak{M}}) \to H^{-1}(G, J_K)$$

is trivial.

Clearly strong Scholz admissibility implies Scholz admissibility. Moreover if $\mathfrak{M}$ is Scholz admissible over k, then the above proposition implies

$$(k \cap N_{K/k}(J_{K\mathfrak{M}}))/N_{K/k}(K_{(\mathfrak{M})}) \cong H^{-1}(G, C_K) \cong H^{-3}(G, \mathbb{Z}). \tag{2.16}$$

P. Heider [4] called $\mathfrak{M}$ a Scholz conductor when (2.16) is valid. We adopt the above definition for consistency with the local case with slightly modified terminology.

For the relationship between the global case and the local case we have the following.

Proposition 2.9. *Let K/k be a Galois extension. Let $\mathfrak{M} = \prod_{\mathfrak{P}} \mathfrak{P}^{e(\mathfrak{P})}$ be a Galois modulus of K/k. Let $\mathfrak{M}_{\mathfrak{P}}$ be the $\mathfrak{P}$-component of $\mathfrak{M}$. For a prime $\mathfrak{P}$ not dividing $\mathfrak{M}$ we set $\mathfrak{M}_{\mathfrak{P}} = \mathfrak{P}^0 = 1$.*

Then $\mathfrak{M}$ is global strong Scholz admissible if and only if $\mathfrak{M}_{\mathfrak{P}}$ is local Scholz admissible in $K_{\mathfrak{P}}/k_{\mathfrak{p}}$ for ever prime $\mathfrak{P}$ of K.

Proof. Let $\mathfrak{P}$ be a prime of K and $\mathfrak{p} = k \cap \mathfrak{P}$. Let $G(\mathfrak{P})$ denote the decomposition group of $\mathfrak{P}$ in K/k. Since a local Scholz admissible modulus is divided by the local Galois conductor, the assumption of our proposition implies that the global Galois conductor $\mathfrak{f}_{K/k}$ divides $\mathfrak{M}$. Therefore if $\mathfrak{P}$ does not divide $\mathfrak{M}$, then $\mathfrak{P}$ is unramified in $K_{\mathfrak{P}}/k_{\mathfrak{p}}$, and $H^{-1}(G(\mathfrak{P}), K^*_{\mathfrak{P}}) = 1$. Hence using semilocal theory we have

$$H^{-1}(G, J_{K\mathfrak{M}}) \simeq \sum_{\mathfrak{P}|\mathfrak{M}} H^{-1}(G(\mathfrak{P}), U^{(e(\mathfrak{P}))}_{K_{\mathfrak{P}}}),$$

where the sum is taken over non conjugate primes $\mathfrak{P}$ of K over k with $\mathfrak{P}|\mathfrak{M}$. Now let $\mathfrak{P}$ be a prime of K with $\mathfrak{P}|\mathfrak{M}$. Then we obtain the following commutative diagram:

$$\begin{array}{ccccccc} & & H^{-1}(G(\mathfrak{P}), U^{(e(\mathfrak{P}))}_{K_{\mathfrak{P}}}) & \xrightarrow{1^{\#}} & H^{-1}(G(\mathfrak{P}), K^*_{\mathfrak{P}}) & & \\ & & \downarrow & & \downarrow & & \\ H^{-1}(G, K^*_{(\mathfrak{M})}) & \longrightarrow & H^{-1}(G, J_{K\mathfrak{M}}) & \longrightarrow & H^{-1}(G, J_K) & \longrightarrow & H^{-1}(G, C_K) \end{array}$$

Since $\mathfrak{M}_{\mathfrak{P}} = \mathfrak{P}^{e(\mathfrak{P})}$ is Scholz admissible by assumption, it follows

$$1^{\#}(H^{-1}(G(\mathfrak{P}), U^{(e(\mathfrak{P}))}_{K_{\mathfrak{P}}})) = 1.$$

Hence $H^{-1}(G, J_{K\mathfrak{M}}) \to H^{-1}(G, J_K)$ is trivial. Thus $\mathfrak{M}$ is strong Scholz admissible.

Conversely assume $\mathfrak{M}$ is strong Scholz admissible. Since $\mathfrak{M}$ contains the all ramified primes in K/k, we know by semilocal theory

$$H^{-1}(G, J_{K\mathfrak{M}}) \simeq \sum_{\mathfrak{P}|\mathfrak{M}} H^{-1}(G(\mathfrak{P}), U^{(e(\mathfrak{P}))}_{K_{\mathfrak{P}}}),$$

and

$$H^{-1}(G, J_K) \simeq \sum_{\mathfrak{P}|\mathfrak{M}} H^{-1}(G(\mathfrak{P}), K^*_{\mathfrak{P}}),$$

where the sum is taken over non conjugate primes $\mathfrak{P}$ of K over k with $\mathfrak{P}|\mathfrak{M}$. Therefore from the triviality $H^{-1}(G, J_{K\mathfrak{M}}) \to H^{-1}(G, J_K)$ we obtain for any prime $\mathfrak{P}|\mathfrak{M}$

$$H^{-1}(G(\mathfrak{P}), U^{(e(\mathfrak{P}))}_{K_{\mathfrak{P}}}) \to H^{-1}(G(\mathfrak{P}), K^*_{\mathfrak{P}})$$

is trivial. Moreover since in the case where $\mathfrak{P}$ does not divide $\mathfrak{M}$ it holds $H^{-1}(G(\mathfrak{P}), K_{\mathfrak{P}}^*) = 1$. This completes the proof. □

From this proposition we know that to obtain a Scholz admissible modulus it suffices to obtain local Scholz admissible moduli.

3. Local Results

In this section we deal with local case. Let $\mathfrak{p}$ be a finite prime of an algebraic number field, and let k denote the completion of the field with respect to $\mathfrak{p}$. Let K/k be a Galois extension of finite degree, and let L/k be a Galois subextension of K. Let $\mathfrak{P}$ and $\mathfrak{P}'$ denote the primes of K and L over $\mathfrak{p}$, respectively. Let $G = Gal(K/k)$ and $H = Gal(K/L)$.

We first state a lemma which is a fundamental tool in this section.

Lemma 3.1. *Let $G = Gal(K/k)$ and $H = Gal(K/L)$. Let $\mathfrak{P}^j$ be a modulus of K. Assume $\mathfrak{P}^j$ is Galois admissible over k. Then*

$$H^{-1}(H, U_K^{(j)}) \xrightarrow{Cor} H^{-1}(G, U_K^{(j)}) \xrightarrow{N_{K/L}} H^{-1}(G/H, U_L^{(u(j))}) \to 1,$$

is exact, where $u(j) = u_{K/L}(j)$ defined by (2.2).

Proof. Since $\mathfrak{P}^j$ is Galois admissible in K/k, so is in K/L. Hence by lemma 2.3 we have $N_{K/L}(U_K^{(j)}) = U_L^{(u(j))}$. Then the assertion follows from [3, Proposition 6] or a direct computation of the above sequence. □

3.1. Tamely ramified case.

If K/k is at most tamely ramified, i.e., $V_{K/k}(1) = \{1\}$, then $H^{-1}(G, U_K^{(j)})$ is easily determined as follows.

Lemma 3.2. (1) *If K/k is unramified, then $H^{-1}(G, U_K^{(j)}) = 1$, for $j \geq 0$.*
(2) *If K/k is at most tamely ramified, then $H^{-1}(G, U_K^{(j)}) = 1$, for $j \geq 1$.*

Proof. (1) is well known, so we omit the proof (e.g. see [7, III(1.4)]).

(2) This result is deduced from [1, lemma 5]. But here we give a simple proof using lemma 3.1.

By definition 2.1 $\mathfrak{P}^j$ is Galois admissible in K/k for $j \geq 1$. Let K_T denote the inertia field of K/k and let $H = Gal(K/K_T)$. Then the above lemma gives the following exact sequence.

$$H^{-1}(H, U_K^{(j)}) \longrightarrow H^{-1}(G, U_K^{(j)}) \longrightarrow H^{-1}(G/H, U_{K_T}^{(u(j))}) \to 1$$

Since K_T/k is unramified, it follows $H^{-1}(G/H, U_{K_T}^{(u(j))}) = \{1\}$ by (1), so it remains to prove $H^{-1}(H, U_K^{(j)}) = \{1\}$. Since K/k is at most tamely ramified, K/K_T is a cyclic extension; therefore we can use the Herbrand quotient. Since we know $Q(U_K^{(j)}) = 1$

by class field theory, it suffices to prove $H^0(H, U_K^{(j)}) = K_T \cap U_K^{(j)}/N_{K/K_T}U_K^{(j)} = \{1\}$. Now by lemma 2.2 we know $N_{K/K_T}U_K^{(j)} = U_{K_T}^{(u(j))}$, where $u(j) = u_{K/K_T}(j)$. Moreover in this case the Hasse's function ϕ_{K/K_T} is given by $\phi(i) = ni$ for $i \geq 0$, where $n = [K : K_T]$. Hence for $a \in K_T$ it holds that $a \equiv 1 \mod \mathfrak{P}^j$ if and only if $a \equiv 1 \mod \mathfrak{P}'^{u(j)}$, where $\mathfrak{P}' = \mathfrak{P} \cap K_T$. This proves $K_T \cap U_K^{(j)}/N_{K/k}U_K^{(j)} = U_{K_T}^{(u(j))}/U_{K_T}^{(u(j))} = \{1\}$. □

From Lemma 3.1 and 3.2 we have the following, which reduces the problem to the strongly ramified case.

Proposition 3.3. *Let K/k be a Galois extension. Let $\mathfrak{M} = \mathfrak{P}^j$ be a modulus of K/k. Then we have the following.*

(1) *Let K_T denote the inertia field of K/k. Assume $\mathfrak{M}$ is Scholz admissible over K_T. Then $\mathfrak{M}$ is Scholz admissible over k.*
(2) *Let K_V denote the ramification field of K/k, and assume $K_V \neq K$. Assume $\mathfrak{M}$ is Scholz admissible over K_V. Then $\mathfrak{M}$ is Scholz admissible over k.*

Proof. Since (1) is proved quite similarly to (2), and the statement of (2) is somewhat more involved than (1), we only give a proof of (2).

(2) Let $H_V = Gal(K/K_V)$. Since $K_V \neq K$ and $\mathfrak{M} = \mathfrak{P}^j$ is Galois admissible over K_V by assumption, we see $j > 0$. On the other hand, K_V/k is at most tamely ramified, so that $\mathfrak{M}$ is also Galois admissible over k. Hence by lemma 3.1 we have the following:

$$\begin{array}{ccccc} H^{-1}(H_V, U_K^{(j)}) & \longrightarrow & H^{-1}(G, U_K^{(j)}) & \longrightarrow & H^{-1}(G/H_V, U_{K_V}^{(u(j))}) \to 1 \\ {\scriptstyle 1^\#}\downarrow & & {\scriptstyle 1^\#}\downarrow & & \\ H^{-1}(H_V, K^*) & \xrightarrow{Cor} & H^{-1}(G, K^*), & & \end{array}$$

where $u(j) = u_{K/K_V}(j)$. Since K_V/k is at most tamely ramified and $j > 0$, we see $H^{-1}(G/H_V, U_{K_V}^{(u(j))}) = \{1\}$ by lemma 3.2 (2). Hence we have

$$Cor(1^\#(H^{-1}(H_V, U_K^{(j)})) = 1^\#(H^{-1}(G, U_K^{(j)})).$$

Since $\mathfrak{M}$ is Scholz admissible over K_V, it follows $1^\#(H^{-1}(H_V, U_K^{(j)})) = \{1\}$, so $1^\#(H^{-1}(G, U_K^{(j)})) = \{1\}$. This proves (2). □

In particular we have the following, which essentially gives an alternative proof of [8, Theorem 31].

Corollary 3.4.

(1) *If K/k is tamely ramified, then the Scholz conductor of K/k is $\mathfrak{p}$.*
(2) *If the first ramification group is cyclic, then the Galois conductor is the Scholz conductor.*

Proof. Let $\mathfrak{P}^j$ be a Galois admissible modulus of K/k.

(1) If K/k is tamely ramified, then K/K_T is a cyclic extension. Hence $\mathfrak{P}^j$ is Scholz admissible over K_T by (2.12). Thus the assertion follows from Proposition 3.3 (1).

(2) is obtained from Proposition 3.3 (2) similarly to (1). □

3.2. **Strongly ramified case.** In this subsection we deal with strongly ramified cases.

Let n be a positive integer, and p a prime number. Let K/k be a totally strongly ramified cyclic extension of degree p^n with $G = <\sigma> = Gal(K/k)$. Let $\mathfrak{P}$ and $\mathfrak{p}$ be the primes of K and k, respectively. Let Π and π be prime elements of K and k, respectively.

Let $r_1, r_2, \cdots$ be the ramification numbers of $\mathfrak{P}$ with respect to K/k. Then by the assumption we know $r_1 > 0$. Since $V_{K/k}(r_j)/V_{K/k}(r_j+1)$ is elementary and G is cyclic, $V_{K/k}(r_j)$ is generated by $\sigma^{p^{j-1}}$, i.e., $V_{K/k}(r_j) = <\sigma^{p^{j-1}}>$, in particular, the last ramification number is r_n. Thus

$$\begin{aligned} G = V_{K/k}(r_1) &\supsetneq V_{K/k}(r_1+1) = \cdots = V_{K/k}(r_2) \\ &\supsetneq V_{K/k}(r_2+1) = \cdots = V_{K/k}(r_3) \\ &\cdots \\ &\supsetneq V_{K/k}(r_{n-1}+1) = \cdots = V_{K/k}(r_n) \supsetneq V_{K/k}(r_n+1) = \{1\}. \end{aligned}$$

Here $\phi^{-1}_{K/k}(r_i)$ are integers, and, in fact we know $\phi^{-1}_{K/k}(r_i) = u_{K/k}(r_i)$ by class field theory. Moreover, since $\#(G/V(r_{i+1})) = p^i$, r_{i+1} are written as the form

(3.1) $$r_{i+1} = r_i + q_i \cdot p^i \quad \text{for} \quad i = 1, 2, \cdots, n-1$$

with some integers q_i. Further the conductor theorem implies that the conductor of K/k is $\mathfrak{p}^{u_{K/k}(r_n)+1}$, where r_n denotes the last ramification number.

Now we have the following theorem, which describes a basic situation for vanishing of $H^{-1}(G, U_K^{(s)})$.

Theorem 3.5. *Let the notations and the assumptions be as above. Let K/k be a totally strongly ramified cyclic extension of prime power degree p^n. Then for $s \geq 1$*

$$1^\# : H^{-1}(G, U_K^{(r_n+s)}) \to H^{-1}(G, U_K^{(s)})$$

is trivial.

For the proof, let us prepare several lemmas.

Let A be a complete set of representatives of $\mathfrak{O}_k/\mathfrak{p}$ with $0 \in A$, and let $\{\Pi'_i\}_{i=0}^\infty$ be a set of elements of $\mathfrak{O}_K$ with $\mathrm{ord}_\mathfrak{P}\Pi'_i = i$ and $\Pi'_0 = 1$. With these notations we have the following.

Lemma 3.6. *Let α be a non zero element of $\mathfrak{O}_K$, and let m be a positive integer. Then α is written of form :*

$$\alpha = a_0\Pi'_{i_0}(1 + a_{i_1}\Pi'_{i_1})(1 + a_{i_2}\Pi'_{i_2}) \cdots (1 + a_{i_{m'}}\Pi'_{i_{m'}})u,$$

where $u \in U_K^{(m)}$ and $a_i \in A$ with $a_i \neq 0$, $0 \leq m' < m$, and $0 < i_1 < i_2 < \cdots < i_{m'}$.

Proof. Since $\mathfrak{p}$ is totally ramified, it holds $\mathfrak{O}_K/\mathfrak{P} \simeq \mathfrak{O}_k/\mathfrak{p}$. So we can write

$$\alpha = a_0\Pi'_{i_0}u_1 \quad \text{with } a_0 \in A, u_1 \in U_K^{(1)}.$$

Next let $u_j \in U_K^{(j)}$ and write

$$u_j = 1 + a_j\Pi'_j + v \quad \text{with } a_j \in A, v \in \mathfrak{P}^{(j+1)}.$$

Put

$$u_{j+1} = \frac{u_j}{1 + a_j\Pi'_j} = 1 + \frac{v}{1 + a_j\Pi'_j},$$

then $u_{j+1} \in U_K^{(j+1)}$ and $u_j = (1 + a_j\Pi'_j)u_{j+1}$, which proves the lemma. □

For non-negative integer $i = 0, 1, 2, \cdots, n$ let $\sigma_i = \sigma^{p^i}$, and put

$$\Pi_i = \Pi^{1+\sigma+\sigma^2+\cdots+\sigma^{p^i-1}}.$$

Then $\Pi_0 = \Pi$, $\Pi_n = N_{K/k}\Pi \in k$ and

$$\text{ord}_{\mathfrak{P}}(\Pi_i) = p^i. \tag{3.2}$$

Further we have

$$\Pi_i^{\sigma-1} = \Pi^{(\sigma+\sigma^2+\cdots+\sigma^{p^i})-(1+\sigma+\sigma^2+\cdots+\sigma^{p^i-1})} = \Pi^{\sigma^{p^i}-1} = \Pi^{\sigma_i-1}.$$

Now we define $\{\Pi'_i\}_{i=0}^{\infty}$ using the above $\{\Pi_i\}_{i=0}^{n}$ as follows. For a positive integer i , let j denote the largest integer such that $p^j|i$, and put $t = i/p^j$. Let

$$\Pi'_i = \begin{cases} \Pi_j^t & \text{if } i > 0 \text{ and } 0 \leq j < n, \\ \Pi_n^{i/p^n} & \text{if } i > 0 \text{ and } n \leq j, \\ 1 & \text{if } i = 0. \end{cases} \tag{3.3}$$

Since this $\{\Pi'_i\}_{i=0}^{\infty}$ satisfies the assumption in Lemma 3.6 from (3.2), we have the following.

Corollary 3.7. *Let α be a non zero element of $\mathfrak{O}_K$, and let m be a positive integer. Then α is written of form :*

$$\alpha = a_0\Pi_{i_0}^{t_0}(1 + a_{i_1}\Pi_{i_1}^{t_1})(1 + a_{i_2}\Pi_{i_2}^{t_2}) \cdots (1 + a_{i_{m'}}\Pi_{i_{m'}}^{t_{m'}})u$$

for some integer $0 \leq m' < m$, where $u \in U_K^{(m)}$ and $a_i \neq 0 \in A$ with $0 < p^{i_1}t_1 < p^{i_1}t_{i_1} < \cdots < p^{i_{m'}}t_{i_{m'}}$.

Moreover, if $p|t_j$*, then* $i_j = n$ *and, in particular,* $\Pi_{i_j}^{t_j} \in k$.

Lemma 3.8. *Let* t *be a positive integer with* $p \not| t$*. Then for* $i = 0, 1, 2, \cdots, n-1$ *we have the following.*

(1) $\mathrm{ord}_{\mathfrak{P}}(\Pi_i^{t(\sigma-1)} - 1) = r_{i+1}$.

(2) *If* $a \in U_k$*, then* $\mathrm{ord}_{\mathfrak{P}}((1 + a\Pi_i^t)^{\sigma-1} - 1) = r_{i+1} + p^i t$.

Moreover, in the case $i = n$*, without the assumption* $p \not| t$*, it holds that* $\Pi_i^{t(\sigma-1)} = (1 + a\Pi_i^t)^{\sigma-1} = 1$.

Proof. (1) Since $\sigma_i = \sigma^{p^i}$ generates $V(r_{i+1})$, we can write

$$\Pi_i^{\sigma-1} = \Pi^{\sigma_i - 1} = 1 + \beta\Pi^{r_{i+1}} \quad \text{for } \beta \in U_k.$$

Therefore we see

$$\Pi_i^{t(\sigma-1)} = (1 + \beta\Pi^{r_{i+1}})^t = 1 + t\beta\Pi^{r_{i+1}} + \gamma\Pi^{2r_{i+1}} \quad \text{for } \gamma \in \mathfrak{O}_k,$$

so from $p \not| t$ we have $\mathrm{ord}_{\mathfrak{P}}(\Pi_i^{t(\sigma-1)} - 1) = r_{i+1}$.

(2) Using (1) the result follows from

$$(1 + a\Pi_i^t)^{\sigma-1} = \frac{1 + a\Pi_i^{t\sigma}}{1 + a\Pi_i^t} = 1 + \frac{a(\Pi_i^{t(\sigma-1)} - 1)}{1 + a\Pi_i^t}\Pi_i^t.$$

□

Lemma 3.9. *Let* t*,* t'*,* s*, and* s' *be positive integers, and assume* $p \not| t$ *and* $p \not| t'$*. For* $a, b \in U_k$*, set* $\alpha = 1 + a\Pi_s^t$ *and* $\beta = 1 + b\Pi_{s'}^{t'}$*. If* $t \neq t'$ *or* $s \neq s'$*, then* $\mathrm{ord}_{\mathfrak{P}}(\alpha^{\sigma-1} - 1) \neq \mathrm{ord}_{\mathfrak{P}}(\beta^{\sigma-1} - 1)$.

Proof. (1) Assume $t \neq t'$ and $s = s'$. Then by Lemma 3.8

$$\mathrm{ord}_{\mathfrak{P}}(\alpha^{\sigma-1} - 1) = r_{s+1} + p^s t \neq r_{s'+1} + p^{s'} t' = \mathrm{ord}_{\mathfrak{P}}(\beta^{\sigma-1} - 1),$$

which proves our case.

(2) Next, let $s < s'$ and assume

$$\mathrm{ord}_{\mathfrak{P}}(\alpha^{\sigma-1} - 1) = r_{s+1} + p^s t = r_{s'+1} + p^{s'} t' = \mathrm{ord}_{\mathfrak{P}}(\beta^{\sigma-1} - 1).$$

Then by (3.1) we can write

$$r_{s'+1} = r_{s+1} + p^{s+1} u$$

for some integer u, so

$$r_{s+1} + p^s t = r_{s'+1} + p^{s'} t' = r_{s+1} + p^{s+1} u + p^{s'} t'.$$

Thus we have $t = pu + p^{s'-s} t'$, which contradicts $p \not| t$. Hence if $s < s'$, then $\mathrm{ord}_{\mathfrak{P}}(\alpha^{\sigma-1} - 1) \neq \mathrm{ord}_{\mathfrak{P}}(\beta^{\sigma-1} - 1)$. □

Lemma 3.10. *Let* $\alpha^{\sigma-1}, \beta^{\sigma-1} \in U_K^{(1)}$ *and assume* $\mathrm{ord}_{\mathfrak{P}}(\alpha^{\sigma-1}-1) \neq \mathrm{ord}_{\mathfrak{P}}(\beta^{\sigma-1}-1)$*, then*

$$\mathrm{ord}_{\mathfrak{P}}((\alpha\beta)^{\sigma-1}-1) = \min\{\mathrm{ord}_{\mathfrak{P}}(\alpha^{\sigma-1}-1), \mathrm{ord}_{\mathfrak{P}}(\beta^{\sigma-1}-1)\}.$$

Proof. From

$$(\alpha\beta)^{\sigma-1}-1 = (\alpha^{\sigma-1}-1)\beta^{\sigma-1} + (\beta^{\sigma-1}-1)$$

the result follows. □

We are now ready to prove our theorem.

Proof of Theorem 3.5. Let $x \in U_K^{(r_n+s)}$ be a representative of an element in $H^{-1}(G, U_K^{(r_n+s)})$. Then x satisfies $N_{K/k}x = 1$. Since $H^{-1}(G, K^*) = 1$, we can write $x = \alpha^{\sigma-1}$ for some $\alpha \in K^*$.

We may assume $\alpha \in \mathfrak{O}_K$ since $(\pi\alpha)^{\sigma-1} = u^{\sigma-1}$, where π is a prime element of k. Now we apply Corollary 3.7 to α with $m = r_n + s$. Then we can write

$$\alpha = a_0 \Pi_{i_0}^{t_0} v u,$$
$$v = (1 + a_{i_1}\Pi_{i_1}^{t_1})(1 + a_{i_2}\Pi_{i_2}^{t_2}) \cdots (1 + a_{i_{m'1}}\Pi_{i_{m'}}^{t_{m'}}),$$

where $u \in U_K^{(r_n+s)}$ and $a_i \in A \subset k$ with $0 < p^{i_1}t_{i_1} < p^{i_2}t_{i_2} < \cdots < p^{i_{m'}}t_{i_{m'}}$ and $0 \leq m' < m$. In the above expression, if $i_j = n$ then $(1 + a_{i_j}\Pi_{i_j}^{t_j})^{\sigma-1} = 1$, so we may assume $i_j < n$, in particular, $p \not| t_j$ for $j > 0$. Since $u^{\sigma-1} \in U_K^{(r_n+s)}$, it follows $(\Pi_{i_0}^{t_0} v)^{\sigma-1} \in U_K^{(r_n+s)}$. Here if $i_0 = n$ or $t_0 = 0$, then $(\Pi_{i_0}^{t_0})^{\sigma-1} = 1$; and in this case we have $x = \alpha^{\sigma-1} = v^{\sigma-1}u^{\sigma-1}$.

Now we claim $i_0 = n$ or $t_0 = 0$. Indeed, suppose $i_0 < n$ and $t_0 > 0$. If $p|t_0$, then $i_0 = n$. So we have $p \not| t_0$. Then Lemma 3.8 implies $\mathrm{ord}_{\mathfrak{P}}(\Pi_{i_0}^{t_0(\sigma-1)} - 1) = r_{i_0+1}$, Moreover since $p \not| t_j$ for $i = i_j$ it follows

$$\mathrm{ord}_{\mathfrak{P}}((1 + a_j\Pi_{i_j}^{t_j})^{\sigma-1} - 1) = r_{i_j+1} + p^{i_j}t_j.$$

Now by Lemma 3.9 we know $\mathrm{ord}_{\mathfrak{P}}((1 + a_j\Pi_{i_j}^{t_j})^{\sigma-1} - 1)$ are different each other since $0 < p^{i_1}t_{i_1} < p^{i_2}t_{i_2} < \cdots < p^{i_{m'}}t_{i_{m'}}$. Thus using Lemma 3.10 we have

$$\mathrm{ord}_{\mathfrak{P}}(v^{\sigma-1} - 1) = \min_{i_j = i_1, \cdots, i_{m'}} \{r_{i_j+1} + p^{i_j}t_j\}.$$

Moreover it holds

$$r_{i_0} \neq r_{i_j+1} + p^{i_j}t_j. \tag{3.4}$$

In fact if $r_{i_0} = r_{i_j+1} + p^{i_j}t_j$ for some i_j , then $i_0 > i_j + 1$, so by (3.1) r_{i_0} is written as the form

$$r_{i_0} = r_{i_j+1} + p^{i_j+1}w.$$

Hence $p|t_j$, which contradicts the assumption.

Thus using (3.4) from Lemma 3.10 we see

$$\mathrm{ord}_{\mathfrak{P}}((\Pi_{i_0}^{t_0} v)^{\sigma-1} - 1) = \min_{i_j = i_1, \cdots, i_{m'}} \{r_{i_j+1} + p^{i_j} t_j, r_{i_0}\} \leq r_{r_0} \leq r_n,$$

which contradicts $(\Pi_{i_0}^{t_0} v)^{\sigma-1} \in U_K^{(r_n+s)}$. This proves $i_0 = n$ or $t_0 = 0$, and $x = \alpha^{\sigma-1} = v^{\sigma-1} u^{\sigma-1}$.

Hence we have $v^{\sigma-1} \in U_K^{(r_n+s)}$. On the other hand the above argument shows $\mathrm{ord}_{\mathfrak{P}}(v^{\sigma-1} - 1) = \min\{r_{i_j+1} + p^{i_j} t_j\}$. Thus $\min\{r_{i_j+1} + p^{i_j} t_j\} \geq r_n + s$. Here we note $r_{i_j+1} \leq r_n$, so we have $p^{i_j} t_j \geq s$. This shows $v \in U_K^{(s)}$, so $vu \in U_K^{(s)}$. Set $w = vu$, then $x = (w)^{\sigma-1}$ with $w = vu \in U_K^{(s)}$. This means $x = (w)^{\sigma-1}$ represents the trivial class in $H^{-1}(G, U_K^{(s)})$, which proves the theorem. □

Using the argument used in the proof of the above we have the following, which states counter results in the strongly ramified case to Lemma 3.2. For simplicity we deal with the case where K/k is cyclic of degree p.

Proposition 3.11. *Let K/k be a strongly ramified cyclic extension of degree p. Let r denote the last ramification number of K/k. Let s be a positive integer such that $s > r$ and $s \not\equiv r \mod p$. Then $H^{-1}(G, U_K^{(s)}) \neq \{1\}$.*

Proof. In the proof we use the above notation with the case $n = 1$.

Put $t = s - r$. Then $p \not| t$ by the assumption. Let $\alpha = 1 + \Pi^t$ and $\beta = \alpha^{\sigma-1}$. Then $N_{K/k}(\beta) = 1$ and $\mathrm{ord}_{\mathfrak{P}}(\beta - 1) = r + t = s$ by Lemma 3.8, so β represents an element $\bar{\beta} \in H^{-1}(G, U_K^{(s)})$. Suppose β represents 1 in $H^{-1}(G, U_K^{(s)})$. Then we can write $\beta = \gamma^{\sigma-1}$ for some element $\gamma \in U_K^{(s)}$, where $\mathrm{ord}_{\mathfrak{P}}(\gamma - 1) = s$ since $\mathrm{ord}_{\mathfrak{P}}(\gamma^{\sigma-1} - 1) = s$. Here with the notation of Corollary 3.7 γ is written of the form $a_0(1 + a_1 \Pi_{i_1}^{t_1}) u$ for $u \in U_K^{(s+1)}$ and $p^{i_1} t_1 = s$. Hence if $p | s$, then $\gamma^{\sigma-1} = u^{\sigma-1} \in U_K^{(s+1)}$. On the other hand if $p \not| s$, then by Lemma 3.8 $(1 + a_1 \Pi_{i_1}^{t_1}) \in U_K^{(s+r)}$. So in both cases we have $\beta = \gamma^{\sigma-1} \in U_K^{(s+1)}$, which contradicts $\mathrm{ord}_{\mathfrak{P}}(\beta - 1) = s$. Thus β represents non-trivial element of $H^{-1}(G, U_K^{(s)})$.

□

We now ready to prove our main theorem, which gives a relationship between Scholz admissible moduli of K/L and K/k. This theorem gives a way to get an estimate for Scholz conductor of K/k from the ramification in K/k, in particular, this result implies the existence of Scholz conductor in general.

Theorem 3.12. *Let $K \supset L \supset k$ be a finite Galois tower. Assume L/k is a totally strongly ramified cyclic extension of prime power degree p^n. Let $\mathfrak{P}$, $\mathfrak{P}'$, and $\mathfrak{p}$ be the primes of K, L, and k, respectively. Let $\mathfrak{P}'^s$ be a Scholz admissible modulus of L with respect to K/L, and let r_n be the last ramification number of $\mathfrak{P}'$ in L/k. Let i be the integer satisfying $ip^{n-1} < s \leq (i+1)p^{n-1}$. Put $t = u_{L/k}(r_n + ip^{n-1}) + 1$. Then $\mathfrak{p}^t$ is a Scholz admissible modulus with respect to K/k.*

Proof. Let $H = Gal(K/L)$. Since $V_{L/k}(r_n+1) = \{1\}$, we see by the Herbrand's theorem (2.10) that $V_{K/k}(v_{K/L}(r_n+1))H/H = V_{L/k}(r_n+1) = \{1\}$, i.e., $V_{K/k}(v_{K/L}(r_n+1)) \subset H$. Moreover by assumption $\mathfrak{P}'^s$ is Scholz admissible with respect to K/L, in particular, Galois admissible with respect to K/L, this means $V_{K/L}(v_{K/L}(s)) = \{1\}$. Hence we have $\{1\} = V_{K/L}(v_{K/L}(s)) = V_{K/k}(v_{K/L}(s)) \cap H$. On the other hand, since $r_n \geq p^{n-1}$, it follows $r_n + ip^{n-1} \geq s$, so $v_{K/k}(t) = \phi_{K/k}(t-1)+1 = \phi_{K/k}(u_{L/k}(r_n+ip^{n-1}))+1 \geq \phi_{K/L}(r_n+ip^{n-1})+1 = v_{K/L}(r_n+ip^{n-1}+1) \geq \max(v_{K/L}(s), v_{K/L}(r_n+1))$, we have $V_{K/k}(v_{K/k}(t)) \subset V_{K/k}(v_{K/L}(s)) \cap H = \{1\}$. This proves that $\mathfrak{p}^t$ is a Galois admissible modulus with respect to K/k.

Let v, v', and v'' denote $v_{K/k}$, $v_{K/L}$, and $v_{L/k}$, respectively. Here we note $u_{K/L}(v(t)) = v''(t) = v_{L/k}(u_{L/k}(r_n+ip^{n-1})+1) = \phi_{L/k}(u_{L/k}(r_n+ip^{n-1}))+1 \geq r_n+ip^{n-1}+1$. Then using Lemma 3.1 we have the following commutative diagram with exact rows.

$$
\begin{array}{ccccc}
H^{-1}(H, U_K^{(v(t))}) & \to & H^{-1}(G, U_K^{(v(t))}) & \xrightarrow{N_{K/L}} & H^{-1}(G/H, U_L^{(v''(t))}) \to 1 \\
 & & & & \downarrow 1^{\#} \\
\downarrow 1^{\#} & & \downarrow 1^{\#} & & H^{-1}(G/H, U_L^{(r_n+ip^{n-1}+1)}) \\
 & & & & \downarrow 1^{\#} \\
H^{-1}(H, U_K^{(v'(s))}) & \to & H^{-1}(G, U_K^{(v'(s))}) & \xrightarrow{N_{K/L}} & H^{-1}(G/H, U_L^{(s)}) \to 1 \\
\downarrow 1^{\#} & & \downarrow 1^{\#} & & \downarrow 1^{\#} \\
H^{-1}(H, U_K^{(v'(ip^{n-1}+1))}) & \to & H^{-1}(G, U_K^{(v'(ip^{n-1}+1))}) & \xrightarrow{N_{K/L}} & H^{-1}(G/H, U_L^{(ip^{n-1}+1)}) \\
\downarrow 1^{\#} & & \downarrow 1^{\#} & & \downarrow 1^{\#} \\
H^{-1}(H, K^*) & \xrightarrow{Cor} & H^{-1}(G, K^*) & \to & H^{-1}(G/H, L^*)
\end{array}
$$

Now we must prove

$$1^{\#} : H^{-1}(G, U_K^{(v(t))}) \to H^{-1}(G, K^*)$$

is trivial. Let A be an element of $H^{-1}(G, U_K^{(v(t))})$, and represent A by an element $\alpha \in U_K^{(v(t))}$ with $N_{K/k}(\alpha) = 1$. Put $\beta = N_{K/L}(\alpha)$, then $\beta \in U_L^{(v''(t))}$ and $N_{L/k}(\beta) = 1$. So β represents an element B of $H^{-1}(G/H, U_L^{(r_n+ip^{n-1}+1)})$. Therefore by Theorem 3.5 we see $B = 1$ in $H^{-1}(G/H, U_L^{(ip^{n-1}+1)})$, i.e., β is written of the form $\beta = \gamma^{\sigma-1}$ for an element γ of $U_L^{(ip^{n-1}+1)}$, where σ denotes a generator of a cyclic group G/H.

Now by Corollary (3.7), γ is written of the form $\gamma = uu'$, where

$$u = (1 + a_{i_1}\Pi_{i_1}^{t_1})(1 + a_{i_2}\Pi_{i_2}^{t_2}) \cdots (1 + a_{i_j}\Pi_{i_j}^{t_j}) \cdots (1 + a_{i_{m'}}\Pi_{i_{m'}}^{t_{m'}}),$$

and $u' \in U_L^{(r_n+ip^{n-1}+1)}$. Since $u' \in U_L^{(r_n+ip^{n-1}+1)}$, it follows $u^{\sigma-1} \in U_L^{(r_n+ip^{n-1}+1)}$. Here we may assume $p \not| t_j$ since if $p|t_j$, then $(1 + a_{i_j}\Pi_{i_j}^{t_j})^{\sigma-1} = 1$. Then using a similar argument in the proof Theorem 3.5 we have

$$\mathrm{ord}_{\mathfrak{P}}(u^{\sigma-1} - 1) = \min\{r_{i_j+1} + p^{i_j}t_j\} \geq r_n + ip^{n-1} + 1.$$

Hence

$$r_n + ip^{n-1} + 1 \leq r_{i_j+1} + p^{i_j}t_j.$$

Here if $i_j + 1 = n$, then $r_{i_j+1} = r_n$ and $ip^{n-1} + 1 \leq p^{i_j}t_j$, so $i + 1 \leq t_j$. Thus in this case we have

$$s \leq (i+1)p^{n-1} \leq p^{i_j}t_j.$$

Next, if $i_j + 1 < n$, then , since $r_n = r_{n-1} + qp^{n-1}$, we have

$$s \leq (i+1)p^{n-1} \leq r_{n-1} - r_{i_j+1} + qp^{n-1} + ip^{n-1} + 1 \leq p^{i_j}t_j.$$

Thus we have $u \in U_L^{(s)}$, and $\gamma = uu' \in U_L^{(s)}$.

Now by the assumption we know $\mathfrak{P}'^s$ is Galois admissible with respect to K/L. Therefore $N_{K/L}U_K^{(v'(s))} = U_L^{(s)}$ by Lemma 2.3, thus we can choose an element δ of $U_K^{(v'(s))}$ such that $N_{K/L}\delta = \gamma$. Let σ' denote an extension of σ to an element of G. Then $N_{K/k}(\alpha/\delta^{\sigma'-1}) = N_{L/k}(\beta/\gamma^{\sigma-1}) = 1$, therefore $\alpha/\delta^{\sigma'-1}$ represents an element of $H^{-1}(G, U_K^{(v'(s))})$, which is actually equal to $1^{\#}(A)$ as an element of $H^{-1}(G, U_K^{(v'(s))})$.

Moreover since $N_{K/L}(\alpha/\delta^{\sigma'-1}) = \beta/\gamma^{\sigma-1} = 1$, we see $\alpha/\delta^{\sigma'-1}$ also represents an element of $H^{-1}(H, U_K^{(v'(s))})$, hence $1^{\#}(A) \in H^{-1}(G, U_K^{(v'(s))})$ is represented by an element of $H^{-1}(H, U_K^{(v'(s))})$. Now by assumption we know

$$1^{\#} : H^{-1}(H, U_K^{(v'(s))}) \to H^{-1}(H, K^*)$$

is trivial. Thus we see $1^{\#}(A) = 1$ in $H^{-1}(G, K^*)$, which proves the assertion. □

Corollary 3.13. *Let the assumptions and the notations be the same as above. Assume $s \leq p^{n-1}$. Then the conductor of L/k is also the Scholz conductor of K/k.*

Proof. Let r_n denote the last ramification number of L/k. Then by the conductor theorem we know that the conductor of L/k is $\mathfrak{p}^{u_{L/k}(r_n)+1}$. But the assumption implies $i = 0$ with the above notation, hence by our theorem $\mathfrak{p}^t$ is Scholz conductor of K/k for $t = u_{L/k}(r_n) + 1$. □

We conclude this paper by giving an example of this corollary.

Let p be an odd prime, and put $K = \mathbb{Q}_p(\zeta)$, where $\zeta = \zeta_{p^n}$ denotes a primitive p^n-th root of unity. Then

(3.5) the Scholz conductor of $K/\mathbb{Q}_p$ is p^n,

since $K/\mathbb{Q}_p$ is cyclic extension.

Next, we consider the case $K = \mathbb{Q}_2(\zeta)$. where $\zeta = \zeta_{2^{n+2}}$ denotes a primitive 2^{n+2}-th root of unity. Put $L = \mathbb{Q}_2(\zeta+\zeta^{-1})$. Then $L/\mathbb{Q}_2$ is a cyclic extension of degree 2^n. Moreover K/L is a quadratic extension with $Gal(K/L) =< \tau >$, where τ denotes the complex conjugate map. Let $\mathfrak{P}$ and $\mathfrak{P}'$ denote the prime of K and the prime of L, respectively. Then, since $\mathrm{ord}_{\mathfrak{P}}(\zeta^{\tau} - \zeta) = \mathrm{ord}_{\mathfrak{P}}(\zeta^{-1} - \zeta) = \mathrm{ord}_{\mathfrak{P}}(\zeta^{-1}(1 - \zeta^2)) = 2$, we have the last ramification number of K/L is 1 and the conductor of K/L is $\mathfrak{P}'^2$, that is $s = 2$ with the above notation. In particular, if $n \geq 2$, then it holds $s \leq 2^{n-1}$. Thus we have the following.

Corollary 3.14. *If $n \geq 2$, the Scholz conductor of $\mathbb{Q}_2(\zeta_{2^{n+2}})/\mathbb{Q}_2$ is 2^{n+2}.*

Corollary 3.14 and (3.5) give essentially an alternative proof of Theorem 3 of [2].

References

[1] P. Draxl, *SK_1 von Algebren über volstädig diskret Körpern und Galoiskohomologie abelischer Körperweiterungen*, J. Reine Angew. Math., **293/294** (1977), pp. 116–142.

[2] Y. Fröhlich, *On fields of class two* , Proc. London Math. Soc. (3), **4**, (1954), pp. 235–256.

[3] Y. Furuta, *On the nilpotent factors of congruent ideal class groups of Galois extensions*, Nagaoya Math. J., **62** (1976), pp. 13–28.

[4] F. P. Heider, *Strahlknoten und Geschlechterkörper mod m*, J. Reine Angew. Math., **320** (1980), pp. 52–67.

[5] S. Iyanaga, *The theory of numbers (North-Holland mathematical library 8)*, North-Holland, Amsterdam, 1975.

[6] W. Jehne, *On knots in algebraic number theory*, J. Reine Angew. Math. **311/312** (1979), pp. 215-254.

[7] J. Neukirch, *Class Field Theory (Grundlehren der mathmatischen Wissenschaften 280)*, Springer-Verlag, Berlin, 1986

[8] S. Shirai, *On the central class field mod $\mathfrak{m}$ of Galois extensions of an algebraic number field*, Nagoya Math. J., **71** (1978), pp. 61-85

[9] A. Scholz. *Totale Normreste, die keine Normen sind, als Erzeuger nichtabelscher Köperweiterungen II*, J. Reine Angew. Math. **182** (1940), pp. 217-234

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