Nihonkai Math. J.
Vol.14(2003), 133-177

Infinitesimal deformation of Galois covering space and its application to Galois closure curves

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Abstract. We develop a general framework of the infinitesimal deformations of finite branched Galois covering spaces of complex dimension one. By using the framework, we discuss the correspondence between the infinitesimal deformations of a branched covering map and those of the Galois closure curve of the map. In particular, we compute Kodaira-Spencer maps for families of Galois closure curves constructed on base spaces which consist of branched covering maps.

1 Introduction

In this article we discuss compact smooth complex manifolds of dimension one, in other words, we discuss curves. Let X, Y be two curves and

$$Y \xrightarrow{\alpha} X \tag{1}$$

a holomorphic branched covering map from the curve Y to the curve X. The branched covering map α induces the following finite extension of fields via pull-back of functions

$$k(X) \overset{\alpha^*}{\hookrightarrow} k(Y), \tag{2}$$

where $k(X)$ (resp. $k(Y)$) denotes the field of all rational functions on X (resp. Y). For a given branched covering map α, the extension (2) may not necessarily be a Galois extension. In the case that the extension (2) is not a Galois extension, we can find uniquely up to isomorphism a curve Z and a holomorphic branched covering map

$$Z \xrightarrow{\beta} Y \tag{3}$$

which give the Galois closure

$$k(X) \overset{\alpha^*}{\hookrightarrow} k(Y) \overset{\beta^*}{\hookrightarrow} k(Z) \tag{4}$$

of the extension (2). The curve Z above is called the *minimal splitting curve* (or, the *Galois closure curve*) of the extension (2) by Yoshihara, Miura and Takahashi in their articles [5][4][8][7][9]. We have interests in describing how the moduli of the complex structure of the Galois closure curve Z varies as we deform the branched covering map (1). In this article we develop a general framework of infinitesimal deformation of a Galois covering space of curves and, by using it, we discuss the correspondence between the infinitesimal deformations of branched covering maps and the infinitesimal deformations of Galois closure curves.

We denote the set of all holomorphic maps of mapping degree d (≥ 1) from a curve Y to a curve X by

$$\mathrm{Map}_d(Y, X). \tag{5}$$

The elements of $\mathrm{Map}_d(Y, X)$ are branched covering maps. The mapspace $\mathrm{Map}_d(Y, X)$ has a complex space structure [6].

And we denote the group of all holomorphic automorphisms of X (resp. Y) by $\mathrm{Aut}(X)$ (resp. $\mathrm{Aut}(Y)$). The groups $\mathrm{Aut}(X)$, $\mathrm{Aut}(Y)$ have finite dimensional complex Lie group structures. An element $(\sigma, \xi) \in \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$ acts on $\alpha \in \mathrm{Map}_d(Y, X)$ via compositions of maps

$$\alpha \to \xi^{-1} \circ \alpha \circ \sigma. \tag{6}$$

Since the extension of fields

$$k(X) \hookrightarrow k(Y) \tag{7}$$

induced by an element $\alpha \in \mathrm{Map}_d(Y, X)$ and that induced by an element $\xi^{-1} \circ \alpha \circ \sigma$ are equivalent as field extensions, they define the same Galois closure curve. Therefore a Galois closure curve should be regarded as to be defined for each equivalence class

$$\mathrm{Map}_d(Y, X)/\,(\mathrm{Aut}(X) \times \mathrm{Aut}(Y)) \tag{8}$$

of maps. The mapspaces

$$\mathrm{Map}_d(Y, X) \quad \text{and} \quad \mathrm{Map}_d(Y, X)/\,(\mathrm{Aut}(X) \times \mathrm{Aut}(Y)) \tag{9}$$

are studied by Namba for several curves X, Y in his book [6]. Readers may consult the book for details of those mapspaces.

As we mentioned in the first paragraph, our interest is to describe the correspondence

$$\mathrm{Map}_d(Y, X)/\,(\mathrm{Aut}(X) \times \mathrm{Aut}(Y)) \xrightarrow{\text{Galois closure}} \{\text{equivalence classes of curves}\}. \tag{10}$$

The question above—how the moduli of the complex structure of the Galois closure curve Z varies as we deform the branched covering map $Y \xrightarrow{\alpha} X$—was originally raised and discussed by Yoshihara in his article [9].

Yoshihara raised and discussed the question above in his article [9] for the case that Y is a degree four smooth projective plane curve $Y \subset P^2$ and X the projective space P^1. In this case the genus of the curve Y is three and the automorphism group $\mathrm{Aut}(Y)$ is at most a finite group. For a point $Q \in Y$, the linear projection $Y \overset{\alpha_Q}{\to} P^1$ with center Q is a mapping degree three holomorphic map. In fact the correspondence of a point $Q \in Y$ to the linear projection $\alpha_Q \in \mathrm{Map}_3(Y, P^1)$ induces the following isomorphism (Namba [6]):

$$Y \cong \mathrm{Map}_3(Y, P^1)/\mathrm{Aut}(P^1) \quad \text{(biholomorphic)}. \tag{11}$$

Yoshihara constructed a non-singular surface S and a holomorphic fiber space

$$\varphi : S \to Y \tag{12}$$

which satisfies that each fiber $\varphi^{-1}(Q)$ on a general —i.e. not flex— point $Q \in Y$ is isomorphic to the Galois closure curve Z_Q of the map α_Q. For a general point $Q \in Y$, the fiber $\varphi^{-1}(Q)$ is a curve of genus ten. By corresponding each point $Q \in Y$ to the fiber $\varphi^{-1}(Q)$, he constructed a holomorphic map Ψ from Y to a compactificaion of the moduli space of genus ten curves

$$\Psi : \ Y \to \overline{\mathcal{M}_{10}}. \tag{13}$$

And by investigating the singular fibers on the flex points $Q \in Y$ of the fiber space (12), he showed that the holomorphic map Ψ above is not a constant map. As an corollary of the result, he proved that the Galois closure curves Z_P and Z_Q are not isomorphic for sufficiently close general points $P, Q \in Y$.

Yoshihara's work above motivated the author to compute directly the differential

$$d\Psi : \ T_Q Y \to T_{[Z_Q]}\mathcal{M}_{10} \tag{14}$$

or equivalently, the *Kodaira-Spencer map*

$$T_Q Y \xrightarrow{\text{K-S}} H^1(Z_Q, \mathcal{O}(TZ_Q)). \tag{15}$$

If we can prove that the Kodaira-Spencer map above is injective, we can arrive at the same conclusion as Yoshihara's by using a quite different way from his. Indeed, as we will see later in Section 4, the Kodaira-Spencer map above is injective, and therefore, the fiber space (12), which forms a family of Galois closure curves, is *effectively parameterized* at the general points $Q \in Y$.

The author found that we can compute the Kodaira-Spencer map above in the general framework of infinitesimal deformation of a Galois covering space, which is the main topic of this article. In the framework, we can also compute Kodaira-Spencer maps for larger classes of curves X, Y, Z. We restrict our concerns to the correspondence between only infinitesimal deformations of both sides of (10) in this article.

For the purpose to discuss the correspondence between infinitesimal deformation of maps and that of Galois closure curves, first we briefly sketch the infinitesimal deformations of maps. The detailed discussion of the subject will be treated later in Section 3.1.

For a map $\alpha \in \mathrm{Map}_d(Y, X)$, we call a section of the pull-back bundle $\alpha^*(TX)$

$$s \in H^0(Y, \mathcal{O}(\alpha^*(TX))) \tag{16}$$

an *infinitesimal deformation* of the map α.

The actions of the automorphism groups $\mathrm{Aut}(X)$, $\mathrm{Aut}(Y)$ on $\mathrm{Map}_d(Y, X)$ induce vector space homomorphisms from the Lie algebras of those groups

$$\mathrm{Lie}\,(\mathrm{Aut}(X)) \cong H^0\,(X, \mathcal{O}\,(TX)), \tag{17}$$
$$\mathrm{Lie}\,(\mathrm{Aut}(Y)) \cong H^0\,(Y, \mathcal{O}\,(TY)), \tag{18}$$

to the vector space $H^0\,(Y, \alpha^*\,(TX))$. Each homomorphism is written as follows: The former is the pull-back of the holomorphic sections

$$\mathcal{P}:\ H^0\,(X, \mathcal{O}\,(TX)) \longrightarrow H^0\,(Y, \mathcal{O}\,(\alpha^*\,(TX))). \tag{19}$$

And the latter is the homomorphism of cohomology groups

$$\mathcal{Q}:\ H^0\,(Y, \mathcal{O}\,(TY)) \xrightarrow{d\alpha} H^0\,(Y, \mathcal{O}\,(\alpha^*\,(TX))), \tag{20}$$

induced by the homomorphism of sheaves

$$\mathcal{O}\,(TY) \xrightarrow{d\alpha} \mathcal{O}\,(\alpha^*\,(TX)). \tag{21}$$

We call an element of the quotient vector space by the images of these maps above

$$H^0\,(Y, \mathcal{O}\,(\alpha^*\,(TX)))\,/\,\,(\mathrm{Im}\,\mathcal{P} + \mathrm{Im}\,\mathcal{Q}) \tag{22}$$

an *equivalence class of infinitesimal deformation*. The vector space (22) has fundamental importance for our infinitesimal deformation theory.

In the case above discussed by Yoshihara, the curve Y is of genus three and therefore it satisfies

$$\mathrm{Lie}\,(\mathrm{Aut}(Y)) = H^0\,(Y, \mathcal{O}\,(TY)) = 0. \tag{23}$$

This implies that $\mathrm{Im}\,\mathcal{Q} = 0$. By using the isomorphism (11), we obtain in this case

$$T_Q Y \cong H^0\left(Y, \mathcal{O}\left(\alpha_Q^*\left(TP^1\right)\right)\right) / \,\mathrm{Im}\,\mathcal{P}. \tag{24}$$

For the purpose to answer the question of our primary concern— how the complex structure of the Galois closure curve Z_α varies as we deform the branched covering map α— which we stated above, we investigate the "inverse problem" of this, which we describe below.

For a branched covering map $Y \xrightarrow{\alpha} X$, we construct uniquely up to isomorphism the Galois closure of the branched covering map α, which consists of a Galois covering space $Z \xrightarrow{\pi} X$, the Galois group of which is a finite group G and a Galois covering space $Z \xrightarrow{\beta} Y$, the Galois group of which is a subgroup $H \subset G$, satisfying $\alpha \circ \beta = \pi$. Then the curves X, Y are written as

$$X = Z/G, \quad Y = Z/H. \tag{25}$$

The inverse problem which we mentioned above is as follows: Let Z be a curve endowed with a finite group G-action so that the quotient map

$$\pi : Z \to X := Z/G \tag{26}$$

should be a Galois covering space with Galois group G. Then, for an arbitrary subgroup $H \subset G$, the quotient map

$$\beta : Z \to Y := Z/H \tag{27}$$

is a Galois covering space with Galois group H. Under the situation above, we can construct a branched covering map $Y \xrightarrow{\alpha} X$ by setting

$$\alpha \circ \beta = \pi. \tag{28}$$

(Note that the Galois covering spaces $Z \xrightarrow{\pi} X$ and $Z \xrightarrow{\beta} Y$ are not supposed to be the Galois closure of the branched covering map α.) The problem is how the branched covering map α varies as we deform the complex structure of the curve Z in a G-invariant way in which the moduli of the curves X, Y should not be changed.

For the purpose to solve the problem, we proceed as follows. Under the identification

$$X = Z/G, \quad Y = Z/H, \tag{29}$$

we observe the phenomena which occur as we deform the complex structure of the curve Z in a G-invariant way. We denote the almost complex structures of the curves X, Y, Z by J_X, J_Y, J_Z respectively. Let $J_Z(t)$ be a smooth family of G-invariant almost complex structures on the curve Z depending on a real parameter $t \in (-\varepsilon, \varepsilon)$ satisfying the following Condition Σ:

(Condition Σ)

The initial value satisfies $J_Z(0) = J_Z$ and $J_Z(t)$ identically equals the original almost complex structures J_Z on some neighborhood of the ramification points $r_1, r_2, \ldots, r_N \in Z$ of the Galois covering space $Z \xrightarrow{\pi} X$.

Since every almost complex structure is integrable on a complex manifold of dimension one, and therefore, since it defines a complex structure of the manifold, the smooth family of G-invariant almost complex structures $J_Z(t)$ above defines a *deformation of the complex structure* of the curve Z. We denote the curve Z with the almost complex structure $J_Z(t)$ by $Z(t)$.

For a point $p \in Z$ which is not a ramification point of the Galois covering space $Z \xrightarrow{\pi} X$, the homomorphisms of tangent spaces below are isomorphisms:

$$d\pi : T_pZ \xrightarrow{\cong} T_{\pi(p)}X, \tag{30}$$

$$d\beta : T_pZ \xrightarrow{\cong} T_{\beta(p)}Y. \tag{31}$$

Therefore we can regard $J_Z(t)$ also as a smooth family of almost complex structures $J_X(t)$ on the curve X by setting

$$J_X(t) := d\pi \circ J_Z(t) \circ (d\pi)^{-1} \quad \text{on } T_{\pi(p)}X. \tag{32}$$

Since $J_X(t)$ defined above identically equals the original almost complex structure J_X on some neighborhood of the branch points $b_i \in X$ of the Galois covering space $Z \xrightarrow{\pi} X$ by the property of Condition Σ, it defines a smooth family of almost complex structures on the curve X. The similar argument shows that we can regard $J_Z(t)$ also as a smooth family of almost complex structures $J_Y(t)$ on the curve Y by setting

$$J_Y(t) := d\beta \circ J_Z(t) \circ (d\beta)^{-1} \quad \text{on } T_{\beta(p)}Y. \tag{33}$$

In this way a G-invariant deformation of the complex structure of the curve Z satisfying Condition Σ induces deformations of those of the curves X, Y.

Next we set

$$\eta_Z := \left.\frac{d}{dt} J_Z(t)\right|_{t=0} \tag{34}$$

and we can regard η_Z as an G-invariant Dolbaut form $\eta_Z \in \Lambda^{0,1}(Z, TZ)_G$, and we see that it satisfies the following Condition $\tilde{\Sigma}$:

(Condition $\tilde{\Sigma}$)

η_Z vanishes on some neighborhood of the ramification points $r_1, r_2, \ldots, r_N \in Z$ of the Galois covering space $Z \xrightarrow{\pi} X$.

Needless to say, the homomorphisms of cotangent space are also isomorphisms for a point $p \in Z$ which is not a ramification point,

$$d\pi^* : T^*_{\pi(p)}X \xrightarrow{\cong} T^*_pZ, \tag{35}$$

$$d\beta^* : T^*_{\beta(p)}Y \xrightarrow{\cong} T^*_pZ. \tag{36}$$

Therefore we can apply a similar argument as with $J_Z(t)$ to the Dolbaut form η_Z satisfying Condition $\tilde{\Sigma}$, to see that we can regard η_Z also as an element of $\eta_X \in \Lambda^{0,1}(X, TX)$, and as an element of $\eta_Y \in \Lambda^{0,1}(X, TY)$ by using the isomorphisms of the tangent spaces and those of cotangent spaces above. Then it is not difficult to check that the following identities hold:

$$\left.\frac{d}{dt} J_X(t)\right|_{t=0} = \eta_X, \quad \left.\frac{d}{dt} J_Y(t)\right|_{t=0} = \eta_Y. \tag{37}$$

We should note that, for any $\eta_Z \in \Lambda^{0,1}(Z, TZ)_G$ satisfying Condition $\tilde{\Sigma}$, we can construct a smooth family of G-invariant almost complex structures $J_Z(t)$ depending a real parameter $t \in (-\varepsilon, \varepsilon)$ satisfying Condition Σ. Furthermore, we will show in Section 2, Lemma 1, that, for any class $a \in H^1(Z, \mathcal{O}(TZ))_G$, we can find a G-invariant Dolbaut form $\eta_Z \in \Lambda^{0,1}(Z, TZ)_G$ which satisfies

$$[\eta_Z] = a \tag{38}$$

and which satisfies Condition $\tilde{\Sigma}$.

As we will see later, we can construct a homomorphism of cohomology groups

$$H^1(Z, \mathcal{O}(TZ))_G \xrightarrow{\mathfrak{I}_X} H^1(X, \mathcal{O}(TX)) \tag{39}$$

so that the following diagram should commute:

$$\begin{array}{ccc} H^1(Z, \mathcal{O}(TZ))_G & \xrightarrow{\mathfrak{I}_X} & H^1(X, \mathcal{O}(TX)) \\ \uparrow & & \uparrow \\ \eta_Z \in \Lambda^{0,1}(Z, TZ)_G & \longrightarrow & \eta_X \in \Lambda^{0,1}(X, TX) \end{array} \tag{40}$$

The diagram describes how the complex structure of the curve X varies via the correspondence $\eta_Z \to \eta_X$ which we defined above (Section 2, Theorem 1). And in a similar way, we can construct a homomorphism of cohomology groups

$$H^1(Z, \mathcal{O}(TZ))_G \xrightarrow{\mathfrak{I}_Y} H^1(Y, \mathcal{O}(TY)) \tag{41}$$

so that the following diagram should commute:

$$\begin{array}{ccc} H^1(Z, \mathcal{O}(TZ))_G & \xrightarrow{\mathfrak{I}_Y} & H^1(Y, \mathcal{O}(TY)) \\ \uparrow & & \uparrow \\ \eta_Z \in \Lambda^{0,1}(Z, TZ)_G & \longrightarrow & \eta_Y \in \Lambda^{0,1}(Y, TY) \end{array} \tag{42}$$

The diagram describes how the complex structure of the curve Y varies via the correspondence $\eta_Z \to \eta_Y$ which we defined above.

For the purpose to obtain the infinitesimal deformations of the complex structure of Z which do not change the moduli of X, Y, we observe the kernels of the homomorphisms (39) and (41). We set subspaces $K_X \subset H^1(Z, \mathcal{O}(TZ))_G$ and $K_Y \subset H^1(Z, \mathcal{O}(TZ))_G$ as follows:

$$K_X := \mathrm{Ker}\left(H^1(Z, \mathcal{O}(TZ))_G \xrightarrow{\mathfrak{I}_X} H^1(X, \mathcal{O}(TX))\right), \tag{43}$$

$$K_Y := \mathrm{Ker}\left(H^1(Z, \mathcal{O}(TZ))_G \xrightarrow{\mathfrak{I}_Y} H^1(Y, \mathcal{O}(TY))\right). \tag{44}$$

Under the definition above, the subspace

$$K_X \cap K_Y \subset H^1(Z, \mathcal{O}(TZ))_G \tag{45}$$

corresponds to the infinitesimal deformations of the branched covering map $Y \xrightarrow{\alpha} X$ as we see below.

Since the smooth family of almost complex structures $J_X(t)$ of the curve X equivalent as the original almost complex structure J_X there exists a smooth family of C^∞-automorphisms $f_t : X \to X$ which satisfies

$$J_X(t) = (f_t)_*^{-1} \circ J_X \circ (f_t)_* . \tag{46}$$

And the similar argument shows that there exists a smooth family of C^∞-automorphisms $g_t : Y \to Y$ which satisfies

$$J_Y(t) = (g_t)_*^{-1} \circ J_Y \circ (g_t)_* . \tag{47}$$

Under the situation above, for a given $a \in K_X \cap K_Y$, we can find $J_Z(t)$, a smooth family of G-invariant almost complex structures on the curve Z depending on a real parameter $t \in (-\varepsilon, \varepsilon)$ satisfying Condition Σ and satisfying that the cohomology class of the differential

$$\eta_Z := \frac{d}{dt} J(t)\bigg|_{t=0} \tag{48}$$

coincides with $a \in K_X \cap K_Y$, and we can construct a smooth family of holomorphic Galois covering spaces $\pi_t : Z(t) \to X$ with Galois group G by setting

$$\pi_t := f_t \circ \pi, \tag{49}$$

and we can construct a smooth family of holomorphic Galois covering spaces $\beta_t : Z(t) \to Y$ with Galois group H, by setting

$$\beta_t := g_t \circ \beta. \tag{50}$$

Note that the complex structures of the curves X, Y are unchanged above.

Then the composite of maps

$$\alpha_t := \pi_t \circ \beta_t^{-1} \quad \text{(well-defined)} \tag{51}$$

is a smooth family of branched covering maps $Y \xrightarrow{\alpha_t} X$ with initial value $\alpha_0 = \alpha$.

The minus of the differential of the family α_t at $t = 0$

$$u := -\frac{d}{dt}\bigg|_{t=0} \alpha_t \tag{52}$$

defines a holomorphic section of the pull-back bundle $\alpha^*(TX)$. Namely

$$u \in H^0(Y, \mathcal{O}(\alpha^*(TX))) . \tag{53}$$

Then the equivalence class of u

$$[u] \in H^0(Y, \mathcal{O}(\alpha^*(TX))) \; / \; (\operatorname{Im}\mathcal{P} + \operatorname{Im}\mathcal{Q}) \tag{54}$$

defines a equivalence class of infinitesimal deformations of the branched covering map α.

As we state in the following Proposition 1, which is our main result in this article, that the equivalence class (54) does not depend on the particular choice of the families $J_Z(t), \pi_t, \beta_t$, but only depends on the element

$$a \in K_X \cap K_Y \tag{55}$$

which we took first.

Proposition 1. *We can define a homomorphism*

$$K_X \cap K_Y \to H^0(Y, \mathcal{O}(\alpha^*(TX))) \ / \ (\mathrm{Im}\, \mathcal{P} + \mathrm{Im}\, \mathcal{Q}) \tag{56}$$

so that the following diagram should commute:

$$\begin{array}{ccc} K_X \cap K_Y & \longrightarrow & H^0(Y, \mathcal{O}(\alpha^*(TX))) \ / \ (\mathrm{Im}\, \mathcal{P} + \mathrm{Im}\, \mathcal{Q}) \\ \uparrow & & \uparrow \\ \eta_Z \in \Lambda^{0,1}(Z, TZ)_G & \longrightarrow & u \in H^0(Y, \mathcal{O}(\alpha^*(TX))) \end{array} \tag{57}$$

(Section 3, Theorem 2).

Furthermore, if the Galois covering spaces $Z \xrightarrow{\pi} X, Z \xrightarrow{\beta} Y$ *are the Galois closure of the branched covering map* $Y \xrightarrow{\alpha} X$*, the homomorphism* (56) *above is injective (Section 3, Theorem 3).*

Moreover, under the assumptions above, if the branched covering map $Y \xrightarrow{\alpha} X$ *is of general ramification type, the injective homomorphism* (56) *is surjective and, consequently, it is an isomorphism (Section 3, Theorem 4). (A branched covering map* α *is of general ramification type if all the ramification points* $p_j \in Y$ *of the map* α *are of order two, and there exist no* p_i, p_j $(i \neq j)$ *satisfying* $\alpha(p_i) = \alpha\left(p_j\right)$*.)*

We see that Proposition 1 can do solve "the inverse problem"—how the branched covering map α varies as we deform the complex structure of the curve Z in a G-invariant way in which the moduli of the curves X, Y should not be changed—which we mentioned above. Then, by taking the inverse mapping of homomorphism (56) in Proposition 1, we obtain the following corollary:

Corollary 1. *If the branched covering map* $Y \xrightarrow{\alpha} X$ *is of general ramification type, the Kodaira-Spencer map of our primary concern is the minus of the inverse map the homomorphism* (56)

$$H^0(Y, \mathcal{O}(\alpha^*(TX))) \ / \ (\mathrm{Im}\, \mathcal{P} + \mathrm{Im}\, \mathcal{Q}) \xrightarrow{\cong} K_X \cap K_Y \subset H^1(Z, \mathcal{O}(TZ))_G \tag{58}$$

which is an isomorphism. In particular the Kodaira-Spencer map is injective.

In the case discussed by Yoshihara, the linear projection α_Q with center $Q \in Y$ is of general ramification type if and only if the point $Q \in Y \subset P^2$ is not flex. For a point $Q \in Y$ which is not a flex point, the homomorphism (56) in Proposition 1 is the inverse map of the Kodaira-Spencer map of our primary concern

$$T_Q Y \xrightarrow{\text{K-S}} H^1(Z_Q, \mathcal{O}(TZ_Q)), \tag{59}$$

under the identification

$$H^0\left(Y, \alpha_Q^*\left(TP^1\right)\right) / (\mathrm{Im}\, \mathcal{P}) \cong T_Q Y. \tag{60}$$

Just as in the case discussed by Yoshihara, we often investigate cases where the curve $X = P^1$. In those cases, we have

$$H^1\left(P^1, \mathcal{O}\left(TP^1\right)\right) = 0, \tag{61}$$

therefore we have the following simple corollary by using Corollary 1:

Corollary 2. *For a branched covering map $Y \xrightarrow{\alpha} P^1$ which is of general ramification type and for its Galois closure curve Z_α, the following sequence is exact:*

$$0 \to H^0\left(Y, \alpha^*\left(TP^1\right)\right) / (\operatorname{Im}\mathcal{P} + \operatorname{Im}\mathcal{Q}) \xrightarrow{\text{K-S}} H^1\left(Z_\alpha, \mathcal{O}\left(TZ_\alpha\right)\right)_G \xrightarrow{\mathcal{I}_Y} H^1\left(Y, \mathcal{O}\left(TY\right)\right). \tag{62}$$

We should note that, for a branched covering map α which is *not* of general ramification type, the corresponding homomorphism (56) may not necessarily be surjective. Therefore, in those cases, the inverse map of the homomorphism (56) may not be defined on the whole vector space

$$H^0\left(Y, \mathcal{O}\left(\alpha^*\left(TP^1\right)\right)\right) / (\operatorname{Im}\mathcal{P} + \operatorname{Im}\mathcal{Q}). \tag{63}$$

In the case discussed by Yoshihara, the linear projections with center $Q \in Y \subset P^2$ which are flex points are not of general ramification type. The genera of the Galois closure curves for those maps are strictly less than ten. Although the identification

$$T_QY \cong H^0\left(Y, \mathcal{O}\left(\alpha_Q^*\left(TP^1\right)\right)\right) / (\operatorname{Im}\mathcal{P} + \operatorname{Im}\mathcal{Q}) \tag{64}$$

still holds for such points $Q \in Y$, the Kodaira-Spencer map

$$T_QY \to \text{tangent space of moduli} \tag{65}$$

can not be defined. We restrict our concerns in this article to the case of branched covering maps of general ramification type.

This article is organized as follows.

In Section 2, we discuss, for a curve Z endowed with a finite group G-action so that the quotient map $Z \xrightarrow{\pi} X := Z/G$ is a Galois covering space with Galois group G, infinitesimal deformations of the structure. Our main result of this section is Theorem 1, which provides foundations of our later discussion.

In Section 3, under the situation in Section 2, and for a given subgroup $H \subset G$, we discuss the Galois covering space $Z \xrightarrow{\beta} Y := Z/H$ with Galois group H, and discuss the branched covering map $Y = Z/H \xrightarrow{\alpha} X$. We investigate several homomorphisms of cohomology groups which are needed for our discussion. Our main results in this section are Theorems 2, 3, and 4 which were stated in Proposition 1 above. In particular, by using these theorems, we derive the properties of the Kodaira-Spencer map of our primary concern.

Finally in Section 4, we apply our general framework, which we will develop in Sections 2 and 3, to the case above discussed by Yoshihara to compute the Kodaira-Spencer map which we mentioned above. And we discuss further the correspondence of the infinitesimal deformations of mapping degree three holomorphic maps from an elliptic curve Y to the projective space $X = P^1$ and those of their Galois closure curves. In both cases, we will show explicit computations of the cohomology groups. As we have seen, for a degree four smooth projective plane curve Y in the case discussed by Yoshihara, the map Ω satisfies $\operatorname{Im}\Omega = 0$. In the case where Y is an elliptic curve, however, the map Ω satisfies $\operatorname{Im}\Omega \neq 0$, therefore the investigation for such Y is all the more challenging in this respect.

The author expresses gratitude to Professor Takushirou Ochiai in Tokyo University, who gave the author foundations of differential geometry and of complex geometry. The author would like to thank Professor Hisao Yoshihara in Niigata University, who, after motivating the author to investigate families of Galois closure curves, gave him various helpful suggestions on the subject and gave him steady encouragements. The author also wishes to thank Doctor Takeshi Takahashi in Nagaoka National College of Technology, who shared fruitful discussions about the subject and about various related topics on algebraic geometry as well.

2 Infinitesimal deformation of Galois covering space of curves

In this section we discuss the two general questions, which we will state below, about Galois covering space of curves. The contents of this section are not only foundations of our later discussion but also have their own interests to us.

Let X, Z be two curves and $Z \xrightarrow{\pi} X$ a finite Galois covering space with Galois group G. We denote the branch points of the covering space by $b_i \in X$ $(i = 1, 2, \dots, m)$.

Our first question is how the moduli of the complex structure of the total space Z varies when we translate the branch points b_i $(i = 1, 2, \dots, m)$ on X. The second one is how the moduli of the complex structure of the base space $X = Z/G$ varies when we deform the complex structure of the total space Z in a G-invariant way.

As we will see later in Theorem 1, the two questions above are answered with a single exact sequence of cohomology groups induced by a short exact sequence of sheaves.

2.1 A sheaf homomorphism

For the purpose to answer the two questions above, first we establish a short exact sequence of sheaves (74) in Proposition 2. As we will see later, the long exact sequence of their cohomology groups answers our two questions. In this subsection and in the next subsection we devote ourselves to establishing Proposition 2.

Let TZ denote the tangent bundle of the curve Z. Since the curve Z has a G-action, denoted by $L_g : Z \to Z$ for $g \in G$, its tangent bundle TZ also has the canonical G-action defined as follows:

$$g\left(\mathcal{V}_p\right) := \left(L_g\right)_*\left(\mathcal{V}_p\right) \quad \in T_{gp}Z \quad (\text{for } g \in G \text{ and for } \mathcal{V}_p \in T_pZ). \tag{66}$$

Then the bundle TZ is a G-equivariant bundle with this G-action.

And let $\pi^*(TX)$ denote the pull-back bundle of TX by the map π. The G-action on the curve Z also lifts canonically to the bundle $\pi^*(TX)$, and it becomes the G-equivariant bundle. For a holomorphic vector bundle E, we denote the sheaf of local holomorphic sections of the bundle by $\mathcal{O}(E)$. The differential $d\pi$ of the map $Z \xrightarrow{\pi} X$ induces the following homomorphism of sheaves:

$$\mathcal{O}(TZ) \xrightarrow{d\pi} \mathcal{O}(\pi^*(TX)). \tag{67}$$

It is not difficult to check that the homomorphism of sheaves above is injective. Let Coker $(d\pi)$ denote the cokernel of the homomorphism of sheaves above, then we obtain the following exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}(TZ) \xrightarrow{d\pi} \mathcal{O}(\alpha^*(TX)) \longrightarrow \mathrm{Coker}(d\pi) \longrightarrow 0. \tag{68}$$

Next we investigate the direct image sheaves of the three sheaves above. Since the functor which correspond a sheaf $\mathcal{A}$ on Z to its direct image sheaf $\pi_*\mathcal{A}$ on X is exact for a finite branched covering space $Z \xrightarrow{\pi} X$, we obtain the following exact sequence of sheaves:

$$0 \longrightarrow \pi_*\mathcal{O}(TZ) \longrightarrow \pi_*\mathcal{O}(\pi^*(TX)) \longrightarrow \pi_*\,\mathrm{Coker}(d\pi) \longrightarrow 0. \tag{69}$$

Since the two vector bundle TZ and $\pi^*(TX)$ are G-equivariant bundles, the direct image sheaves $\pi_*\mathcal{O}(TZ)$ and $\pi_*\mathcal{O}(\pi^*(TX))$ have G-action, that is, for any open set $U \subset X$, the module of the local sections on U of each direct image sheaf is a G-module.

Since the homomorphism of vector bundles

$$TZ \xrightarrow{d\pi} \pi^*(TX) \tag{70}$$

commutes with the G-action, the homomorphism of sheaves (69) also commutes with the G-action.

Since every representation of a finite group G over the field of complex numbers $\mathbf{C}$ is completely reducible, and therefore, since the functor which correspond a sheaf $\mathcal{B}$ which is defined over $\mathbf{C}$ and which is endowed with a finite group G-action to the sub-sheaf $\mathcal{B}_G$ of G-invariant sections is exact, we obtain the following exact sequence of sheaves:

$$0 \to (\pi_*\mathcal{O}(TZ))_G \to (\pi_*\mathcal{O}(\pi^*(TX)))_G \to (\pi_*\,\mathrm{Coker}(d\pi))_G \to 0 \tag{71}$$

Here above it is not difficult to check that the following identity holds:

$$(\pi_*\mathcal{O}(\pi^*(TX)))_G \cong \mathcal{O}(TX). \tag{72}$$

Moreover, we claim the following proposition:

Proposition 2. *In the exact sequence of sheaves* (71), *the identity*

$$(\pi_* \operatorname{Coker}(d\pi))_G \cong \bigoplus_i T_{b_i}X \tag{73}$$

holds, where $\bigoplus_i T_{b_i}X$ *denotes the skyscraper sheaf which has non-zero stalk* $T_{b_i}X$ *only on the branch points* $b_i \in X$ $(i = 1, 2, \ldots, m)$. *Therefore we obtain the following exact sequence of sheaves by using* (71)*:*

$$0 \longrightarrow (\pi_* \mathcal{O}(TZ))_G \longrightarrow \mathcal{O}(TX) \longrightarrow \bigoplus_i T_{b_i}X \longrightarrow 0. \tag{74}$$

The proof is rather long. We write it in the following Section 2.2.

2.2 Proof of Proposition 2

In order to prove Proposition 2, we observe the homomorphism of sheaves

$$\begin{array}{ccccc} 0 & \longrightarrow & (\pi_* \mathcal{O}(TZ))_G & \xrightarrow{d\pi} & (\pi_* \mathcal{O}(\pi^*(TX)))_G \\ & & & & \| \\ & & & & \mathcal{O}(TX) \end{array} \tag{75}$$

at stalk lèvel. Since the homomorphism of vector bundles (70) is an fiber isomorphism on each point $p \in Z$ which is not a ramification point, the homomorphism of the stalks on such a point $p \in X$ is an isomorphism:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\pi_* \mathcal{O}(TZ))_{G,p} & \underset{\cong}{\xrightarrow{d\pi}} & (\pi_* \mathcal{O}(\pi^*(TX)))_{G,p} & \longrightarrow & 0. \\ & & & & \| & & \\ & & & & \mathcal{O}(TX)_p & & \end{array} \tag{76}$$

Next we observe the homomorphism of the stalks on a branch point $b \in X$:

$$\begin{array}{ccccc} 0 & \longrightarrow & (\pi_* \mathcal{O}(TZ))_{G,b} & \longrightarrow & (\pi_* \mathcal{O}(\pi^*(TX)))_{G,b} \,. \\ & & & & \| \\ & & & & \mathcal{O}(TX)_b \end{array} \tag{77}$$

For a branch point $b \in X$, we set the corresponding ramification points

$$\pi^{-1}(b) = \{r_1, r_2, \ldots, r_M\} \subset Z. \tag{78}$$

We denote the isotropy subgroup at a ramification point $r_i \in Z$ by $H_i \subset G$. Each subgroup H_i is isomorphic to the cyclic group $\mathfrak{C}_n$ $(nM = \operatorname{Ord} G)$ and is conjugate to each other in the group G.

2.2.1 Local property of cyclic coverings

For the purpose of investigating the homomorphism of the stalks (77), first we observe the following homomorphism of sheaves on X at the stalks on a ramification point $r_1 \in Z$:

$$0 \longrightarrow \mathcal{O}(TZ)_{r_1} \longrightarrow \mathcal{O}(\pi^*(TX))_{r_1}. \tag{79}$$

Both of the stalks $\mathcal{O}(TZ)_{r_1}, \mathcal{O}(\pi^*(TX))_{r_1}$ are H_1-vector spaces. The following proposition holds for the H_1-invariant subspaces of them:

Proposition 3. (*local property of cyclic coverings*). *The following identity holds:*

$$\begin{tikzcd} 0 \arrow[r] & \left(\mathcal{O}(TZ)_{r_1}\right)_{H_1} \arrow[r, "d\pi"] & \left(\mathcal{O}(\pi^*(TX))_{r_1}\right)_{H_1} \arrow[r] \arrow[d, equal] & T_bX \arrow[r] & 0. \\ & & \mathcal{O}(TX)_b & & \end{tikzcd} \tag{80}$$

Proof. We use a local holomorphic coordinate function z around the ramification point $r_1 \in Z$ which satisfies the following properties:

(1) There is a local holomorphic coordinate function x around the branched point $b = \pi(r_1) \in X$ satisfying $x(b) = 0$ and satisfying $\pi^*(x) = z^n$. (Consequently $z(r_1) = 0$.)

(2) The generator $\sigma \in H(\cong \mathfrak{C}_n)$ acts on the local coordinate function z via pull-back of function as follows:

$$\sigma(z) = \zeta z \quad (\zeta := e^{2\pi i/n}). \tag{81}$$

By using the local coordinate system above, first we observe the action of H_1 on $\mathcal{O}(TZ)_{r_1}$ and we determin its H_1-invariant subspace. The action of $\sigma \in \mathfrak{C}_n$ on the base element $\frac{\partial}{\partial z} \in \mathcal{O}(TZ)_{r_1}$ is computed as follows:

$$\sigma\left(\frac{\partial}{\partial z}\right) = \zeta^{-1}\frac{\partial}{\partial z}. \tag{82}$$

Therefore, for an convergent power series $f = f(z) \in \mathcal{O}_{Z,r_1}$, the generator $\sigma \in \mathfrak{C}_n$ acts on the element $v = f\frac{\partial}{\partial z} \in \mathcal{O}(TZ)_{r_1}$ as follows:

$$\sigma\left(f\frac{\partial}{\partial z}\right) = (\sigma f)\zeta^{-1}\frac{\partial}{\partial z}. \tag{83}$$

Hence the element $v = f\frac{\partial}{\partial z} \in \mathcal{O}(TZ)_{r_1}$ is invariant under the action of $\sigma \in \mathfrak{C}_n$ if and only if f satisfies

$$\sigma(f) = \zeta f. \tag{84}$$

Then f can be written as

$$f = z\pi^*(g) \tag{85}$$

with some convergent power series $g = g(x) \in \mathcal{O}_{X,b}$. Consequently, we see that an element $v \in \mathcal{O}(TZ)_{r_1}$ is H_1-invariant if and only if it can be written as

$$v = z\pi^*(g)\frac{\partial}{\partial z} \tag{86}$$

with some convergent power series $g = g(x) \in \mathcal{O}_{X,b}$.

Next we observe the image of the homomorphism $\mathcal{O}(TZ)_{r_1} \xrightarrow{d\pi} \mathcal{O}(\pi^*(TX))_{r_1}$. The image of the base element $\frac{\partial}{\partial z} \in \mathcal{O}(TZ)_{r_1}$ is computed as follows:

$$d\pi\left(\frac{\partial}{\partial z}\right) = nz^{n-1}\pi^*\left(\frac{\partial}{\partial x}\right). \tag{87}$$

Therefore the image of a H_1-invariant element (86) is computed as follows:

$$d\pi(v) = nz^n\pi^*(g)\pi^*\left(\frac{\partial}{\partial x}\right) \tag{88}$$

$$= n\pi^*\left(xg\frac{\partial}{\partial x}\right). \tag{89}$$

Thus the image of $\left(\mathcal{O}(TZ)_{r_1}\right)_{H_1} \xrightarrow{d\pi} \mathcal{O}(TX)_b$ coincides with the submodule $M_1 \subset \mathcal{O}(TX)_b$ which consists of all sections which take zero at the point $b \in X$. Since the identity

$$\mathcal{O}(TX)_b / M_1 \cong T_bX \tag{90}$$

holds, the cokernel of

$$\left(\mathcal{O}(TZ)_{r_1}\right)_{H_1} \xrightarrow{d\pi} \mathcal{O}(TX)_b \tag{91}$$

is canonically isomorphic to T_bX, which is our desired result.

This completes the proof of Proposition 3. □

2.2.2 Induced representation

Now we return to the observation of the homomorphism of the following sheaves at stalk level:

$$\begin{array}{ccccc} 0 & \longrightarrow & (\pi_*\mathcal{O}(TZ))_G & \longrightarrow & (\pi_*\mathcal{O}(\pi^*(TX)))_G\,. \\ & & & & \| \\ & & & & \mathcal{O}(TX) \end{array} \tag{92}$$

In what follows, we prove that the following exact sequence of the stalks holds for a branch point $b \in X$:

$$0 \longrightarrow (\pi_*\mathcal{O}(TZ))_{G,b} \longrightarrow \mathcal{O}(TX)_b \longrightarrow T_bX \longrightarrow 0. \tag{93}$$

There are the following direct sum decompositions of the modules:

$$\pi_* \mathcal{O}(TZ)_b \cong \bigoplus_{j=1}^{M} \mathcal{O}(TZ)_{r_j}, \tag{94}$$

$$\pi_* \mathcal{O}(\pi^*(TX))_b \cong \bigoplus_{j=1}^{M} \mathcal{O}(\pi^*(TX))_{r_j}. \tag{95}$$

As we will see below, the modules above are isomorphic to the G-modules induced by the H_1-modules. The stalk $\mathcal{O}(TZ)_{r_1}$ of the sheaf $\mathcal{O}(TZ)$ on the curve Z is a H_1-vector space. The stalk $\pi_* \mathcal{O}(TZ)_b$ is a G-vector space induced by the H_1-vector space $\mathcal{O}(TZ)_{r_1}$. To be more precise, it is not difficult to check that there exists the canonical isomorphism

$$\pi_* \mathcal{O}(TZ)_b \cong \mathbf{C}G \otimes_{\mathbf{C}H_1} \mathcal{O}(TZ)_{r_1}, \tag{96}$$

where $\mathbf{C}G$ (resp. $\mathbf{C}H_1$) denotes the group algebra of G (resp. H_1) over the field of complex numbers $\mathbf{C}$. The same argument is true for the stalk of the sheaf $\mathcal{O}(\pi^*(TX))$. Then we also obtain the following isomorphism:

$$\pi_* \mathcal{O}(\pi^*(TX))_b \cong \mathbf{C}G \otimes_{\mathbf{C}H_1} \mathcal{O}(\pi^*(TX))_{r_1}. \tag{97}$$

By taking the G-invariant subspaces of the identities (96) and (97), we obtain the following identities:

$$(\pi_* \mathcal{O}(TZ)_b)_G \cong \left(\mathcal{O}(TZ)_{r_1}\right)_{H_1}, \tag{98}$$

$$(\pi_* \mathcal{O}(\pi^*(TX))_b)_G \cong \left(\mathcal{O}(\pi^*(TX))_{r_1}\right)_{H_1} \cong \mathcal{O}(TX)_b. \tag{99}$$

Then we obtain the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\pi_* \mathcal{O}(TZ)_b)_G & \xrightarrow{d\pi} & (\pi_* \mathcal{O}(\pi^*(TX))_b)_G = \mathcal{O}(TX)_b & \longrightarrow & T_bX & \longrightarrow & 0 \\
 & & \Vert & & \Vert & & \Vert & & \\
0 & \longrightarrow & \left(\mathcal{O}(TZ)_{r_1}\right)_{H_1} & \xrightarrow{d\pi} & \left(\mathcal{O}(\pi^*(TX))_{r_1}\right)_{H_1} = \mathcal{O}(TX)_b & \longrightarrow & T_bX & \longrightarrow & 0
\end{array} \tag{100}$$

The exact sequence of the upper row in the diagram above, which is our desired result, is obtained by the exact sequence of the lower row, which we already obtained in Proposition 3.

In this way, we obtain the following exact sequence for a branch point $b \in X$:

$$0 \longrightarrow (\pi_* \mathcal{O}(TZ))_{G,b} \longrightarrow \mathcal{O}(TX)_b \longrightarrow T_bX \longrightarrow 0. \tag{101}$$

Hence $(\pi_* \operatorname{Coker}(d\pi))_G$ in the exact sequence of sheaves (71) is canonically isomorphic to the skyscraper sheaf $\bigoplus_i T_{b_i}X$.

This completes the proof of Proposition 2.

2.3 G-invariant deformation of Z

The exact sequence of sheaves (74) in Proposition 2 induces the following exact sequence of their cohomology groups:

$$0 \longrightarrow H^0(X, (\pi_*\mathcal{O}(TZ))_G) \longrightarrow H^0(X, \mathcal{O}(TX)) \longrightarrow \bigoplus_i T_{b_i}X \xrightarrow{\ \delta\ } H^1(X, (\pi_*\mathcal{O}(TZ))_G) \longrightarrow H^1(X, \mathcal{O}(TX)) \longrightarrow 0. \tag{102}$$

Now we explain the fact that the cohomology group $H^j(X, (\pi_*\mathcal{O}(TZ))_G)$ $(j = 0, 1)$ is isomorphic to the cohomology group $H^j(Z, \mathcal{O}(TZ))_G$. Since the finite group G is acting on the curve Z, the group also acts on the cohomology groups $H^j(Z, \mathcal{O}(TZ))$ $(j = 0, 1)$. Then the finite group G has linear representations on the cohomology groups $H^j(Z, \mathcal{O}(TZ))$. We denote the subspace of $H^j(Z, \mathcal{O}(TZ))$ which consists of all G-invariant elements by $H^j(Z, \mathcal{O}(TZ))_G$. Since every representation of a finite group over the field of complex numbers $\mathbf{C}$ is completely reducible, the following identity holds:

$$H^j(X, (\pi_*\mathcal{O}(TZ))_G) \cong H^j(X, (\pi_*\mathcal{O}(TZ)))_G. \tag{103}$$

Furthermore, since the cohomology groups of a sheaf $\mathcal{C}$ on Z and those of its direct image sheaf $\pi_*\mathcal{C}$ are isomorphic for a finite branched covering space $Z \xrightarrow{\pi} X$, the following identity holds:

$$H^j(X, (\pi_*\mathcal{O}(TZ)))_G \cong H^j(Z, \mathcal{O}(TZ))_G. \tag{104}$$

Finally we obtain the following identity which we mentioned above:

$$H^j(X, (\pi_*\mathcal{O}(TZ))_G) \cong H^j(Z, \mathcal{O}(TZ))_G \quad (j = 0, 1). \tag{105}$$

We denote the subspace of all G-invariant element in $\Lambda^{0,*}(Z, TZ)$ by $\Lambda^{0,*}(Z, TZ)_G$. Since the Dolbaut complex $\Lambda^{0,*}(Z, TZ)$ is a $C^\infty(X)$-module and it defines a fine resolution of the sheaf $(\pi_*\mathcal{O}(TZ))$, it is not difficult check, by using the fact that every representation of a finite group G over $\mathbf{C}$ is completely reducible, that the complex $\Lambda^{0,*}(Z, TZ)_G$ defines a fine resolution of the sheaf $(\pi_*\mathcal{O}(TZ))_G$. The identity (105) can be obtained also by using this fine resolution.

In what follows we identify $H^1(X, (\pi_*\mathcal{O}(TZ))_G)$ with $H^1(Z, \mathcal{O}(TZ))_G$. At that time, we can rewrite the exact sequence of cohomology groups (102) as follows:

$$0 \longrightarrow H^0(Z, \mathcal{O}(TZ))_G \longrightarrow H^0(X, \mathcal{O}(TX)) \longrightarrow \bigoplus_i T_{b_i}X \xrightarrow{\ \delta\ } H^1(Z, \mathcal{O}(TZ))_G \xrightarrow{\ \jmath_X\ } H^1(X, \mathcal{O}(TX)) \longrightarrow 0. \tag{106}$$

In the following Sections 2.3.1 and 2.3.2, we explain how the elements of $H^1(Z, \mathcal{O}(TZ))_G$ are related to the G-invariant deformations of the curve Z.

2.3.1 Deformation

We denote the almost complex structures of the curves X, Z by J_X, J_Z respectively. Let $J_Z(t)$ denote a smooth family of G-invariant almost complex structures on the curve Z depending on a real parameter $t \in (-\varepsilon, \varepsilon)$ satisfying the following Condition Σ:

(Condition Σ)

The initial value satisfies $J_Z(0) = J_Z$ and $J_Z(t)$ identically equals the original almost complex structures J_Z on some neighborhood of the ramification points $r_1, r_2, \ldots, r_N \in Z$ of the Galois covering space $Z \xrightarrow{\pi} X$.

Since every almost complex structure is integrable on the complex manifold of dimension one, and therefore, since it defines a complex structure of the manifold, the smooth family of G-invariant almost complex structures $J_Z(t)$ above defines a *deformation of the complex structure* of the curve Z.

For a point $p \in Z$ which is not a ramification point of the Galois covering space $Z \xrightarrow{\pi} X$, the homomorphism of tangent spaces below is isomorphism:

$$d\pi : \ T_p Z \xrightarrow{\cong} T_{\pi(p)} X. \tag{107}$$

Therefore we can regard $J_Z(t)$ also as a smooth family of almost complex structures $J_X(t)$ on the curve X by setting

$$J_X(t) := d\pi \circ J_Z(t) \circ (d\pi)^{-1} \quad \text{on } T_{\pi(p)} X. \tag{108}$$

Since $J_X(t)$ defined above identically equals the original almost complex structure J_X on some neighborhood of the branch points $b_i \in X$ by the property of Condition Σ, it defines a smooth family of almost complex structures on the curve X. In this way a G-invariant deformation of the complex structure of the curve Z satisfying Condition Σ induces a deformation of that of the curve X.

2.3.2 Infinitesimal deformation

For $J_Z(t)$, a smooth family of G-invariant almost complex structures on the curve Z depending on a real parameter $t \in (-\varepsilon, \varepsilon)$ satisfying Condition Σ, we set

$$\eta_Z := \frac{d}{dt} J_Z(t) \Big|_{t=0} \tag{109}$$

and we can regard η_Z as an element of $\Lambda^{0,1}(Z, TZ)_G$. The Dolbaut form $\eta_Z \in \Lambda^{0,1}(Z, TZ)_G$ is G-invariant and satisfies the following Condition $\tilde{\Sigma}$:

(Condition $\tilde{\Sigma}$)

η_Z vanishes on some neighborhood of the ramification points $r_1, r_2, \ldots, r_N \in Z$ of the Galois covering space $Z \xrightarrow{\pi} X$.

Needless to say, the homomorphism of cotangent spaces is also an isomorphism for each point $p \in Z$ which is not a ramification point

$$d\pi^* : T^*_{\pi(p)}X \overset{\cong}{\to} T^*_p Z. \tag{110}$$

Therefore we can apply a similar argument as with $J_Z(t)$ to the Dolbaut form η_Z satisfying Condition $\tilde{\Sigma}$, to see that we can regard η_Z also as an element of $\eta_X \in \Lambda^{0,1}(X, TX)$, by using the isomorphisms of the tangent spaces and those of cotangent spaces. Then it is not difficult to check that the following diagram commutes:

$$\begin{array}{ccc} H^1(Z, \mathcal{O}(TZ))_G & \xrightarrow{\mathfrak{J}_X} & H^1(X, \mathcal{O}(TX)) \\ \uparrow & & \uparrow \\ \eta_Z \in \Lambda^{0,1}(Z, TZ)_G & \longrightarrow & \eta_X \in \Lambda^{0,1}(X, TX) \end{array} \tag{111}$$

As we described in Section 2.3.1, a smooth family of G-invariant almost complex structures on the curve Z satisfying Condition Σ induces that of the curve X, denoted by $J_X(t)$. It is not difficult to check that the following identity holds:

$$\left.\frac{d}{dt}J_X(t)\right|_{t=0} = \eta_X. \tag{112}$$

We should note that, for any $\eta_Z \in \Lambda^{0,1}(Z, TZ)_G$ satisfying Condition $\tilde{\Sigma}$, we can construct, conversely, a smooth family of G-invariant almost complex structures $J_Z(t)$ depending on a real parameter $t \in (-\varepsilon, \varepsilon)$ satisfying Condition Σ, as we see below.

We can regard the value of η_Z at a point $p \in Z$ as an endomorphism on the tangent space T_pZ which we denote by $A_p \in \mathrm{End}\left(T_pZ\right)$. The endomorphism A_p satisfies

$$J_{Zp}A_p + A_pJ_{Zp} = 0. \tag{113}$$

Set

$$J_Z(t)_p := J_{Zp}e^{-J_{Zp}A_pt} \in \mathrm{End}\left(T_pZ\right). \tag{114}$$

Then $J_Z(t)$ is our desired one. In this way a Dolbaut form $\eta_Z \in \Lambda^{0,1}(Z, TZ)_G$ satisfying Condition $\tilde{\Sigma}$ generates a smooth family of G-invariant almost complex structures $J_Z(t)$ depending a real parameter $t \in (-\varepsilon, \varepsilon)$ satisfying Condition Σ.

Furthermore, for any class $a \in H^1(Z, \mathcal{O}(TZ))_G$, we can find a G-invariant Dolbaut form $\eta_Z \in \Lambda^{0,1}(Z, TZ)_G$ which satisfies

$$[\eta_Z] = a \tag{115}$$

and which satisfies Condition $\tilde{\Sigma}$ by the following Lemma 1:

Lemma 1. (*representative satisfying Condition $\tilde{\Sigma}$*).

(1) *Let C be a curve and E a holomorphic vector bundle over C. The following fact holds for the cohomology group $H^1(C, \mathcal{O}(E))$. For a given cohomology class $a \in H^1(C, \mathcal{O}(E))$, and for given finite points $r_1, r_2, \ldots, r_N \in C$, there exists a Dolbaut form $\xi \in \Lambda^{0,1}(C, \mathcal{O}(E))$ which is a representative of the given class a and which is identically zero on some neighborhood of the given points $\{r_1, r_2, \ldots, r_N\} \subset C$.*

(2) *Assume further in the situation above that the curve C is endowed with a finite group G-action so that the quotient map $C \to C/G$ should be a Galois covering space with Galois group G and with the ramification points $r_1, r_2, \ldots, r_N \in C$. And assume that the holomorphic vector bundle E is a G-equivariant bundle over the curve C. Then for any G-invariant Dolbaut form $\eta \in \Lambda^{0,1}(C, \mathcal{O}(E))_G$, we can find a G-invariant Dolbaut form $\xi \in \Lambda^{0,1}(C, \mathcal{O}(E))_G$ which defines the same cohomology class as η in $H^1(C, \mathcal{O}(E))_G$ and which is identically zero on some neighborhood of the ramification points $\{r_1, r_2, \ldots, r_N\} \subset C$.*

Proof. We prove that, for a given Dolbaut form $\eta \in \Lambda^{0,1}(C, E)$, there exists a Dolbaut form $\xi \in \Lambda^{0,1}(C, E)$ which defines the same cohomology class as η and which is identically zero on some neighborhood of the given points $\{r_1, r_2, \ldots, r_N\} \subset C$.

Let W_i be a small disk the center of which is each given point $r_i \in C$. We may assume that the holomorphic vector bundle E restricted to each W_i is holomorphically trivial. Then, by the Dolbaut lemma (or $\bar{\partial}$-Poincaré lemma, c.f. [3]), there exists $s_i \in \Lambda^{0,0}(W_i, E)$ which satisfies

$$\eta|_{W_i} = \bar{\partial} s_i. \tag{116}$$

By taking a smaller disk $V_i \subset W_i$ for each point r_i if necessary, we may assume that there exists a C^∞ global section $s \in \Lambda^{0,0}(C, E)$ the restriction of which to each disk V_i coincides with s_i above:

$$s|_{V_i} = s_i|_{V_i}. \tag{117}$$

Then the element

$$\xi := \eta - \bar{\partial} s \in \Lambda^{0,1}(C, E) \tag{118}$$

takes zero on all V_i's and defines the same class as η in the cohomology group $H^1(C, \mathcal{O}(E))$, which is our desired one. This completes the proof of (1).

Assume that the curve C is endowed with an finite group G-action which satisfies the conditions in (2). Then, after averaging by the G-action, we may assume that the section $s \in \Lambda^{0,0}(Z, TZ)_G$ above is G-invariant, which assures that ξ above is also G-invariant. This completes the proof of (2). □

2.4 Theorem 1

Now we answer the two questions which we posed in the beginning of this section— how the moduli of the complex structure of the total space Z varies when we translate the branch points b_i $(i = 1, 2, \ldots, m)$ on X, and how the moduli of the complex structure of the base space

$X = Z/G$ varies when we deform the complex structure of the total space Z in a G-invariant way.

A translations of the branch points b_i $(i = 1, 2, \ldots, m)$ on X is understood as to give a section of the skyscraper sheaf $\bigoplus_i T_{b_i}X$, in other words, to give tangent vectors

$$v_i \in T_{b_i}X. \tag{119}$$

And a G-invariant infinitesimal deformation of the curve Z is understood as to give an element of $H^1(Z, TZ)_G$.

We state the following theorem:

Theorem 1. (1) *The infinitesimal change of the moduli of the complex structure of the total space Z induced by a translation of the branch points $b_i \in X$ coincides with the connecting homomorphism*

$$\bigoplus_i T_{b_i}X \xrightarrow{\delta} H^1(Z, \mathcal{O}(TZ))_G \tag{120}$$

of the exact sequence of cohomology groups (106) (*or equivalently,* (102)).

(2) *An element of $H^1(Z, \mathcal{O}(TZ))_G$ is an infinitesimal deformation of the moduli of the complex structure of total space Z which is compatible with the G-action. And the corresponding infinitesimal change of the moduli of the complex structure of the base space X is described by the homomorphism*

$$H^1(Z, \mathcal{O}(TZ))_G \xrightarrow{\mathfrak{I}_X} H^1(X, \mathcal{O}(TX)). \tag{121}$$

Proof. For given $v_i \in T_{b_i}X$'s, we construct $J_Z(t)$, a smooth family of almost complex structure of the curve Z, and a smooth family of Galoîs covering spaces

$$Z(t) := (Z, J_Z(t)) \xrightarrow{\pi_t} X \tag{122}$$

depending on a real parameter $t \in (-\varepsilon, \varepsilon)$ which satisfies

$$\left.\frac{d}{dt} b_i(t)\right|_{t=0} = v_i \in T_{b_i}X \tag{123}$$

for the branch points $b_i(t)$.

First we can find an element $\mathcal{V} \in \Lambda^{0,0}(X, TX)$ which satisfies the following conditions:

(1) The value of $\mathcal{V}$ at each branch point $b_i \in X$ equals the given vector v_i.

$$\mathcal{V}_{b_i} = v_i \in T_{b_i}X \tag{124}$$

(2) For each branch point $b_i \in X$, there exists an open neighborhood U_i of the point on which $\mathcal{V}$ is holomorphic.

We can regard $\mathcal{V} \in \Lambda^{0,0}(X, TX)$ as a real vector field $\mathcal{V} \in \mathfrak{X}(X)$, and we denote the one-parameter transformation group generated by the vector field by $\varphi_t : X \to X$. The composite of the maps

$$\pi_t := \varphi_t \circ \pi : Z \to X \tag{125}$$

is G-invariant and it is a C^∞ Galois covering space. The ramification points of the Galois covering space (125) are $r_1, r_2, \ldots, r_N \in Z$, which are not changed from those of the Galois covering space $Z \xrightarrow{\pi} X$. And the branch points of the Galois covering space (125) are $\varphi_t(b_1), \varphi_t(b_2), \ldots, \varphi_t(b_n)$. It should be noted that, for sufficiently small $t \in (-\varepsilon, \varepsilon)$, the map π_t is holomorphic on some neighborhood of each ramification point $r_i \in Z$.

For each point $p \in Z$ which is not a ramification point of the Galois covering space $Z \xrightarrow{\pi} X$, the homomorphism of the tangent spaces

$$T_pZ \xrightarrow{d\pi_t} T_{\pi(p)}X \tag{126}$$

is an isomorphism. Therefore, for such points, we can "pull-back" the almost complex structure J_X on $T_{\pi(p)}X$ to T_pZ by setting

$$J_Z(t) := (d\pi_t)^{-1} \circ J_X \circ d\pi_t. \tag{127}$$

Since the map π_t is holomorphic on some neighborhood of the ramification points, the almost complex structure $J_Z(t)$ defined above coincides with the original almost complex structure on such neighborhoods. Hence $J_Z(t)$ defines a smooth family of G-invariant almost complex structures on the curve Z satisfying Condition Σ. We denote the curve Z with almost complex structure $J_Z(t)$ by $Z(t)$. The projection

$$Z(t) \xrightarrow{\pi_t} X \tag{128}$$

is a smooth family of holomorphic Galois covering spaces depending on a real parameter $t \in (-\varepsilon, \varepsilon)$.

Set

$$\eta_Z := \left.\frac{d}{dt} J_Z(t)\right|_{t=0}, \tag{129}$$

and we can identify η_Z as an element of $\Lambda^{0,1}(Z, TZ)_G$ satisfying Condition $\tilde{\Sigma}$. The Dolbaut form η_Z is determined by $\mathcal{V} \in \Lambda^{0,0}(X, TX)$ via the following procedure. We set

$$\eta_X := \bar{\partial}\mathcal{V} \in \Lambda^{0,1}(X, TX), \tag{130}$$

and we can identify η_X with η_Z via the isomorphisms of the tangent spaces and the cotangent spaces which we mentioned before. The procedure above coincides with the definition of the connecting homomorphism

$$\bigoplus_i T_{b_i}X \xrightarrow{\delta} H^1(X, (\pi_* \mathcal{O}(TZ))_G) \tag{131}$$

of the exact sequence of the cohomology groups. This completes the proof of the assertion (1) in Theorem 1.

The assertion (2) in Theorem 1 has already been proved. □

Remark 1. For the case where X is the projective space P^1, the exact sequence of cohomology groups (106) becomes the following simpler one:

$$\begin{tikzcd}
 & H^1(Z, \mathcal{O}(TZ))_G \arrow[rr, "\mathfrak{I}_X"] & & 0. \\
0 \arrow[r] & H^0(Z, \mathcal{O}(TZ))_G \arrow[r] & H^0\left(P^1, \mathcal{O}\left(TP^1\right)\right) \arrow[r] & \bigoplus_i T_{b_i}P^1 \arrow[ull, bend right, "\delta"']
\end{tikzcd} \tag{132}$$

And we know

$$\dim \mathrm{Lie}\left(\mathrm{Aut}\left(P^1\right)\right) = \dim H^0\left(P^1, \mathcal{O}\left(TP^1\right)\right) \tag{133}$$

$$= 3. \tag{134}$$

The exact sequence (132) will be used later in Section 4 to compute the dimension of the vector space $H^1(Z, \mathcal{O}(TZ))_G$.

3 Further investigation and main results

In the previous section, we discussed the infinitesimal deformations of a Galois covering space

$$Z \xrightarrow{\pi} X \tag{135}$$

with Galois group G, and we established Theorem 1. Under the situation above, for a subgroup $H \subset G$, the curve $Y := Z/H$ is a compact smooth curve and the quotient map

$$Z \xrightarrow{\beta} Y \tag{136}$$

is a Galois covering with Galois group H. At that time we can construct the unique branched covering map

$$Y \xrightarrow{\alpha} X \tag{137}$$

which satisfies $\alpha \circ \beta = \pi$.

In this section we investigate the infinitesimal deformations of the Galois covering spaces (135) and (136). We will see that an infinitesimal deformation of the Galois covering spaces (135) which preserves the moduli of the curves X, Y induces an infinitesimal deformation of the branched covering map $Y \xrightarrow{\alpha} X$ which is defined above. This procedure leads us to Theorem 2, which answers the "inverse problem"—how the branched covering map α varies as we deform the complex structure of the curve Z in a G-invariant way in which the moduli of the curves X, Y should not be changed— which we mentioned in Introduction. And, in particular, in the case where the Galois covering spaces (135) and (136) are the Galois closure of the branched covering map $Y \xrightarrow{\alpha} X$, we obtain Theorem 3. And, furthermore, if the branched covering map α is of *general ramification type*, we obtain Theorem 4. By using these theorems, we finally obtain the Kodaira-Spencer map of our primary concern.

3.1 Vector space of infinitesimal deformations of maps

First we discuss how we deal with the infinitesimal deformations of the branched covering map $Y \xrightarrow{\alpha} X$ in this subsection.

For a map $\alpha \in \mathrm{Map}_d(Y, X)$, as in Introduction, we call a section of pull-back bundle $\alpha^*(TX)$

$$s \in H^0(Y, \mathcal{O}(\alpha^*(TX))) \tag{138}$$

an *infinitesimal deformation* of map α. The mapspace $\mathrm{Map}_d(Y, X)$ has a complex space structure. And if the condition

$$H^1(Y, \mathcal{O}(\alpha^*(TX))) = 0 \tag{139}$$

holds, the mapspace is non-singular at the point $\alpha \in \mathrm{Map}_d(Y, X)$ (c.f. [6]). (Note that the condition (139) is a *sufficient* condition for the mapspace to be non-singular. It is not a *necessary* condition.) All of the curves we will discuss later in Section 4 satisfy the condition (139). If the mapspace $\mathrm{Map}_d(Y, X)$ is non-singular at a point $\alpha \in \mathrm{Map}_d(Y, X)$, the tangent space of the mapspace at the point α can be identified with the cohomology group

$$H^0(Y, \mathcal{O}(\alpha^*(TX))). \tag{140}$$

At that time, in particular, there exists a smooth family of holomorphic maps of mapping degree d

$$f_t : Y \to X \tag{141}$$

depending on a real parameter $t \in (-\varepsilon, \varepsilon)$ satisfying

$$\left.\frac{d}{dt} f_t\right|_{t=0} = s \tag{142}$$

for an arbitrary $s \in H^0(Y, \mathcal{O}(\alpha^*(TX)))$.

As we mentioned in Introduction, the actions of the automorphism groups $\mathrm{Aut}(X)$, $\mathrm{Aut}(Y)$ on $\mathrm{Map}_d(Y, X)$ induce vector space homomorphism from the Lie algebra of those group above

$$\mathrm{Lie}(\mathrm{Aut}(X)) \cong H^0(X, \mathcal{O}(TX)) \tag{143}$$

$$\mathrm{Lie}(\mathrm{Aut}(Y)) \cong H^0(Y, \mathcal{O}(TY)) \tag{144}$$

to the vector space $H^0(Y, \alpha^*(TX))$. Each homomorphism is written as follows: The former is the pull-back of the holomorphic sections

$$\mathcal{P} : H^0(X, \mathcal{O}(TX)) \longrightarrow H^0(Y, \mathcal{O}(\alpha^*(TX))). \tag{145}$$

And the latter is the homomorphism of cohomology groups

$$\mathcal{Q} : H^0(Y, \mathcal{O}(TY)) \xrightarrow{d\alpha} H^0(Y, \mathcal{O}(\alpha^*(TX))), \tag{146}$$

induced by the homomorphism of sheaves

$$\mathcal{O}(TY) \xrightarrow{d\alpha} \mathcal{O}(\alpha^*(TX)). \tag{147}$$

As we will see later, the homomorphisms $\mathcal{P}$ and $\mathcal{Q}$ are both injective. We call an element of the quotient vector space

$$H^0(Y, \mathcal{O}(\alpha^*(TX))) / (\operatorname{Im}\mathcal{P} + \operatorname{Im}\mathcal{Q}). \tag{148}$$

an *equivalence class of infinitesimal deformation.* The vector space (148) has fundamental importance for our infinitesimal deformation theory.

Remark 2. We will investigate later in Section 4 families of Galois closure curves on such a base space that the tangent space of it is isomorphic to the vector space (148).

For later use, we investigate the homomorphisms $\mathcal{P}$ and $\mathcal{Q}$ as homomorphisms of cohomology groups in the following Sections 3.1.1 and 3.1.2.

3.1.1 Sheaf $\mathcal{S}_A$

First we define a sheaf $\mathcal{S}_A$ as follows. The pull-back operation of the local sections of the bundle TX to the bundle $\alpha^*(TX)$ defines the following homomorphism of sheaves on X:

$$\mathcal{O}(TX) \xrightarrow{\text{pull-back}} \alpha_*\mathcal{O}(\alpha^*(TX)). \tag{149}$$

It is not difficult to check that the homomorphism above is injective and then we define the sheaf $\mathcal{S}_A$ as the cokernel of the homomorphism above

$$\mathcal{S}_A := \operatorname{Coker}(\text{pull-back}: \ \mathcal{O}(TX) \to \alpha_*\mathcal{O}(\alpha^*(TX))). \tag{150}$$

The exact sequence of the sheaves

$$0 \longrightarrow \mathcal{O}(TX) \xrightarrow{\text{pull-back}} \alpha_*\mathcal{O}(\alpha^*(TX)) \longrightarrow \mathcal{S}_A \longrightarrow 0 \tag{151}$$

induces the following long exact sequence of their cohomology groups:

$$\begin{array}{l} 0 \longrightarrow H^0(X, \mathcal{O}(TX)) \xrightarrow{\mathcal{Q}'} H^0(X, \alpha_*\mathcal{O}(\alpha^*(TX))) \longrightarrow H^0(X, \mathcal{S}_A) \xrightarrow{\delta} \\ \longrightarrow H^1(X, \mathcal{O}(TX)) \longrightarrow H^1(X, \alpha_*\mathcal{O}(\alpha^*(TX))) \longrightarrow H^1(X, \mathcal{S}_A) \longrightarrow 0. \end{array} \tag{152}$$

Under the identification of the cohomology groups

$$H^j(X, \alpha_*\mathcal{O}(\alpha^*(TX))) \cong H^j(Y, \mathcal{O}(\alpha^*(TX))) \quad (j = 0, 1), \tag{153}$$

the homomorphism $\mathcal{Q}$ coincides with the homomorphism $\mathcal{Q}'$ in the long exact sequence of the cohomology groups above.

We discuss here a fine resolution of the sheaf $\mathcal{S}_A$. The Dolbaut complex $\Lambda^{0,*}(X, TX)$ is a $C^\infty(X)$-module and it defines a fine resolution of the sheaf $\mathcal{O}(TX)$. And the Dolbaut complex

$\Lambda^{0,*}(Y, \alpha^*(TX))$ is a $C^\infty(X)$-module and it defines a fine resolution of the sheaf $\alpha_* \mathcal{O}(\alpha^*(TX))$. Therefore we see that

$$\Lambda^{0,*}(Y, \alpha^*(TX)) \;/\; \Lambda^{0,*}(X, TX) \tag{154}$$

is a $C^\infty(X)$-module and that it defines a fine resolution of the sheaf $\mathcal{S}_A$, which we will use later in the proof of Theorem 2.

3.1.2 Sheaf $\mathcal{S}_{\text{jets}}$

Next we define a sheaf $\mathcal{S}_{\text{jets}}$ as follows. The differential $d\alpha$ of the map $Y \xrightarrow{\alpha} X$ defines the following homomorphism of vector bundles:

$$TY \xrightarrow{d\alpha} \alpha^*(TX). \tag{155}$$

This induces the following homomorphism of sheaves on Y:

$$\mathcal{O}(TY) \xrightarrow{d\alpha} \mathcal{O}(\alpha^*(TX)). \tag{156}$$

Moreover, this induces the following homomorphism of the direct image sheaves on X of them:

$$\alpha_*\mathcal{O}(TY) \xrightarrow{d\alpha} \alpha_*\mathcal{O}(\alpha^*(TX)). \tag{157}$$

It is not difficult to check that the homomorphism above is injective and then we define the sheaf $\mathcal{S}_{\text{jets}}$ as the cokernel of the homomorphism

$$\mathcal{S}_{\text{jets}} := \operatorname{Coker}(d\alpha : \ \alpha_*\mathcal{O}(TY) \to \alpha_*\mathcal{O}(\alpha^*(TX))). \tag{158}$$

As we see below, the sheaf $\mathcal{S}_{\text{jets}}$ is a skyscraper sheaf which has non-zero stalks only on the branch points $a_i \in X$ of the branched covering space $Y \xrightarrow{\alpha} X$. It is not difficult to check that for each point $p \in Y$ which is not a ramification point of $Y \xrightarrow{\alpha} X$, the homomorphism (156) is an isomorphism. And when the point $p \in Y$ is a ramification point of order k of the branch covering map $Y \xrightarrow{\alpha} X$, the homomorphism (156) of the stalk at the point $p \in Y$

$$\mathcal{O}(TY)_p \xrightarrow{d\alpha} \mathcal{O}(\alpha^*(TX))_p \tag{159}$$

is injective and the image of the homomorphism $d\alpha$ is the submodule of sections which takes zero of order $k-1$ at the point $p \in Y$, which we denote by M_{k-1}. Therefore the cokernel of the homomorphism (156) is a skyscraper sheaf which has non-zero stalks only on the ramification points $p_j \in Y$. And the stalk of a ramification point $p \in Y$ is the *jet* of order $k-1$ of the pull-back bundle $\alpha^*(TX)$ at the point. Namely

$$\mathcal{O}(\alpha^*(TX))_p \,/\, \operatorname{Im} d\alpha \cong j_p^{k-1}(\alpha^*(TX)) \tag{160}$$

$$:= \mathcal{O}(\alpha^*(TX))_p \,/ M_{k-1}. \tag{161}$$

Hence the sheaf $\mathcal{S}_{\text{jets}}$ is a skyscraper sheaf on X which has non-zero stalks only on the branch points $a_i \in X$. And the stalk at a branch point $a \in X$ of the branched covering space $Y \xrightarrow{\alpha} X$ is the direct sum of the modules (160) of the ramification points $p_1, p_2, \dots, p_K \in Y$ which satisfy $\alpha(p_i) = a$.

As we mentioned in Introduction, we define the following definition about branched covering maps:

Definition 1. A branched covering map $Y \xrightarrow{\alpha} X$ is of *general ramification type* if all the ramification points $p_j \in Y$ of the branched covering map are of order two, and there exist no p_i, p_j $(i \neq j)$ satisfying $\alpha(p_i) = \alpha\left(p_j\right)$.

One may see without difficulty that the following identity holds for a branched covering map $Y \xrightarrow{\alpha} X$ with branch points $a_i \in X$ which is of general ramification type:

$$\mathcal{S}_{\text{jets}} \cong \bigoplus_i T_{a_i}X. \tag{162}$$

By the definition of the sheaf $\mathcal{S}_{\text{jets}}$, the following exact sequence of sheaves holds:

$$0 \longrightarrow \alpha_* \mathcal{O}(TY) \xrightarrow{d\alpha} \alpha_* \mathcal{O}(\alpha^*(TX)) \longrightarrow \mathcal{S}_{\text{jets}} \longrightarrow 0. \tag{163}$$

The exact sequence of sheaves (163) induces a following exact sequence of their cohomology groups:

$$\begin{tikzcd}
0 \arrow[r] & H^0(X, \alpha_*\mathcal{O}(TY)) \arrow[r, "d\alpha"] & H^0(X, \alpha_*\mathcal{O}(\alpha^*(TX))) \arrow[r] & H^0\left(X, \mathcal{S}_{\text{jets}}\right) \arrow[dll, "\delta"'] \\
 & H^1(X, \alpha_*\mathcal{O}(TY)) \arrow[r] & H^1(X, \alpha_*\mathcal{O}(\alpha^*(TX))) \arrow[r] & 0.
\end{tikzcd} \tag{164}$$

Under the identification of the cohomology groups (153), the homomorphism Ω coincides with the homomorphism $d\alpha$ in the long exact sequence of cohomology groups (164).

3.2 Deformation of Z preserving the moduli of X, Y

In this subsection, first we observe how the complex structures of the curve X, Y vary when we deform the complex structure of the curve Z in a G-compatible way. The infinitesimal deformations of the complex structure of the curve Z are described by the cohomology group $H^1(Z, \mathcal{O}(TZ))$. And among them, the G-invariant ones are described by its G-invariant subspace $H^1(Z, \mathcal{O}(TZ))_G$. Next we investigate the subspaces $K_X \subset H^1(Z, \mathcal{O}(TZ))_G$ and $K_Y \subset H^1(Z, \mathcal{O}(TZ))_G$ the elements of which do not change the moduli of the curve X, Y respectively.

For the purpose to investigate the correspondence of the infinitesimal deformations of the complex structures of the curves X, Y, Z which we mentioned above, we study the following exact sequences of sheaves.

First, in order to investigate how the complex structures of the curve X varies, we observe the exact sequence

$$0 \longrightarrow (\pi_*\mathcal{O}(TZ))_G \longrightarrow \mathcal{O}(TX) \longrightarrow \bigoplus_i T_{b_i}X \longrightarrow 0 \tag{165}$$

which we established in Proposition 2 and the exact sequence of their cohomology groups

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,(\pi_*\mathcal{O}(TZ))_G) \arrow[r] & H^0(X,\mathcal{O}(TX)) \arrow[r] & \bigoplus_i T_{b_i}X \arrow[dll, "\delta"] \\
& H^1(X,(\pi_*\mathcal{O}(TZ))_G) \arrow[r, "\mathfrak{I}_X"] & H^1(X,\mathcal{O}(TX)) \arrow[r] & 0,
\end{tikzcd} \tag{166}$$

the properties of which we discussed in Theorem 1. For our purpose to obtain the infinitesimal deformations which do not change the moduli of the curve X, we define a vector space $K_X \subset H^1(X,(\pi_*\mathcal{O}(TZ))_G)$ as follows:

$$K_X := \mathrm{Ker}\left(H^1(X,(\pi_*\mathcal{O}(TZ))_G) \xrightarrow{\mathfrak{I}_X} H^1(X,\mathcal{O}(TX))\right) \tag{167}$$

$$= \mathrm{Im}\left(\bigoplus_i T_{b_i}X \xrightarrow{\delta} H^1(X,(\pi_*\mathcal{O}(TZ))_G)\right). \tag{168}$$

Recall that we pointed out in Section 2.3 that the following identification of cohomology groups holds:

$$H^1(X,(\pi_*\mathcal{O}(TZ))_G) \cong H^1(Z,\mathcal{O}(TZ))_G. \tag{169}$$

3.2.1 Sheaf $\mathcal{S}_\mathrm{B}$

Next, in order to investigate how the complex structures of the curve Y varies, we define a sheaf $\mathcal{S}_\mathrm{B}$ on X as follows:

$$\mathcal{S}_\mathrm{B} := \mathrm{Coker}\,(d\beta : (\pi_*\mathcal{O}(TZ))_G \to \alpha_*\mathcal{O}(TY)). \tag{170}$$

The exact sequence of sheaves

$$0 \longrightarrow (\pi_*\mathcal{O}(TZ))_G \xrightarrow{d\beta} \alpha_*\mathcal{O}(TY) \longrightarrow \mathcal{S}_\mathrm{B} \longrightarrow 0 \tag{171}$$

induces the following long exact sequence of their cohomology groups:

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,(\pi_*\mathcal{O}(TZ))_G) \arrow[r] & H^0(X,\alpha_*\mathcal{O}(TY)) \arrow[r] & H^0(X,\mathcal{S}_\mathrm{B}) \arrow[dll, "\delta"] & \\
& H^1(X,(\pi_*\mathcal{O}(TZ))_G) \arrow[r, "\mathfrak{I}_Y"] & H^1(X,\alpha_*\mathcal{O}(TY)) \arrow[r] & H^1(X,\mathcal{S}_\mathrm{B}) \arrow[r] & 0.
\end{tikzcd} \tag{172}$$

As is the same as the case with X, for our purpose to obtain the infinitesimal deformations which do not change the moduli of the curve Y, we define a vector space $K_Y \subset H^1(X, (\pi_*\mathcal{O}(TZ))_G)$ as follows:

$$K_Y := \mathrm{Ker}\Big(H^1(X, (\pi_*\mathcal{O}(TZ))_G) \xrightarrow{\mathcal{J}_Y} H^1(X, \mathcal{O}(TY))\Big) \tag{173}$$

$$= \mathrm{Im}\Big(H^0(X, \mathcal{S}_B) \xrightarrow{\delta} H^1(X, (\pi_*\mathcal{O}(TZ))_G)\Big). \tag{174}$$

We discuss here a fine resolution of the sheaf $\mathcal{S}_B$. As we mentioned in Section 2.3, the Dolbaut complex $\Lambda^{0,*}(Z, TZ)_G$ is a $C^\infty(X)$-module and it defines a fine resolution of the sheaf $(\pi_*\mathcal{O}(TZ))_G$. And the Dolbaut complex $\Lambda^{0,*}(Y, TY)$ is a $C^\infty(X)$-module and it defines a fine resolution of the sheaf $\alpha_*\mathcal{O}(TY)$. Therefore we see that

$$\Lambda^{0,*}(Y, TY) \;/\; \Lambda^{0,*}(Z, TZ)_G \tag{175}$$

is a $C^\infty(X)$-module and that it defines a fine resolution of the sheaf $\mathcal{S}_B$, which we will use later in the proof of Theorem 2.

We will see in the following Section 3.2.2 that the elements of the vector space

$$K_X \cap K_Y \subset H^1(X, (\pi_*\mathcal{O}(TZ))_G) \tag{176}$$

are related to the infinitesimal deformations which do not change the moduli of the curve $X = Z/G$ and $Y = Z/H$.

3.2.2 Construction of smooth family

For an element

$$a \in K_X \subset H^1(Z, \mathcal{O}(TZ))_G, \tag{177}$$

we can find a G-invariant Dolbaut form $\eta_Z \in \Lambda^{0,1}(Z, TZ)_G$ which is a representative of the cohomology class a and which satisfies Condition $\tilde{\Sigma}$ by virtue of Lemma 1. As we mentioned in the previous section, the G-invariant Dolbaut form η_Z generate a smooth family of G-invariant almost complex structures on the curve Z depending on a real parameter $t \in (-\varepsilon, \varepsilon)$ satisfying Condition Σ. We denote the curve Z with the almost complex structure $J_Z(t)$ by $Z(t)$. Furthermore, we know that the Dolbaut form η_Z can be identified with an Dolbaut form $\eta_X \in \Lambda^{0,1}(X, TX)$ by using the isomorphisms of the tangent spaces and the cotangent spaces. Since $[\eta_Z] \in K_X$, the cohomology class of the Dolbaut form η_X in $H^1(X, \mathcal{O}(TX))$ is zero. Then there exists an element $\mathcal{V}_X \in \Lambda^{0,0}(X, TX)$ satisfying

$$\eta_X = \bar{\partial}\mathcal{V}_X. \tag{178}$$

The element $\mathcal{V}_X$, identified with a real vector field on the curve X, generates an one-parameter transformation group $f_t : X \to X$ depending on a real parameter $t \in (-\varepsilon, \varepsilon)$. Then the composite of maps

$$\pi_t := f_t \circ \pi \tag{179}$$

is a smooth family of holomorphic Galois covering space with Galois group G

$$\pi_t : Z(t) \to X. \tag{180}$$

Note that the complex structure of the curve X is unchanged above.

The similar argument is true for an element

$$a \in K_Y \subset H^1(Z, \mathcal{O}(TZ))_G. \tag{181}$$

Take η_Z as above. Then η_Z can be identified with an Dolbaut form $\eta_Y \in \Lambda^{0,1}(Y, TY)$ by using the isomorphisms of the tangent spaces and the cotangent spaces. Since $[\eta_Z] \in K_Y$, the cohomology class of the Dolbaut form η_Y in $H^1(Y, \mathcal{O}(TY))$ is zero. Then there exists an element $\mathcal{V}_Y \in \Lambda^{0,0}(Y, TY)$ satisfying

$$\eta_Y = \bar{\partial}\mathcal{V}_Y. \tag{182}$$

The element $\mathcal{V}_Y$, identified with a real vector field on the curve Y, generates an one-parameter transformation group $g_t : Y \to Y$ depending on a real parameter $t \in (-\varepsilon, \varepsilon)$. Then the composite of maps

$$\beta_t := g_t \circ \beta \tag{183}$$

is a smooth family of holomorphic Galois covering space with Galois group H

$$\beta_t : Z(t) \to Y. \tag{184}$$

Note that the complex structure of the curve Y is unchanged above.

Combining the procedures about X and Y above, we see that, for an element

$$a \in K_X \cap K_Y \subset H^1(Z, \mathcal{O}(TZ))_G, \tag{185}$$

we can construct both smooth families of Galois covering spaces $\pi_t : Z(t) \to X$ and $\beta_t : Z(t) \to Y$. And then we can construct a smooth family of branched covering maps $Y \xrightarrow{\alpha_t} X$ with initial value $\alpha_0 = \alpha$ by setting

$$\alpha_t := \pi_t \circ \beta_t^{-1} \quad \text{(well-defined).} \tag{186}$$

The minus of the differential of the family α_t at $t = 0$

$$u := -\left.\frac{d}{dt}\right|_{t=0} \alpha_t \tag{187}$$

defines a holomorphic section of the pull-back bundle $\alpha^*(TX)$. Namely

$$u \in H^0(Y, \mathcal{O}(\alpha^*(TX))). \tag{188}$$

Finally the equivalence class of u

$$[u] \in H^0(Y, \mathcal{O}(\alpha^*(TX))) \ / \ (\operatorname{Im}\mathcal{P} + \operatorname{Im}\mathcal{Q}) \tag{189}$$

defines a equivalence class of infinitesimal deformations of the branched covering map α.

We show in the following Theorem 2, which is our main theorem in this article, that the equivalence class (189) does not depend on the particular choice of the families $J_Z(t), \pi_t, \beta_t$, but only depends on the element

$$a \in K_X \cap K_Y \subset H^1(Z, \mathcal{O}(TZ))_G \tag{190}$$

which we took first.

3.3 Main results

Theorem 2. *We can define a homomorphism*

$$K_X \cap K_Y \to H^0(X, \alpha_*\mathcal{O}(\alpha^*(TX))) \,/\, (\operatorname{Im}\mathcal{P} + \operatorname{Im}\mathcal{Q}), \tag{191}$$

the minus of which describes how the branched covering map $Y \xrightarrow{\alpha} X$ varies when we deform the complex structure of the curve Z in a G-invariant way in which the moduli of the curves $X = Z/G$ and that of the curve $Y = Z/H$ are not changed.

Proof. By using the homomorphism between the two exact sequences of sheaves (151) and (171)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}(TX) & \xrightarrow{\text{pull-back}} & \alpha_*\mathcal{O}(\alpha^*(TX)) & \longrightarrow & \mathcal{S}_A & \longrightarrow & 0 \\
 & & {\scriptstyle d\pi}\uparrow & & {\scriptstyle d\alpha}\uparrow & & \uparrow & & \\
0 & \longrightarrow & (\pi_*\mathcal{O}(TZ))_G & \xrightarrow{d\beta} & \alpha_*\mathcal{O}(TY) & \longrightarrow & \mathcal{S}_B & \longrightarrow & 0,
\end{array} \tag{192}$$

we can write the following commutative diagram of cohomology groups: (The vertical and horizontal lines are reversed for convenience.)

$$\begin{array}{ccc}
H^1(X, \alpha_*\mathcal{O}(TY)) & & \\
\uparrow{\scriptstyle \mathcal{J}_Y} & & \\
H^1(X, (\pi_*\mathcal{O}(TZ))_G) & \xrightarrow{\mathcal{J}_X} & H^1(X, \mathcal{O}(TX)) \\
{\scriptstyle \delta_1}\uparrow & & {\scriptstyle \delta_2}\uparrow \\
H^0(X, \mathcal{S}_B) & \xrightarrow{\rho} & H^0(X, \mathcal{S}_A) \\
\uparrow & & {\scriptstyle \varphi}\uparrow \\
H^0(X, \alpha_*\mathcal{O}(TY)) & \xrightarrow{\mathcal{Q}} & H^0(X, \alpha_*\mathcal{O}(\alpha^*(TX))) \\
 & & {\scriptstyle \mathcal{P}}\uparrow \\
 & & H^0(X, \mathcal{O}(TX))
\end{array} \tag{193}$$

We define a homomorphism

$$K_X \cap K_Y \to H^0(X, \alpha_* \mathcal{O}(\alpha^*(TX))) / (\operatorname{Im} \mathcal{P} + \operatorname{Im} \mathcal{Q}) \tag{194}$$

as follows: For an element $a \in K_X \cap K_Y$, we can find an element $b \in H^0(X, \mathcal{S}_B)$ satisfying $\delta_1(b) = a$, because $a \in K_Y$. Set $c = \rho(b) \in H^0(X, \mathcal{S}_A)$, then we can find an element $d \in H^0(X, \alpha_* \mathcal{O}(\alpha^*(TX)))$ satisfying $\varphi(d) = c$, because $\delta_2(c) = 0$. The element $d \in H^0(X, \alpha_* \mathcal{O}(\alpha^*(TX)))$ defines an equivalence class $[d]$ in the vector space

$$H^0(X, \alpha_* \mathcal{O}(\alpha^*(TX))) / (\operatorname{Im} \mathcal{P} + \operatorname{Im} \mathcal{Q}). \tag{195}$$

It is not difficult check that the correspondence

$$a \in K_X \cap K_Y \to [d] \in H^0(X, \alpha_* \mathcal{O}(\alpha^*(TX))) / (\operatorname{Im} \mathcal{P} + \operatorname{Im} \mathcal{Q}) \tag{196}$$

described above is well-defined by chasing the diagram (193).

Let $J_Z(t)$ denote a smooth family of G-invariant almost complex structures on the curve Z depending on a real parameter $t \in (-\varepsilon, \varepsilon)$ satisfying Condition Σ. We denote the curve Z with the almost complex structure $J_Z(t)$ by $Z(t)$. And we set $\eta_Z \in \Lambda^{0,1}(Z, TZ)_G$ as follows:

$$\eta_Z := \left. \frac{d}{dt} J_Z(t) \right|_{t=0}. \tag{197}$$

The Dolbaut form η_Z satisfies Condition $\tilde{\Sigma}$. Let $\pi_t : Z(t) \to X$ be a smooth family of holomorphic Galois covering space with Galois group G satisfying $\pi_0 = \pi$. And let $\beta_t : Z(t) \to Y$ be a smooth family of holomorphic Galois covering space with Galois group H satisfying $\beta_0 = \beta$. Note that, for any $a \in H^1(Z, \mathcal{O}(TZ))_G$, we have already shown that there exist such families that satisfies

$$[\eta_Z] = a \in H^1(Z, \mathcal{O}(TZ))_G. \tag{198}$$

In what follows, however, we proceed without using the particular families which we constructed before.

Under the situation which we described above, we can define a smooth family of branched covering maps $\alpha_t : Y \to X$ satisfying $\alpha_0 = \alpha$, by setting

$$\alpha_t := \pi_t \circ (\beta_t)^{-1} \quad \text{(well-defined).} \tag{199}$$

In what follows, we show that the following diagram commutes:

$$\begin{array}{ccc} K_X \cap K_Y & \longrightarrow & H^0(Y, \mathcal{O}(\alpha^*(TX))) \ / \ (\operatorname{Im} \mathcal{P} + \operatorname{Im} \mathcal{Q}) \\ \uparrow & & \uparrow \\ \eta_Z \in \Lambda^{0,1}(Z, TZ)_G & \longrightarrow & u = -\left.\frac{d}{dt}\alpha_t\right|_{t=0} \in H^0(Y, \mathcal{O}(\alpha^*(TX))) \end{array} \tag{200}$$

First we define maps $f_t : X \to X$ by setting

$$f_t := \pi \circ \pi_t^{-1} \quad \text{(well-defined).} \tag{201}$$

The maps f_t's are C^∞ on X and, moreover, holomorphic on some neighborhood of the branch points of the Galois covering space $Z \xrightarrow{\pi} X$ for sufficiently small $t \in (-\varepsilon, \varepsilon)$, and then they define a family of C^∞-automorphisms of X. Similarly we can define a family of C^∞-automorphisms $g_t : Y \to Y$ by setting

$$g_t := \beta \circ \beta_t^{-1} \quad \text{(well-defined).} \tag{202}$$

Set real vector fields $\mathcal{V}_X \in \mathfrak{X}(X)$ and $\mathcal{V}_Y \in \mathfrak{X}(Y)$ as follows:

$$\mathcal{V}_X = \frac{d}{dt} f_t \bigg|_{t=0}, \quad \mathcal{V}_Y = \frac{d}{dt} g_t \bigg|_{t=0}. \tag{203}$$

These vector fields are also regarded as $\mathcal{V}_X \in \Lambda^{0,0}(X, TX)$ and $\mathcal{V}_Y \in \Lambda^{0,0}(Y, TY)$. Then it is not difficult to check that $\mathcal{V}_Y$ defines a cohomology class

$$[\mathcal{V}_Y] \in H^0(X, \mathcal{S}_{\mathrm{B}}) \tag{204}$$

by using the fine resolution of the sheaf $\mathcal{S}_{\mathrm{B}}$ (175). The cohomology class $[\mathcal{V}_Y]$ corresponds to the element $b \in H^0(X, \mathcal{S}_{\mathrm{B}})$ which we defined in the diagram (193). And we set $s \in \Lambda^{0,0}(Y, \alpha^*(TX))$ as

$$s := d\alpha(\mathcal{V}_Y). \tag{205}$$

It is not difficult to check that s defines a cohomology class

$$[s] \in H^0(X, \mathcal{S}_{\mathrm{A}}) \tag{206}$$

by using the fine resolution of the sheaf $\mathcal{S}_{\mathrm{A}}$ (154). The cohomology class $[s] \in H^0(X, \mathcal{S}_{\mathrm{A}})$ corresponds to the element $c \in H^0(X, \mathcal{S}_{\mathrm{A}})$ which we defined in the diagram (193). Since the cohomology class of the element $\alpha^*(\mathcal{V}_X) \in \Lambda^{0,0}(Y, \alpha^*(TX))$ in $H^0(X, \mathcal{S}_{\mathrm{A}})$ is zero, the difference

$$s - \alpha^*(\mathcal{V}_X) \tag{207}$$

defines the same cohomology class as s in $H^0(X, \mathcal{S}_{\mathrm{A}})$. It is not difficult to check that the difference above defines a holomorphic section of the pull-back bundle $\alpha^*(TX)$. Namely

$$s - \alpha^*(\mathcal{V}_X) \in H^0(Y, \alpha^*(TX)). \tag{208}$$

The element $s - \alpha^*(\mathcal{V}_X) \in H^0(X, \alpha_*\mathcal{O}(\alpha^*(TX)))$ corresponds to the element $d \in H^0(X, \alpha_*\mathcal{O}(\alpha^*(TX)))$ which we defined in the diagram (193).

Finally it is not difficult to check that the minus of the differential of the family α_t at $t = 0$ satisfies

$$-\frac{d}{dt}\alpha_t \bigg|_{t=0} = s - \alpha^*(\mathcal{V}_X) \in H^0(Y, \mathcal{O}(\alpha^*(TX))), \tag{209}$$

and the equivalence class

$$[s - \alpha^* (\mathcal{V}_X)] \in H^0 (Y, \alpha^* (TX)) / (\operatorname{Im} \mathcal{P} + \operatorname{Im} \mathcal{Q}) \tag{210}$$

coincides with the element

$$[d] \in H^0 (X, \alpha_* \mathcal{O} (\alpha^* (TX))) / (\operatorname{Im} \mathcal{P} + \operatorname{Im} \mathcal{Q}) \tag{211}$$

which we defined before, under the identification of the cohomology groups below:

$$H^0 (Y, \alpha^* (TX)) \cong H^0 (X, \alpha_* \mathcal{O} (\alpha^* (TX))) . \tag{212}$$

This completes the proof. □

3.4 Galois closure case

In this subsection we discuss the case where the Galois covering spaces $Z \xrightarrow{\pi} X, Z \xrightarrow{\beta} Y$ are the Galois closure of the branched covering map $Y \xrightarrow{\alpha} X$.

Combining the exact sequences of sheaves (165), (163), (151) and (170), we obtain the following commutative diagram of homomorphisms of sheaves:

$$\begin{tikzcd}
 & 0 & 0 & & \\
 & \bigoplus_i T_{b_i}X \arrow[u] \arrow[r, "\gamma"] & \mathcal{S}_{\text{jets}} \arrow[u] & & \\
0 \arrow[r] & \mathcal{O}(TX) \arrow[u] \arrow[r, "\text{pull-back}"] & \alpha_* \mathcal{O}(\alpha^*(TX)) \arrow[u] \arrow[r] & \mathcal{S}_{\mathrm{A}} \arrow[r] & 0 \\
0 \arrow[r] & (\pi_* \mathcal{O}(TZ))_G \arrow[u, "d\pi"] \arrow[r, "d\beta"] & \alpha_* \mathcal{O}(TY) \arrow[u, "d\alpha"] \arrow[r] & \mathcal{S}_{\mathrm{B}} \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd} \tag{213}$$

First we observe the homomorphism of sheaves

$$\bigoplus_i T_{b_i}X \xrightarrow{\gamma} \mathcal{S}_{\text{jets}} \tag{214}$$

which is defined as in the diagram (213). As we mentioned in Section 3.1.2, the sheaf $\mathcal{S}_{\text{jets}}$ is a skyscraper sheaf which has non-zero stalks only on the branch points of the branched covering space $Y \xrightarrow{\alpha} X$. And the stalk on the branch point $b \in X$ is the direct sum of jets at the corresponding ramification points $p_1, p_2, \ldots, p_K$ $(\alpha(p_j) = b)$ of the pull-back bundle $\alpha^* (TX)$. Namely

$$\left(\mathcal{S}_{\text{jets}}\right)_b \cong \bigoplus_j^K j^*_{p_j} (\alpha^* (TX)) . \tag{215}$$

In the case where the Galois covering spaces $Z \xrightarrow{\pi} X, Z \xrightarrow{\beta} Y$ are the Galois closure of the branched covering space $Y \xrightarrow{\alpha} X$, the set of branch points $\{b_1, b_2, \ldots, b_m\} \subset X$ of the branched covering map $Z \xrightarrow{\pi} X$ coincides with that of the branched covering space $Y \xrightarrow{\alpha} X$ which we denoted by $\{a_1, a_2, \ldots, a_L\} \subset X$. Furthermore, in that case, the homomorphism (214) is injective, as we see below. We observe the homomorphism at stalk level for a branch point $b \in X$ and for the corresponding ramification points $p_1, p_2, \ldots, p_K \in Y$ $(\alpha(p_j) = b)$

$$T_bX \to \bigoplus_j^K j_{p_j}^* (\alpha^* (TX)) . \tag{216}$$

Each homomorphism $T_bX \to j_{p_j}^* (\alpha^* (TX))$ is determined as the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(TY)_{p_i,H_i} & \xrightarrow{d\alpha} & \mathcal{O}(TX)_b & \longrightarrow & T_bX & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \text{pull-back}} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}(TY)_{p_j} & \xrightarrow{d\alpha} & \mathcal{O}(\alpha^*(TX))_{p_j} & \longrightarrow & j_{p_j}^*(\alpha^*(TX)) & \longrightarrow & 0 \end{array} \tag{217}$$

Here above, we see that the down arrows are all injective and, therefore, we see that the homomorphism of sheaves (214) is injective. Moreover, we also see, under the assumption above, that the homomorphism of sheaves (214) is surjective if and only if the branched covering map $Y \xrightarrow{\alpha} X$ is of general ramification type.

By summarizing the arguments above about the sheaf $\mathcal{S}_{\text{jets}}$, we obtain the following proposition.

Proposition 4. *If the Galois covering spaces $Z \xrightarrow{\pi} X, Z \xrightarrow{\beta} Y$ are the Galois closure of the branched covering map $Y \xrightarrow{\alpha} X$ the homomorphism of sheaves*

$$\bigoplus_i T_{b_i}X \xrightarrow{\gamma} \mathcal{S}_{\text{jets}} \tag{218}$$

in the diagram (213) *is injective.*

Furthermore, the homomorphism is surjective if and only if the branched covering map $Y \xrightarrow{\alpha} X$ is of general ramification type. In that case, we have the isomorphism

$$\mathcal{S}_{\text{jets}} \cong \bigoplus_i T_{b_i}X. \tag{219}$$

And we can also obtain the following proposition by chasing the diagram (213):

Proposition 5. *If the homomorphism*

$$\bigoplus_i T_{b_i}X \xrightarrow{\gamma} \mathcal{S}_{\text{jets}} \tag{220}$$

is injective, the homomorphism

$$\begin{array}{c} \mathcal{S}_{\mathrm{A}} \\ \uparrow \\ \mathcal{S}_{\mathrm{B}} \end{array} \tag{221}$$

in the diagram (213) *is also injective.*

By using Proposition 5, we obtain the following theorem:

Theorem 3. *If the Galois covering spaces $Z \xrightarrow{\pi} X, Z \xrightarrow{\beta} Y$ are the Galois closure of the branched covering map $Y \xrightarrow{\alpha} X$, the homomorphism in Theorem 2*

$$K_X \cap K_Y \to H^0(X, \alpha_* \mathcal{O}(\alpha^*(TX))) / (\operatorname{Im} \mathcal{P} + \operatorname{Im} \mathcal{Q}) \tag{222}$$

is injective.

Proof. It is not difficult to check that the assertion of the theorem follows from the fact that the homomorphism

$$H^0(X, \mathcal{S}_{\mathrm{B}}) \xrightarrow{\rho} H^0(X, \mathcal{S}_{\mathrm{A}}) \tag{223}$$

in the diagram (213) is injective under the assumption. □

Let $\mathcal{S}_{\mathrm{C}}$ denote the cokernel of the homomorphism (218). Then we have the following exact sequence of sheaves:

$$0 \longrightarrow \bigoplus_i T_{b_i}X \longrightarrow \mathcal{S}_{\mathrm{jets}} \longrightarrow \mathcal{S}_{\mathrm{C}} \longrightarrow 0. \tag{224}$$

By using the former part of Proposition 4 and Proposition 5, we can check that the diagram (213) extends to the following commutative diagram:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \bigoplus_i T_{b_i}X \arrow[r, "\gamma"] \arrow[u] & \mathcal{S}_{\mathrm{jets}} \arrow[r] \arrow[u] & \mathcal{S}_{\mathrm{C}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{O}(TX) \arrow[r, "\text{pull-back}"] \arrow[u] & \alpha_*\mathcal{O}(\alpha^*(TX)) \arrow[r] \arrow[u] & \mathcal{S}_{\mathrm{A}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & (\pi_*\mathcal{O}(TZ))_G \arrow[r, "d\beta"] \arrow[u, "d\pi"] & \alpha_*\mathcal{O}(TY) \arrow[r] \arrow[u, "d\alpha"] & \mathcal{S}_{\mathrm{B}} \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{225}$$

Finally we discuss the case where the branched covering map $Y \xrightarrow{\alpha} X$ is of general ramification type. At that time, by using the latter part of Proposition 4, we have $\mathcal{S}_C = 0$. Then we obtain the following theorem:

Theorem 4. *If the Galois covering spaces $Z \xrightarrow{\pi} X, Z \xrightarrow{\beta} Y$ are the Galois closure of the branched covering map $Y \xrightarrow{\alpha} X$ which is of general ramification type, the injective homomorphism in Theorem 3*

$$K_X \cap K_Y \to H^0(X, \alpha_* \mathcal{O}(\alpha^*(TX))) / (\operatorname{Im}\mathcal{P} + \operatorname{Im}\mathcal{Q}) \tag{226}$$

is surjective and, consequently, it is an isomorphism.

Proof. Under the assumption of the theorem, we have

$$\mathcal{S}_B \cong \mathcal{S}_A. \tag{227}$$

It is not difficult to check that the assertion of the theorem follows from the fact that the homomorphism

$$H^0(X, \mathcal{S}_B) \xrightarrow{\rho} H^0(X, \mathcal{S}_A) \tag{228}$$

in the diagram (193) is an isomorphism. □

Since the homomorphism (226) is an isomorphism under the assumption of Theorem 4, we can define its inverse map

$$H^0(X, \alpha_* \mathcal{O}(\alpha^*(TX))) / (\operatorname{Im}\mathcal{P} + \operatorname{Im}\mathcal{Q}) \to K_X \cap K_Y \subset H^1(Z, \mathcal{O}(TZ))_G. \tag{229}$$

Now we consider a set of smooth families depending on a real parameter $t \in (-\varepsilon, \varepsilon)$ which consist of a smooth family of branched covering maps $Y \xrightarrow{\alpha_t} X$ of general ramification type, a smooth family of Galois covering spaces $Z \xrightarrow{\pi_t} X$ with Galois group G, and a smooth family of Galois covering spaces $Z(t) \xrightarrow{\beta_t} Y$ with Galois group H. And we assume that they satisfy $\alpha_t \circ \beta_t = \pi_t$ for all $t \in (-\varepsilon, \varepsilon)$ and that the Galois covering spaces $Z(t) \xrightarrow{\pi_t} X, Z(t) \xrightarrow{\beta_t} Y$ are the Galois closure of the branched covering maps $Y \xrightarrow{\alpha_t} X$ for each $t \in (-\varepsilon, \varepsilon)$. Then the correspondence between the infinitesimal deformation of the complex structure of the Galois closure $Z(t)$ and the equivalence class of the infinitesimal deformation of the branched covering map $Y \xrightarrow{\alpha_t} X$ is completely determined by Theorem 4. Therefore we obtain that the minus of the inverse map (229) is the Kodaira-Spencer map of our primary concern, under the identifications

$$H^1(Z, \mathcal{O}(TZ))_G \subset H^1(Z, \mathcal{O}(TZ)) \quad \text{(subspace)}, \tag{230}$$
$$H^0(X, \alpha_* \mathcal{O}(\alpha^*(TX))) \cong H^0(Y, \mathcal{O}(\alpha^*(TX))). \tag{231}$$

4 Applications to examples

In this final section of this article we apply our general framework which we have developed in previous sections for the purpose of describing the Kodaira-Spencer map (226) to some cases to show explicit computations.

The first case is that the curve Y is a degree four smooth projective plane curve and X the projective space P^1 which was discussed by Yoshihara in his article [9]. And the second one is that the curve Y is an elliptic curve and $X = P^1$. We discuss infinitesimal deformations of mapping degree three holomorphic maps and those of their Galois closure curves.

As we have seen, for a degree four smooth projective plane curve Y, the map $\mathcal{Q}$ satisfied $\operatorname{Im}\mathcal{Q} = 0$. In the case where Y is an elliptic curve, however, the map $\mathcal{Q}$ satisfies $\operatorname{Im}\mathcal{Q} \neq 0$, therefore the investigation for such Y is all the more challenging in this respect.

In both cases, we will show explicit computations of the cohomology groups to describe the Kodaira-Spencer map of our primary interest.

4.1 Degree four smooth projective plane curve case

In this subsection we discuss the case that the curve Y is a degree four smooth projective plane curve $Y \subset P^2$ and the curve X the projective space P^1, which Yoshihara discussed in his article [9]. In this case the genus of the curve Y is three. We know that

$$\operatorname{Lie}(\operatorname{Aut}(Y)) = 0 \tag{232}$$

and that $\operatorname{Aut}(Y)$ is at most a finite group.

For the curve Y, we have

$$\operatorname{Map}_d(Y, P^1) = \emptyset \quad (d = 1, 2), \tag{233}$$

and the holomorphic maps of mapping degree three, which we discuss below, give the gonality.

It is not difficult to check

$$\dim H^0\left(Y, \mathcal{O}\left(\alpha^*\left(TP^1\right)\right)\right) = 4, \tag{234}$$

$$\dim H^1\left(Y, \mathcal{O}\left(\alpha^*\left(TP^1\right)\right)\right) = 0, \tag{235}$$

for any element $\alpha \in \operatorname{Map}_3(Y, P^1)$. Therefore, the mapspace $\operatorname{Map}_3(Y, P^1)$ is non-singular at any point $\alpha \in \operatorname{Map}_3(Y, P^1)$ and the dimension of its tangent space is four. And since $\dim \operatorname{Im}\mathcal{P} = 3, \dim \operatorname{Im}\mathcal{Q} = 0$, we obtain

$$\dim H^0\left(Y, \mathcal{O}\left(\alpha^*\left(TP^1\right)\right)\right) \ / \ (\operatorname{Im}\mathcal{P} + \operatorname{Im}\mathcal{Q}) = 1 \quad (= \dim Y). \tag{236}$$

We can correspond each point $Q \in Y$ to the linear projection $Y \overset{\alpha_Q}{\to} P^1$ with center $Q \in Y$. This correspondence induce the following isomorphism (c.f. [6]):

$$Y \cong \operatorname{Map}_3(Y, P^1)/\operatorname{Aut}(P^1) \quad \text{(biholomorphic)}. \tag{237}$$

For a general—i.e. not flex—point $Q \in Y$, the linear projection $Y \overset{\alpha_Q}{\to} P^1$ is of general ramification type, and has ten ramification points of order two. We discuss below the Galois closure curves of such points $Q \in Y$. The Galois closure curve Z_Q for such a point $Q \in Y$ is constructed as a Galois covering space

$$Z_Q \overset{\beta}{\to} Y \tag{238}$$

which has ten branch points on Y, and the Galois group H of which is the cyclic group of order two. Then the composite of maps

$$\pi := \alpha_Q \circ \beta : \ Z_Q \to X \tag{239}$$

is a Galois covering space, the Galois group G of which is the symmetric group of degree three. Under the situation above, the identities

$$Z_Q/G \cong P^1, \quad Z_Q/H \cong Y, \tag{240}$$

hold and the Galois covering spaces (238) and (239) are the Galois closure of the map α_Q. By using the Riemann-Hurwitz formula, we see that the genus of the Galois closure curves Z_Q is ten. The Galois covering space $Z_Q \overset{\pi_Q}{\to} P^1$ has thirty ramification points on Z_Q and has ten branch points on P^1. Since the genus of the curve Z_Q is ten, we have

$$\dim H^1\left(Z_Q, \mathcal{O}\left(TZ_Q\right)\right) = 27. \tag{241}$$

By using the exact sequence of cohomology groups in Remark 1 of Theorem 1

$$\begin{tikzcd} 0 \arrow[r] & H^0\left(Z_Q, \mathcal{O}\left(TZ_Q\right)\right)_G \arrow[r] & H^0\left(P^1, \mathcal{O}\left(TP^1\right)\right) \arrow[r] & \bigoplus_{i=1}^{10} T_{b_i}P^1 \arrow[r, "\delta"] & H^1\left(Z_Q, \mathcal{O}\left(TZ_Q\right)\right)_G \arrow[r] & 0, \end{tikzcd} \tag{242}$$

we obtain the dimension of its G-invariant subspace

$$\dim H^1\left(Z_Q, \mathcal{O}\left(TZ_Q\right)\right)_G = 7. \tag{243}$$

And we know

$$\dim H^1\left(Y, \mathcal{O}\left(TY\right)\right) = 6. \tag{244}$$

Then, under the identification

$$H^0\left(Y, \alpha^*\left(TP^1\right)\right) / \left(\operatorname{Im} \mathcal{P}\right) \cong T_Q Y, \tag{245}$$

we finally obtain the exact sequence

$$0 \longrightarrow T_Q Y \xrightarrow{\text{K-S}} H^1\left(Z_Q, \mathcal{O}\left(TZ_Q\right)\right)_G \xrightarrow{\mathfrak{J}_Y} H^1\left(Y, \mathcal{O}\left(TY\right)\right) \tag{246}$$

by using Corollary 2 in Introduction. Since the dimensions of the vector spaces above are

$$\dim T_Q Y = 1, \quad \dim H^1\left(Z_Q, \mathcal{O}\left(TZ_Q\right)\right)_G = 7, \quad \dim H^1\left(Y, \mathcal{O}\left(TY\right)\right) = 6, \tag{247}$$

we see the homomorphism $\mathfrak{J}_Y$ is surjective.

In particular, as we have seen above, the Kodaira-Spencer map is injective. Then we obtain the same conclusion as Yoshihara's in his article [9]—for sufficiently close general points $P, Q \in Y$, the corresponding Galois closure curves Z_P, Z_Q are not isomorphic—in the quite different way from his.

4.2 Elliptic curve case

In this subsection we discuss the case that the curve Y is an elliptic curve and X the projective space P^1.

In this case we have

$$\dim H^0(Y, \mathcal{O}(TY)) = 1, \tag{248}$$

and it is not difficult to check $\dim \operatorname{Im} \Omega = 1$. First we prove the following proposition:

Proposition 6. *For any elliptic curve and for any branched covering map $Y \xrightarrow{\alpha} P^1$, we have $\operatorname{Im} \mathcal{P} \cap \operatorname{Im} \Omega = 0$. Therefore the sum*

$$\operatorname{Im} \mathcal{P} + \operatorname{Im} \Omega \subset H^0\left(Y, \mathcal{O}\left(\alpha^*\left(TP^1\right)\right)\right) \tag{249}$$

is the direct sum $\operatorname{Im} \mathcal{P} \oplus \operatorname{Im} \Omega$.

Proof. Since P^1 is simply connected, the map α has at least one ramification point $p \in Y$, the order of which we denote by m (≥ 2). In what follows, we use a coordinate y on Y which comes from the universal covering $\mathbf{C} \to Y$ satisfying $y(p) = 0$, and an inhomogeneous coordinate x on P^1 satisfying $x \circ \alpha(p) = 0$. The map α is written as a convergent power series with the first term y^m around the point $p \in Y$. Namely

$$\alpha^*(x) = y^m + (\text{higher order terms}). \tag{250}$$

Since an element of $H^0\left(P^1, \mathcal{O}\left(TP^1\right)\right)$ is written as $\left(a_0 + a_1 x + a_2 x^2\right) \frac{\partial}{\partial x}$ $(a_0, a_1, a_2 \in \mathbf{C})$, an element of $\operatorname{Im} \mathcal{P}$ is written as

$$\left(a_0 + a_1 \alpha^*(x) + a_2 (\alpha^*(x))^2\right) \alpha^*\left(\frac{\partial}{\partial x}\right). \tag{251}$$

And since an element of $H^0(Y, \mathcal{O}(TY))$ is written as $b\frac{\partial}{\partial y}$ $(b \in \mathbf{C})$, an element of $\operatorname{Im} \Omega$ are written as

$$b\left(my^{m-1} + (\text{higher order terms})\right) \alpha^*\left(\frac{\partial}{\partial x}\right). \tag{252}$$

It is not difficult to check $\operatorname{Im} \mathcal{P} \cap \operatorname{Im} \Omega = 0$ by using the expressions (251) and (252).

This completes the proof of Proposition 6. □

Since, for an elliptic curve Y, we have

$$\operatorname{Map}_1(Y, P^1) = \emptyset, \tag{253}$$

the holomorphic maps of mapping degree two give the gonality. Namba showed in his book [6]

$$\operatorname{Map}_2(Y, P^1) / \left(\operatorname{Aut}(P^1) \times \operatorname{Aut}(Y)\right) = \{\text{one point}\}. \tag{254}$$

This implies that the holomorphic maps of mapping degree two from an elliptic curve Y to the projective space P^1 are all equivalent in the sense (254). Since the extensions of fields $k(P^1) \hookrightarrow k(Y)$ induced by these holomorphic maps of mapping degree two are the second order, all of them are Galois extensions with Galois group the cyclic group of order two. For such maps we regard the curve Y itself to be the Galois closure curve. In these cases, therefore, our question is trivial.

The first mapping degree for which our question is non-trivial is three. It is not difficult to check

$$\dim H^0\left(Y, \mathcal{O}\left(\alpha^*\left(TP^1\right)\right)\right) = 6, \tag{255}$$

$$\dim H^1\left(Y, \mathcal{O}\left(\alpha^*\left(TP^1\right)\right)\right) = 0, \tag{256}$$

for any element $\alpha \in \mathrm{Map}_3(Y, P^1)$. Therefore, the mapspace $\mathrm{Map}_3(Y, P^1)$ is non-singular at any point $\alpha \in \mathrm{Map}_3(Y, P^1)$ and the dimension of its tangent space is six. Combining Proposition 6 and the fact that $\dim \mathrm{Im}\,\mathcal{P} = 3, \dim \mathrm{Im}\,\mathcal{Q} = 1$, we obtain

$$\dim H^0\left(Y, \mathcal{O}\left(\alpha^*\left(TP^1\right)\right)\right) \;/\; (\mathrm{Im}\,\mathcal{P} \oplus \mathrm{Im}\,\mathcal{Q}) = 2 \quad (= \dim P^2). \tag{257}$$

4.2.1 Geometry of embedding $Y \subset P^2$

An elliptic curve Y can be embedded as a degree three smooth projective plane curve $Y \overset{i}{\hookrightarrow} P^2$. And, at that time, we can correspond each point $Q \in P^2 \setminus Y$ to the linear projection $Y \overset{\alpha_Q}{\to} P^1$ with center $Q \in P^2 \setminus Y$. The mapping degree of these maps α_Q is three. In fact, This correspondence induces the isomorphism

$$\left(P^2 \setminus Y\right)/F \cong \mathrm{Map}_3(Y, P^1)/\left(\mathrm{Aut}(P^1) \times \mathrm{Aut}(Y)\right) \tag{258}$$

for some finite group F (c.f. [6]). Then a translation of the center of the linear projection $Q \in P^2 \setminus Y$ induces an infinitesimal deformation of the map α_Q

$$T_Q P^2 \longrightarrow H^0\left(Y, \mathcal{O}\left(\alpha_Q^*\left(TP^1\right)\right)\right) / \mathrm{Im}\,\mathcal{P}. \tag{259}$$

We claim the following proposition as to the correspondence above:

Proposition 7. *For all points $Q \in P^2 \setminus Y$, the correspondence* (259) *induces the following isomorphism between vector spaces of dimension two:*

$$T_Q P^2 \xrightarrow{\cong} H^0\left(Y, \mathcal{O}\left(\alpha_Q^*\left(TP^1\right)\right)\right) / (\mathrm{Im}\,\mathcal{P} \oplus \mathrm{Im}\,\mathcal{Q}). \tag{260}$$

Proof. Assume that we have an embedding $Y \overset{i}{\hookrightarrow} P^2$ and that we have a map α_Q as the composite of the embedding i and the linear projection $\mathrm{pr}_Q : P^2 \setminus Q \to P^1$. Namely

$$\alpha_Q = \mathrm{pr}_Q \circ i. \tag{261}$$

We use a homogeneous coordinate system $(x : y : z)$ on P^2 on which the center of the linear projection $Q \in P^2$ is written as $Q = (0 : 0 : 1)$, and on which the linear projection pr_Q is written as

$$(x : y : z) \in P^2 \overset{\mathrm{pr}_Q}{\to} (x : y) \in P^1. \tag{262}$$

In what follows, we use this homogeneous coordinate system.

The pull-back bundle $\mathrm{pr}_Q^*\left(TP^1\right)$ extends to the whole P^2, which we denote by l, and we have $\deg l = \deg TP^1 = 2$. Then we see that

$$\alpha_Q^*\left(TP^1\right)\Big|_Y \cong l|_Y, \tag{263}$$

and that the restriction of the sections

$$H^0\left(P^2, \mathcal{O}(l)\right) \xrightarrow[\cong]{i^*} H^0\left(Y, \mathcal{O}\left(\alpha^*\left(TP^1\right)\right)\right) \tag{264}$$

is an isomorphism between vector spaces of dimension six. Since the left hand side of the isomorphism above is identified with the vector space of all homogeneous polynomials of degree two on P^2, we can identify $H^0\left(Y, \alpha^*\left(TP^1\right)\right)$ with the same vector space of polynomials.

In what follows we show that there is a direct sum decomposition

$$H^0\left(Y, \mathcal{O}\left(\alpha_Q^*\left(TP^1\right)\right)\right) \cong \mathrm{Im}\,\mathcal{P} \oplus \mathrm{Im}\,\mathcal{Q} \oplus T_Q P^2 \tag{265}$$

by using the identification above.

First we observe an element of $T_Q P^1$ as a translation of the center of the linear projection. We use an inhomogeneous coordinate system defined as

$$u := x/z, \quad v := y/z. \tag{266}$$

A tangent vector

$$\mathcal{V} = a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v} \in T_Q P^2 \quad (a, b \in \mathbf{C}) \tag{267}$$

is identified with a polynomial

$$(ax + by)z. \tag{268}$$

And an element of the subspace $\mathrm{Im}\,\mathcal{P}$, which comes from $\mathrm{Lie}\left(\mathrm{Aut}(P^1)\right)$ is identified with a polynomial with variable x, y (i.e. not containing the variable z). Therefore we obtain

$$\mathrm{Im}\,\mathcal{P} \cap T_Q P^2 = 0. \tag{269}$$

And, since the automorphisms of P^1 and the translations of the center of the linear projection do not change the holomorphic equivalent class of a line bundle $L := \alpha_Q^*(H_{P^1})$, we obtain

$$\left(\mathrm{Im}\,\mathcal{P} \oplus T_Q P^2\right) \cap \mathrm{Im}\,\mathcal{Q} = 0. \tag{270}$$

Therefore we obtain the following direct sum decomposition:

$$H^0\left(Y, \mathcal{O}\left(\alpha_Q^*\left(TP^1\right)\right)\right) \cong \operatorname{Im} \mathcal{P} \oplus \operatorname{Im} \mathcal{Q} \oplus T_Q P^2. \tag{271}$$

In particular we obtain

$$T_Q P^2 \cong H^0\left(Y, \mathcal{O}\left(\alpha_Q^*\left(TP^1\right)\right)\right) / (\operatorname{Im} \mathcal{P} \oplus \operatorname{Im} \mathcal{Q}). \tag{272}$$

□

For an embedding $Y \overset{i}{\hookrightarrow} P^2$, the ramification types of the linear projection $\alpha_Q := \mathrm{pr}_Q \circ i$ with center $Q \in P^2 \setminus Y$ have direct relation with the multi-tangent lines of the embedding $Y \overset{i}{\hookrightarrow} P^2$, as we describe below.

Elementary computations show that the embedding $Y \subset P^2$ has nine flex points, all of which are 1-flex, and that it has corresponding nine multi-tangent lines.

For a point $Q \in P^2 \setminus Y$ which is not on multi-tangent lines, the map α_Q is of general ramification type and has six ramification points of order two. The genus of the Galois closure curve Z_Q for those maps is four. We discuss the Kodaira-Spencer maps for these maps which have general ramification types in the following Section 4.2.2.

For a point $Q \in P^2 \setminus Y$ which is on a single multi-tangent line, the map α_Q has four ramification points of order two and has one ramification point of order three. The genus of the Galois closure curve Z_Q for those maps is three.

For a point $Q \in P^2 \setminus Y$ at which two multi-tangent lines meet. the map α_Q has two ramification points of order two and has two ramification points of order three. The genus of the Galois closure curve for those maps is two.

For a point $Q \in P^2 \setminus Y$ at which three multi-tangent lines meet. the map α_Q has three ramification points of order three. In this case the map α_Q itself is a Galois covering map with Galois group the cyclic group of order three. For such a map we regard the curve Y itself to be the Galois closure curve Z_Q. Such points are called *Galois points* with Galois group $\mathfrak{C}_3$ by Yoshihara, Miura, Takahashi and Duyaguit in their articles [5][4][8][7][2][1][9]. This situation happens only when the elliptic curve Y is of j-invariant = 0. Such elliptic curves are projectively equivalent to the *Fermat cubic curve*

$$x^3 + y^3 + z^3 = 0 \quad \text{(homogeneous coordinates on } P^2\text{)}. \tag{273}$$

By the investigation of possible ramification types of maps, we see that it can not happen that more than three multi-tangent lines meet at one point.

4.2.2 General points on P^2

We discuss the Galois closure curves of the linear projections $\alpha_Q := \mathrm{pr}_Q \circ i$ with center $Q \in P^2 \setminus Y$ which are not on multi-tangent lines of Y.

For such a point $Q \in P^2 \setminus Y$, the linear projection $Y \overset{\alpha_Q}{\to} P^1$ is of general ramification type, and has six ramification points of order two. In a similar way as in the previous subsection, the

Galois closure curve Z_Q for such a point $Q \in P^2 \setminus Y$ is constructed as a Galois covering space

$$Z_Q \xrightarrow{\beta} Y \tag{274}$$

which has six branch points on Y. Then the composite of maps

$$\pi := \alpha_Q \circ \beta : \ Z_Q \to X \tag{275}$$

is a Galois covering space, the Galois group G of which is the symmetric group of degree three. In this case we see that the genus of the Galois closure curves Z_Q is four. The Galois covering space $Z_Q \xrightarrow{\pi_Q} P^1$ has eighteen ramification points on Z_Q and has six branch points on P^1. Since the genus of the curve Z_Q is four, we have

$$\dim H^1(Z_Q, \mathcal{O}(TZ_Q)) = 9. \tag{276}$$

By using the exact sequence of cohomology groups in Remark 1 of Theorem 1

$$0 \longrightarrow H^0(Z, \mathcal{O}(TZ))_G \longrightarrow H^0\left(P^1, \mathcal{O}\left(TP^1\right)\right) \longrightarrow \bigoplus_{i=1}^{6} T_{b_i}P^1 \xrightarrow{\delta} H^1(Z_Q, \mathcal{O}(TZ_Q))_G \longrightarrow 0, \tag{277}$$

we obtain the dimension of its G-invariant subspace

$$\dim H^1(Z_Q, \mathcal{O}(TZ_Q))_G = 3. \tag{278}$$

And we know

$$\dim H^1(Y, \mathcal{O}(TY)) = 1. \tag{279}$$

Then, under the identification

$$H^0\left(Y, \alpha_Q^*\left(TP^1\right)\right) / (\operatorname{Im}\mathcal{P} \oplus \operatorname{Im}\mathcal{Q}) \cong T_QP^2 \tag{280}$$

we finally obtain the exact sequence

$$0 \longrightarrow T_QP^2 \xrightarrow{\text{K-S}} H^1(Z_Q, \mathcal{O}(TZ_Q))_G \xrightarrow{\mathfrak{I}_Y} H^1(Y, \mathcal{O}(TY)) \tag{281}$$

by using Corollary 2 in Introduction. Since the dimensions of the vector spaces above are

$$\dim T_QP^2 = 2, \quad \dim H^1(Z_Q, \mathcal{O}(TZ_Q))_G = 3, \quad \dim H^1(Y, \mathcal{O}(TY)) = 1, \tag{282}$$

we see that the homomorphism $\mathfrak{I}_Y$ is surjective.

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Received September 29, 2003 Revised November 18, 2003