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Finite Model Property for an Intuitionistic Modal Logic

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Abstract

An intuitionistic modal logic that is an intuitionistic bi-modal version of the modal logic **K5** is proved, by means of the filtration method, to enjoy the finite model property.

0 Introduction

The purpose of this note is to show that the intuitionistic modal logic $\mathbf{intK5}_{\Box\Diamond}$, which is defined later, has the finite model property. This forms a supplement to Hasimoto [2], in which the finite model property for some intuitionistic modal logics is proved by means of the filtration method, while that for others including $\mathbf{intK5}_{\Box\Diamond}$is left open. It is peculiar to our proof that, although it is carried out through filtration, yet the filter is a finite set of formulas of an enlarged language, and not of the original language.

A propositional bi-modal language with the modal operators $\Box$ and $\Diamond$ is supposed to be fixed as the original language.

Consult Chagrov–Zakharyaschev [1] for the filtration method.

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1 The intuitionistic modal logic $\mathbf{intK5}_{\Box\Diamond}$

A *Kripke frame* (for intuitionistic bi-modal logics) is a quadruple $\mathcal{F} = \langle W, \lhd, R_\Box, R_\Diamond \rangle$ that satisfies the following properties: W is a nonempty set; $\lhd$ is a partial order on W; while $R_\Box$ and $R_\Diamond$ are binary relations on W such that

(1.1) $\lhd \circ R_\Box \circ \lhd = R_\Box$ and $\lhd^{-1} \circ R_\Diamond \circ \lhd^{-1} = R_\Diamond$.

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A *Kripke frame for* intK5$_{\Box\Diamond}$, in turn, is a Kripke frame $\mathcal{F} = \langle W, \lhd, R_\Box, R_\Diamond \rangle$ with the property

(1.2) $R_\Diamond^{-1} \circ R_\Box \subseteq R_\Box$ and $R_\Box^{-1} \circ R_\Diamond \subseteq R_\Diamond$.

A *point* in a Kripke frame $\mathcal{F} = \langle W, \lhd, R_\Box, R_\Diamond \rangle$ is any element of W; while a *valuation* in $\mathcal{F}$ is a map V that associates a subset $V(p)$ ($\subseteq W$) with each propositional letter p such that

(1.3) if $x \lhd y$ and $x \in V(p)$, then $y \in V(p)$, for every $x, y \in W$.

A *Kripke model* (based on $\mathcal{F}$) is a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$ of a Kripke frame $\mathcal{F}$ and a valuation V in $\mathcal{F}$.

Let $\mathcal{M} = \langle \mathcal{F}, V \rangle$ be a Kripke model based on a Kripke frame $\mathcal{F} = \langle W, \lhd, R_\Box, R_\Diamond \rangle$, and let $x \in W$. By induction on the construction of a formula A, the relation $(\mathcal{M}, x) \models A$ is defined as below:

(i) $(\mathcal{M}, x) \models p$ iff $x \in V(p)$;

(ii) $(\mathcal{M}, x) \models B \wedge C$, iff $(\mathcal{M}, x) \models B$ and $(\mathcal{M}, x) \models C$; and similarly for $(\mathcal{M}, x) \models B \vee C$;

(iii) $(\mathcal{M}, x) \models B \supset C$, iff $x \lhd y$ and $(\mathcal{M}, y) \models B$ imply $(\mathcal{M}, y) \models C$ for every $y \in W$; and similarly for $(\mathcal{M}, x) \models \neg B$;

(iv) $(\mathcal{M}, x) \models \Box B$, iff $x \, R_\Box \, y$ implies $(\mathcal{M}, y) \models B$ for every $y \in W$; and

(v) $(\mathcal{M}, x) \models \Diamond B$, iff $x \, R_\Diamond \, y$ and $(\mathcal{M}, y) \models B$ for some $y \in W$.

It is easily proved, owing to the properties (1.3) and (1.1), by induction on the construction of A that: if $x \lhd y$ and $(\mathcal{M}, x) \models A$, then $(\mathcal{M}, y) \models A$, for every $x, y \in W$.

A formula A is called *valid* in a Kripke frame $\mathcal{F}$, if $(\mathcal{M}, x) \models A$ for every Kripke model $\mathcal{M}$ based on $\mathcal{F}$ and every point x in $\mathcal{F}$.

Then, the intuitionistic modal logic **intK5**$_{\Box\Diamond}$is the bi-modal logic which is characterized by the class of Kripke frames for **intK5**$_{\Box\Diamond}$; that is, a formula is a thesis of **intK5**$_{\Box\Diamond}$, iff it is valid in every Kripke frame for **intK5**$_{\Box\Diamond}$. See Hasimoto [2] for a finite axiomatization of **intK5**$_{\Box\Diamond}$.

2 Modal operators $\Box^\infty$ and $\Diamond^\infty$

For the convenience of filtration, we introduce new modal operators $\Box^\infty$ and $\Diamond^\infty$ such that for every Kripke model $\mathcal{M} = \langle \mathcal{F}, V \rangle$ based on a Kripke frame $\mathcal{F} = \langle W, \lhd, R_\Box, R_\Diamond \rangle$, and every $x \in W$,

(vi) $(\mathcal{M}, x) \models \Box^\infty B$, iff $x \, R_\Box^\infty \, y$ implies $(\mathcal{M}, y) \models B$ for every $y \in W$; and

(vii) $(\mathcal{M}, x) \models \Diamond^{\infty}B$, iff $x\, R^{\infty}_{\Diamond}\, y$ and $(\mathcal{M}, y) \models B$ for some $y \in W$;

where $R^{\infty}_{\Box}$ and $R^{\infty}_{\Diamond}$ are the transitive closures of $R_{\Box}$ and $R_{\Diamond}$, respectively; to put it concretely, $x\, R^{\infty}_{\Box}\, y$ ($x\, R^{\infty}_{\Diamond}\, y$, resp.), iff for some sequence $x_1, x_2, \ldots, x_n$ of elements of W of length $n \geq 2$, $x_k\, R_{\Box}\, x_{k+1}$ ($x_k\, R_{\Diamond}\, x_{k+1}$, resp.) for $k = 1, 2, \ldots, n-1$, and moreover, $x_1 = x$ and $x_n = y$.

Proposition 2.1 *Let $\mathcal{F} = \langle W, \lhd, R_{\Box}, R_{\Diamond}\rangle$ be a Kripke frame for* **intK5**$_{\Box\Diamond}$. *The following properties hold for every $x, y, z \in W$.*

(1) *If $x\, R_{\Box}\, y$, then $x\, R^{\infty}_{\Box}\, y$.*

(2) *If $x\, R_{\Diamond}\, y$, then $x\, R^{\infty}_{\Diamond}\, y$.*

(3) *If $x\, R_{\Box}\, y$ and $y\, R^{\infty}_{\Box}\, z$, then $x\, R^{\infty}_{\Box}\, z$.*

(4) *If $x\, R_{\Box}\, y$ and $x\, R^{\infty}_{\Diamond}\, z$, then $y\, R_{\Diamond}\, z$.*

(5) *If $x\, R_{\Diamond}\, y$ and $y\, R^{\infty}_{\Diamond}\, z$, then $x\, R^{\infty}_{\Diamond}\, z$.*

(6) *If $x\, R_{\Diamond}\, y$ and $x\, R^{\infty}_{\Box}\, z$, then $y\, R_{\Box}\, z$.*

PROOF. (1)–(3) and (5) are evident. To show (4), we suppose $x\, R_{\Box}\, y$ and $x\, R^{\infty}_{\Diamond}\, z$, and derive $y\, R_{\Diamond}\, z$. From the latter assumption, $x_k\, R_{\Diamond}\, x_{k+1}$ ($k = 1, 2, \ldots, n-1$), $x_1 = x$ and $x_n = z$, for some $x_1, x_2, \ldots, x_n$ ($n \geq 2$). Owing to the property (1.2), $x_{k+1}\, R_{\Box}\, y$ and $y\, R_{\Diamond}\, x_{k+1}$ are deduced by induction on k ($k = 1, 2, \ldots, n-1$). So, $y\, R_{\Diamond}\, z$ in particular.

The proof of (6) is similar to that of (4), and so is omitted. ∎

Corollary 2.2 *Let $\mathcal{M} = \langle \mathcal{F}, V\rangle$ be a Kripke model based on a Kripke frame $\mathcal{F} = \langle W, \lhd, R_{\Box}, R_{\Diamond}\rangle$ for* **intK5**$_{\Box\Diamond}$. *The following properties hold for every $x, y \in W$ and every formula B.*

(1) *If $(\mathcal{M}, x) \models \Box^{\infty}B$, then $(\mathcal{M}, x) \models \Box B$.*

(2) *If $(\mathcal{M}, x) \models \Diamond B$, then $(\mathcal{M}, x) \models \Diamond^{\infty}B$.*

(3) *If $x\, R_{\Box}\, y$ and $(\mathcal{M}, x) \models \Box^{\infty}B$, then $(\mathcal{M}, y) \models \Box^{\infty}B$.*

(4) *If $x\, R_{\Box}\, y$ and $(\mathcal{M}, x) \models \Diamond^{\infty}B$, then $(\mathcal{M}, y) \models \Diamond B$.*

(5) *If $x\, R_{\Diamond}\, y$ and $(\mathcal{M}, y) \models \Diamond^{\infty}B$, then $(\mathcal{M}, x) \models \Diamond^{\infty}B$.*

(6) *If $x\, R_{\Diamond}\, y$ and $(\mathcal{M}, y) \models \Box B$, then $(\mathcal{M}, x) \models \Box^{\infty}B$.*

PROOF. Immediately follows from the proposition, item by item. ∎

Digression If $\mathcal{F} = \langle W, \lhd, R_\Box, R_\Diamond \rangle$ is a Kripke frame for **intK5**$_{\Box\Diamond}$, and moreover if $R_\Box = R_\Diamond$, then $R_\Box^\infty = R_\Box^2$ $(= R_\Box \circ R_\Box)$. In fact, the "$\supseteq$"-part is evident, while for the "$\subseteq$"-part, we suppose $x \, R_\Box^\infty \, y$, and derive $x \, R_\Box^2 \, y$. By the assumption, $x_k \, R_\Box \, x_{k+1}$ $(k = 1, 2, \ldots, n-1)$, $x_1 = x$ and $x_n = y$, for some $x_1, x_2, \ldots, x_n$ $(n \geq 2)$. Owing to the property $R_\Box^{-1} \circ R_\Box \subseteq R_\Box$, it follows $x_2 \, R_\Box \, x_{k+1}$ and $x_{k+1} \, R_\Box \, x_2$ by induction on k $(k = 1, 2, \ldots, n-1)$. So, $x_2 \, R_\Box \, y$ in particular. This together with $x \, R_\Box \, x_2$ implies $x \, R_\Box^2 \, y$.

Hence, in the (classical) modal logic **K5**, one can substitute the double necessitation $\Box\Box$ for the new operator $\Box^\infty$, and similarly for $\Diamond^\infty$. In a separate paper, we will show a modified subformula property for **K5** which differs from Takano [3], by combining this observation with the filtration technique developed in the next section.

3 Filtration

To accomplish filtration for the intuitionistic modal logic **intK5**$_{\Box\Diamond}$, we suppose throughout this section, that a formula A_0 (of the original language, namely, without $\Box^\infty$ nor $\Diamond^\infty$) and a Kripke model $\mathcal{M} = \langle \mathcal{F}, V \rangle$ based on a Kripke frame $\mathcal{F} = \langle W, \lhd, R_\Box, R_\Diamond \rangle$ for **intK5**$_{\Box\Diamond}$, such that $(\mathcal{M}, x) \not\models A_0$ for some $x \in W$, are given.

First, put

$$\Sigma = \mathrm{Sub}(A_0) \cup \{\Box^\infty B \mid \Box B \in \mathrm{Sub}(A_0)\} \cup \{\Diamond^\infty B \mid \Diamond B \in \mathrm{Sub}(A_0)\},$$

where $\mathrm{Sub}(A_0)$ denotes the set of subformulas of A_0, and define the equivalence relation $\sim$ on W as follows: $x \sim y$ if and only if

$$(\mathcal{M}, x) \models A \text{ iff } (\mathcal{M}, y) \models A, \text{ for every } A \in \Sigma,$$

and denote by $[x]$ the equivalence class generated by x. Clearly, Σ is a finite set containing A_0, and is closed under subformulas.

Next, define the Kripke frame $\mathcal{F}_\Sigma = \langle W_\Sigma, \lhd_\Sigma, R_{\Box\Sigma}, R_{\Diamond\Sigma} \rangle$ as follows:

(3.1) $W_\Sigma = \{[x] \mid x \in W\}$.

(3.2) $[x] \lhd_\Sigma [y]$, iff $(\mathcal{M}, x) \models A$ implies $(\mathcal{M}, y) \models A$ for every $A \in \Sigma$.

(3.3) $[x] \, R_{\Box\Sigma} \, [y]$, iff all of the following three conditions hold:

1. $(\mathcal{M}, x) \models \Box B$ implies $(\mathcal{M}, y) \models B$ for every $\Box B \in \Sigma$,
2. $(\mathcal{M}, x) \models \Box^\infty B$ implies $(\mathcal{M}, y) \models \Box^\infty B$ for every $\Box B \in \Sigma$, and
3. $(\mathcal{M}, x) \models \Diamond^\infty B$ implies $(\mathcal{M}, y) \models \Diamond B$ for every $\Diamond B \in \Sigma$.

(3.4) $[x] \, R_{\Diamond\Sigma} \, [y]$, iff all of the following three conditions hold:

1. $(\mathcal{M}, y) \models B$ implies $(\mathcal{M}, x) \models \Diamond B$ for every $\Diamond B \in \Sigma$,

2. $(\mathcal{M}, y) \models \Diamond^{\infty} B$ implies $(\mathcal{M}, x) \models \Diamond^{\infty} B$ for every $\Diamond B \in \Sigma$, and
3. $(\mathcal{M}, y) \models \Box B$ implies $(\mathcal{M}, x) \models \Box^{\infty} B$ for every $\Box B \in \Sigma$.

Then, W_Σ is finite, since Σ is finite; while $\lhd_\Sigma, R_{\Box\Sigma}$ and $R_{\Diamond\Sigma}$ are well-defined, since $\Box^{\infty} B \in \Sigma$ iff $\Box B \in \Sigma$, and $\Diamond^{\infty} B \in \Sigma$ iff $\Diamond B \in \Sigma$.

Proposition 3.1 *The quadruple $\mathcal{F}_\Sigma$ forms a Kripke frame.*

PROOF. The relation $\lhd_\Sigma$ clearly forms a partial order on W_Σ, and it is left to check the $\mathcal{F}_\Sigma$-version $\lhd_\Sigma \circ R_{\Box\Sigma} \circ \lhd_\Sigma = R_{\Box\Sigma}$ and $\lhd_\Sigma^{-1} \circ R_{\Diamond\Sigma} \circ \lhd_\Sigma^{-1} = R_{\Diamond\Sigma}$ of (1.1).

To show $\lhd_\Sigma \circ R_{\Box\Sigma} \circ \lhd_\Sigma \subseteq R_{\Box\Sigma}$ first, we suppose $[x](\lhd_\Sigma \circ R_{\Box\Sigma} \circ \lhd_\Sigma)[y]$, and derive $[x]\, R_{\Box\Sigma}\, [y]$. By the assumption, $[x] \lhd_\Sigma [u]$, $[u]\, R_{\Box\Sigma}\, [v]$ and $[v] \lhd_\Sigma [y]$, for some u, v. Hence

(a) $(\forall A \in \Sigma)[(\mathcal{M}, x) \models A \Rightarrow (\mathcal{M}, u) \models A]$;

and

(b) $(\forall \Box B \in \Sigma)[(\mathcal{M}, u) \models \Box B \Rightarrow (\mathcal{M}, v) \models B]$;

(c) $(\forall \Box B \in \Sigma)[(\mathcal{M}, u) \models \Box^{\infty} B \Rightarrow (\mathcal{M}, v) \models \Box^{\infty} B]$;

(d) $(\forall \Diamond B \in \Sigma)[(\mathcal{M}, u) \models \Diamond^{\infty} B \Rightarrow (\mathcal{M}, v) \models \Diamond B]$;

and

(e) $(\forall A \in \Sigma)[(\mathcal{M}, v) \models A \Rightarrow (\mathcal{M}, y) \models A]$.

We must show

(f) $(\forall \Box B \in \Sigma)[(\mathcal{M}, x) \models \Box B \Rightarrow (\mathcal{M}, y) \models B]$;

(g) $(\forall \Box B \in \Sigma)[(\mathcal{M}, x) \models \Box^{\infty} B \Rightarrow (\mathcal{M}, y) \models \Box^{\infty} B]$;

(h) $(\forall \Diamond B \in \Sigma)[(\mathcal{M}, x) \models \Diamond^{\infty} B \Rightarrow (\mathcal{M}, y) \models \Diamond B]$;

and these all are evident.

Conversely, suppose $[x]\, R_{\Box\Sigma}\, [y]$. Since $[x] \lhd_\Sigma [x]$, $[x]\, R_{\Box\Sigma}\, [y]$ and $[y] \lhd_\Sigma [y]$, we have $[x](\lhd_\Sigma \circ R_{\Box\Sigma} \circ \lhd_\Sigma)[y]$; hence $\lhd_\Sigma \circ R_{\Box\Sigma} \circ \lhd_\Sigma \supseteq R_{\Box\Sigma}$.

The proof of $\lhd_\Sigma^{-1} \circ R_{\Diamond\Sigma} \circ \lhd_\Sigma^{-1} = R_{\Diamond\Sigma}$ is similar, and so is omitted. ■

Proposition 3.2 *The Kripke frame $\mathcal{F}_\Sigma$ is that for* **intK5**$_{\Box\Diamond}$.

PROOF. We must show the $\mathcal{F}_\Sigma$-version $R_{\Diamond\Sigma}^{-1} \circ R_{\Box\Sigma} \subseteq R_{\Box\Sigma}$ and $R_{\Box\Sigma}^{-1} \circ R_{\Diamond\Sigma} \subseteq R_{\Diamond\Sigma}$ of (1.2); that is, if $[x]\, R_{\Box\Sigma}\, [y]$ and $[x]\, R_{\Diamond\Sigma}\, [z]$, then $[z]\, R_{\Box\Sigma}\, [y]$ and $[y]\, R_{\Diamond\Sigma}\, [z]$.

So, we suppose $[x]\, R_{\Box\Sigma}\, [y]$ and $[x]\, R_{\Diamond\Sigma}\, [z]$. We have (f)–(h) above as well as

(i) $(\forall \Diamond B \in \Sigma)[(\mathcal{M}, z) \models B \Rightarrow (\mathcal{M}, x) \models \Diamond B]$;

(j) $(\forall\Diamond B \in \Sigma)[(\mathcal{M}, z) \models \Diamond^{\infty} B \Rightarrow (\mathcal{M}, x) \models \Diamond^{\infty} B]$;

(k) $(\forall\Box B \in \Sigma)[(\mathcal{M}, z) \models \Box B \Rightarrow (\mathcal{M}, x) \models \Box^{\infty} B]$.

To derive $[z]\, R_{\Box\Sigma}\, [y]$, we must show

(l) $(\forall\Box B \in \Sigma)[(\mathcal{M}, z) \models \Box B \Rightarrow (\mathcal{M}, y) \models B]$;

(m) $(\forall\Box B \in \Sigma)[(\mathcal{M}, z) \models \Box^{\infty} B \Rightarrow (\mathcal{M}, y) \models \Box^{\infty} B]$;

(n) $(\forall\Diamond B \in \Sigma)[(\mathcal{M}, z) \models \Diamond^{\infty} B \Rightarrow (\mathcal{M}, y) \models \Diamond B]$.

For $\Box B \in \Sigma$, using (k), Corollary 2.2 (1) and (f), successively, we have

$$(\mathcal{M}, z) \models \Box B \Rightarrow (\mathcal{M}, x) \models \Box^{\infty} B \Rightarrow (\mathcal{M}, x) \models \Box B \Rightarrow (\mathcal{M}, y) \models B;$$

and also using Corollary 2.2 (1), (k) and (g), successively, we have

$$(\mathcal{M}, z) \models \Box^{\infty} B \Rightarrow (\mathcal{M}, z) \models \Box B \Rightarrow (\mathcal{M}, x) \models \Box^{\infty} B \Rightarrow (\mathcal{M}, y) \models \Box^{\infty} B.$$

Hence, (l) and (m) hold. For $\Diamond B \in \Sigma$, on the other hand, using (j) and (h), successively, we have

$$(\mathcal{M}, z) \models \Diamond^{\infty} B \Rightarrow (\mathcal{M}, x) \models \Diamond^{\infty} B \Rightarrow (\mathcal{M}, y) \models \Diamond B.$$

Hence (n) holds, too; and so $[z]\, R_{\Box\Sigma}\, [y]$ has been derived.

Derivation of $[y]\, R_{\Diamond\Sigma}\, [z]$ is similar, and so is omitted. ■

Proposition 3.3 *The following properties hold for every $x, y \in W$ and every formulas A, B.*

(1) *If $x \lhd y$, then $[x] \lhd_{\Sigma} [y]$.*

(2) *If $x\, R_{\Box}\, y$, then $[x]\, R_{\Box\Sigma}\, [y]$.*

(3) *If $x\, R_{\Diamond}\, y$, then $[x]\, R_{\Diamond\Sigma}\, [y]$.*

(4) *If $[x] \lhd_{\Sigma} [y]$, $A \in \Sigma$ and $(\mathcal{M}, x) \models A$, then $(\mathcal{M}, y) \models A$.*

(5) *If $[x]\, R_{\Box\Sigma}\, [y]$, $\Box B \in \Sigma$ and $(\mathcal{M}, x) \models \Box B$, then $(\mathcal{M}, y) \models B$.*

(6) *If $[x]\, R_{\Diamond\Sigma}\, [y]$, $\Diamond B \in \Sigma$ and $(\mathcal{M}, y) \models B$, then $(\mathcal{M}, x) \models \Diamond B$.*

PROOF. (1) and (4)–(6) are evident. To show (2), we suppose $x\, R_{\Box}\, y$, and derive $[x]\, R_{\Box\Sigma}\, [y]$. So, we must show (f)–(h) above. Among these, (f) is clear, while (g) and (h) follow from Corollary 2.2 (3) and (4), respectively.

The proof of (3) is similar to that of (2), and so is omitted. ■

Let V_{Σ} be the valuation in $\mathcal{F}_{\Sigma}$ such that for every propositional letter p,

$$(3.5)\ \ V_\Sigma(p) = \begin{cases} \{[x] \mid x \in V(p)\}, & p \in \Sigma; \\ \emptyset, & p \notin \Sigma. \end{cases}$$

Then, $[x] \in V_\Sigma(p)$ iff $x \in V(p)$ for every $x \in W$ and every propositional letter $p \in \Sigma$.

Proposition 3.4 *The map V_Σ forms a valuation in $\mathcal{F}_\Sigma$.*

PROOF. We must check the V_Σ-version

$$\text{if } [x] \lhd_\Sigma [y] \text{ and } [x] \in V_\Sigma(p), \text{ then } [y] \in V_\Sigma(p)$$

of (1.3). But whether $p \in \Sigma$ or not, this can be assured. ■

So, we have defined the Kripke model $\mathcal{M}_\Sigma = \langle \mathcal{F}_\Sigma, V_\Sigma \rangle$ based on the Kripke frame $\mathcal{F}_\Sigma$ for **intK5$_{\Box\Diamond}$**. Although $\mathcal{M}_\Sigma$ forms a filtration of the given Kripke model $\mathcal{M}$ through the finite set Σ, yet Σ is not a set of formulas of the original language, but of the enlarged language.

Proposition 3.5 *The following equivalence holds for every $x \in W$ and every $A \in$ Sub(A_0) :*

$$(\mathcal{M}, x) \models A \quad \textit{iff} \quad (\mathcal{M}_\Sigma, [x]) \models A.$$

PROOF. By the routine induction on the construction of A, utilizing Proposition 3.3. ■

So, $(\mathcal{M}_\Sigma, [x]) \not\models A_0$ for some $x \in W$, in particular.

Since $\mathcal{F}_\Sigma$ contains only a finite number of points, and **intK5$_{\Box\Diamond}$**is finitely axiomatizable, we have obtained the following theorem.

Theorem *The intuitionistic modal logic* **intK5$_{\Box\Diamond}$***has the finite model property, and hence is decidable.* ■

4 Y. Hasimoto's remark

Proposition 3.5 claims the equivalence only for formulas in Sub(A_0), and this suffices for our purpose. But, as Y. Hasimoto pointed out, the equivalence holds always for all formulas in Σ. The proof follows from Proposition 3.3 as well as the following proposition.

Proposition 4.1 *The following properties hold for every $x, y \in W$ and every formula B.*

(1) *If $x\, R^\infty_\Box\, y$, then $[x]\, R^\infty_{\Box\Sigma}\, [y]$.*

(2) *If $x\, R^\infty_\Diamond\, y$, then $[x]\, R^\infty_{\Diamond\Sigma}\, [y]$.*

(3) *If* $[x]\,R^{\infty}_{\Box\Sigma}\,[y]$, $\Box B \in \Sigma$ *and* $(\mathcal{M},x) \models \Box^{\infty}B$, *then* $(\mathcal{M},y) \models B$.

(4) *If* $[x]\,R^{\infty}_{\Diamond\Sigma}\,[y]$, $\Diamond B \in \Sigma$ *and* $(\mathcal{M},y) \models B$, *then* $(\mathcal{M},x) \models \Diamond^{\infty}B$.

PROOF. To show (1), suppose $x\,R^{\infty}_{\Box}\,y$. Then, $x_k\,R_{\Box}\,x_{k+1}$ $(k = 1,2,\ldots,n-1)$, $x_1 = x$ and $x_n = y$, for some $x_1, x_2, \ldots, x_n$ $(n \geq 2)$. By Proposition 3.3 (2), $[x_k]\,R_{\Box\Sigma}\,[x_{k+1}]$ $(k = 1,2,\ldots,n-1)$, and moreover, $[x_1] = [x]$ and $[x_n] = [y]$. Hence $[x]\,R^{\infty}_{\Box\Sigma}\,[y]$.

To show (3), next, suppose $[x]\,R^{\infty}_{\Box\Sigma}\,[y]$, $\Box B \in \Sigma$ and $(\mathcal{M},x) \models \Box^{\infty}B$. By the first assumption, $[x_k]\,R_{\Box\Sigma}\,[x_{k+1}]$ $(k = 1,2,\ldots,n-1)$, $[x_1] = [x]$ and $[x_n] = [y]$, for some $x_1, x_2, \ldots, x_n$ $(n \geq 2)$. It follows $(\mathcal{M},x_k) \models \Box^{\infty}B$ $(k = 1,2,\ldots,n-1)$ by induction on k. So, $(\mathcal{M},x_{n-1}) \models \Box^{\infty}B$ in particular, and hence $(\mathcal{M},x_{n-1}) \models \Box B$ by Corollary 2.2 (1). So, $(\mathcal{M},y) \models B$.

The proof of (2) and (4) is similar to that of (1) and (3), respectively, and so is omitted. ∎

References

[1] A. Chagrov and M. Zakharyaschev, **Modal Logic** (Oxford Logic Guides 35), Oxford, New York (1997).

[2] Y. Hasimoto, *Finite model property for some intuitionistic modal logics*, **Bull. Sect. Logic, Univ. Łódź** 30 (2001), pp. 87–97.

[3] M. Takano, *A modified subformula property for the modal logics* $K5$ *and* $K5D$, **Bull. Sect. Logic, Univ. Łódź** 30 (2001), pp. 115–122.

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