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A REMARK ON PREHOMOGENEOUS ACTIONS OF LINEAR ALGEBRAIC GROUPS

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ABSTRACT. Let G be a connected linear algebraic group acting rationally on an affine variety V defined over an algebraically closed field of characteristic zero, and G' a closed normal subgroup of G such that G/G' is a torus. In this paper, we show that the condition that a G-orbit O in V is decomposed into infinitely many G'-orbits can be characterized by the existence of a certain G-relative invariant on the orbit O. In fact, this is a condition of whether or not G' acts prehomogeneously on O.

INTRODUCTION

Let G be a connected linear algebraic group acting rationally on an affine variety V defined over an algebraically closed field of characteristic zero, and G' a closed normal subgroup of G such that G/G' is a torus. In this paper, we will show that a G-orbit (i.e., a G-homogeneous space) O in V is decomposed into infinitely many G'-orbits if and only if there exist a non-trivial rational character χ of G and a non-constant rational function f on the orbit O such that $\chi|_{G'} = 1$ and $f(g \cdot w) = \chi(g)f(w)$ for any $g \in G$ and $w \in O$. As mentioned in Proposition 1.2, this is equivalent to the condition that O has no open G'-orbit.

Let $\rho : G \to GL(V)$ be a rational representation of a linear algebraic group G on a finite dimensional vector space V. If V is decomposed into a finite union of G-orbits, it must have a unique Zariski dense orbit (we call it a finite prehomogeneous vector space (abbrev. F.P.)). Such F.P.'s were first classified in the case of irreducible ρ (see [6], § 8), and next under the assumption that ρ is the action of $G = G' \times GL(1)^l$ on V which is the composite of a rational representation ρ' of a semisimple algebraic

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group G' and scalar multiplications $GL(1)^l$ on each irreducible component V_i, where $V = V_1 \oplus \cdots \oplus V_l$ is the decomposition of ρ' into irreducible components (see [3]).

Up to now, some class of F.P.'s for semisimple algebraic groups has been classified (see [5]). On the other hand, it is known that the condition that a representation of $SL(d_1) \times \cdots \times SL(d_r)$ associated with an arbitrary quiver of type A_r is not an F.P. can be characterized by the existence of a certain absolute invariant (see [4]). Our theorem (Theorem 2.2) gives a unified understanding of the reason why a finite-dimensional rational representation ρ' of a semisimple algebraic group G' is not an F.P. though the composite of ρ' and scalar multiplications is an F.P.

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1. PRELIMINARIES

We assume that all are defined over an algebraically closed field $\mathbb{K}$ of characteristic zero.

Lemma 1.1. *Let G be a linear algebraic group acting rationally on an affine variety V, and G' a closed subgroup of G. For $v \in V$, let $G \cdot v = \bigsqcup_{\lambda \in \Lambda} G' \cdot v_\lambda$ be the orbital decomposition of the G-orbit $O = G \cdot v$ containing v into G'-orbits. If $\#\Lambda < \infty$, then we have $\dim O = \dim G' \cdot v_\lambda$ for some $\lambda \in \Lambda$.*

Proof. Let $O = G' \cdot v_1 \sqcup \cdots \sqcup G' \cdot v_r$ be the decomposition into its finitely many G'-orbits. Taking the Zariski closure, we have $\overline{O} = \overline{G' \cdot v_1} \cup \cdots \cup \overline{G' \cdot v_r}$. The uniqueness for the decomposition into irreducible components implies that $\dim \overline{O} = \dim \overline{G' \cdot v_\lambda}$ for some v_λ.

In general, we choose an irreducible component U of $\overline{O}$ satisfying $\dim \overline{O} = \dim U$. Considering the decomposition of G into its connected components, we may assume that $U = \overline{G^\circ \cdot w}$ for some $w \in O$ (here G° is the identity component of G). Then, since the orbit O is open in $\overline{O}$, we see that $U \cap O$ is open in the irreducible U. Hence we have $\dim \overline{O} = \dim U = \dim U \cap O \leqq \dim O \leqq \dim \overline{O}$. Consequently we obtain our assertion. $\square$

Proposition 1.2. *Let G be a connected linear algebraic group acting rationally on an affine variety V, and G' a closed normal subgroup of G. For $v \in V$, let*

$G\cdot v=\bigsqcup_{\lambda\in\Lambda} G'\cdot v_\lambda$ be the orbital decomposition of the G-orbit $O=G\cdot v$ containing v into G'-orbits. Then the following seven conditions are equivalent:

(1) *$\#\Lambda<\infty$; i.e., O is decomposed into only finitely many G'-orbits.*
(2) *$\#\Lambda=1$; i.e., O is G'-homogeneous.*
(3) *O has a Zariski open G'-orbit; i.e., O is G'-prehomogeneous.*
(4) *$G=G'H_v$, where we put $H_v=\{g\in G;\ g\cdot v=v\}$.*
(5) $\dim G=\dim G'H_v$.
(6) $\dim G/G'=\dim H_v/H_v\cap G'$.
(7) $\dim O=\dim G'\cdot w$ *for any $w\in O$.*

Proof. Since the canonical surjection $H_v\to G'H_v/G'$ is a morphism of algebraic groups with kernel $H_v\cap G'$, we have $\dim H_v=\dim H_v\cap G'+\dim G'H_v/G'$; i.e., $\dim H_v/H_v\cap G'=\dim G'H_v/G'$. On the other hand, we note that $H_v\cap G'=H_w\cap G'$ for any $w\in O$ because G' is a normal subgroup of G. Thus we have

$$\begin{aligned}
&\dim G-\dim G'H_v\\
&\quad=\dim G-\dim G'-\dim G'H_v+\dim G'\\
&\quad=\dim G/G'-\dim G'H_v/G'\\
&\quad=\dim G/G'-\dim H_v/H_v\cap G'\\
&\quad=\dim G-\dim G'-\dim H_v+\dim H_v\cap G'\\
&\quad=\dim G-\dim H_v-\dim G'+\dim H_w\cap G' &&\text{for any } w\in O\\
&\quad=\dim O-\dim G'\cdot w &&\text{for any } w\in O.
\end{aligned}$$

Hence we obtain (5) $\Longleftrightarrow$ (6) $\Longleftrightarrow$ (7). Suppose that $\dim G\gneqq\dim G'H_v$. Then the same calculation implies that $\dim O\gneqq\dim G'\cdot w$ for any $w\in O$. Therefore Lemma 1.1 implies $\#\Lambda=\infty$, which shows that (1) $\Longrightarrow$ (5).

If $G\supsetneqq G'H_v$, then we have $\dim G\gneqq\dim G'H_v$ because $G'H_v$ is a proper closed subset of the irreducible variety G. Thus we have (5) $\Longleftrightarrow$ (4). Suppose that $G=G'H_v$. Then, for any $w\in O$, we can choose $g'\in G'$ and $h\in H_v$ satisfying $(g'h)\cdot v=w$. Thus we have $g'\cdot v=w$, which shows that (4) $\Longrightarrow$ (2).

Obviously (2) implies (3). To show the converse, we take an arbitrary point w belonging to an open G'-orbit in O. Then $G'H_w$ is open in G, because it is the inverse image of the open orbit $G' \cdot w$ under the morphism $G \to O$, $g \mapsto g \cdot w$. Since G' is a normal subgroup of G, we have $G = G'H_w$, which implies (2). $\square$

2. The existence of relative invariants

Although the next lemma is well known in the theory of prehomogeneous vector spaces, it is important for our theorem; so we will give the proof here (see [2], Proposition 2.11).

For a linear algebraic group G, we denote by $X(G)$ the group of all rational characters of G.

Lemma 2.1. *Let G be a connected linear algebraic group acting rationally on an affine variety V, and $H_v = \{g \in G;\ g \cdot v = v\}$ the isotropy subgroup at $v \in V$. Then, for any $\chi \in H_v{}^{\perp} = \{\chi \in X(G);\ \chi|_{H_v} = 1\}$, there exists a relative invariant corresponding to χ on the orbit $O = G \cdot v$; that is, a rational function f satisfying $f(g \cdot w) = \chi(g) f(w)$ for any $g \in G$ and $w \in O$. (Here we consider the restriction of an element of the function field $\mathbb{K}(\overline{O})$ of the irreducible variety $\overline{O}$ to be a rational function on O.)*

Proof. Take $\chi \in H_v{}^{\perp}$. Then we can choose a regular function $\bar{f}$ on G/H_v satisfying $\bar{f}(gH_v) = \chi(g)$. Since the canonical bijective mapping $\pi : G/H_v \to O$ is a morphism between non-singular irreducible algebraic varieties, we have $\pi^*\big(\mathbb{K}(O)\big) = \mathbb{K}(G/H_v)$ (here note that the field $\mathbb{K}$ is of characteristic zero; see [1], AG. 18.2). Hence π is an isomorphism, and we can uniquely choose a regular function $f \in \mathbb{K}(O) = \mathbb{K}(\overline{O})$ satisfying $\bar{f} = f \circ \pi$. In fact, it is explicitely given by $f(w) = \chi(g)$ for $w = g \cdot v \in O$, which is well-defined and is a relatively G-invariant regular function. $\square$

Theorem 2.2. *In the same situation as in Proposition 1.2, assume that G/G' is a torus. For a point $v \in V$, we put $H_v{}^{\perp} = \{\chi \in X(G);\ \chi|_{H_v} = 1\}$ and $H_v'^{\perp} = \{\chi \in X(G');\ \chi|_{H_v'} = 1\}$ (here $H_v = \{g \in G;\ g \cdot v = v\}$ and $H_v' = H_v \cap G'$). Then the following conditions are equivalent:*

1. *The orbit $O = G \cdot v$ is decomposed into infinitely many G'-orbits; i.e., O is not G'-prehomogeneous.*
2. *The restriction homomorphism $\pi : H_v{}^{\perp} \to H_v'^{\perp}$ is not injective.*

(3) *There exist a non-trivial rational character χ of G and a non-constant rational function f on the orbit O such that $\chi|_{G'} = 1$ and $f(g \cdot w) = \chi(g)f(w)$ for any $g \in G$ and $w \in O$.*

Proof. Since $\ker \varphi_1 = X(G/H_v[G,G]) \simeq {H_v}^{\perp}$, $\ker \varphi_2 = X(G'/H'_v) \simeq {H'_v}^{\perp}$, and $X(H_v) = X(H_v[G,G])$ (here $[G,G]$ is the commutator subgroup of G), the canonical short exact sequences induce the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
1 & \longrightarrow & \ker \varphi_0 & \longrightarrow & {H_v}^{\perp} & \xrightarrow{\pi} & {H'_v}^{\perp} \\
 & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & X(G/G') & \longrightarrow & X(G) & \longrightarrow & X(G') \\
 & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} \\
1 & \longrightarrow & X(H_v/H'_v) & \longrightarrow & X(H_v) & \longrightarrow & X(H'_v).
\end{array}$$

Note that H_v/H'_v ($\simeq G'H_v/G'$) can be regarded as a closed subgroup of the torus G/G', because they are defined over an algebraically closed field of characteristic 0. So we see that they are diagonalizable, and hence we have $\dim G/G' = \operatorname{rank} X(G/G')$ and $\dim H_v/H'_v = \operatorname{rank} X(H_v/H'_v)$. On the other hand, we see that $\operatorname{rank} X(G/G') \gneqq \operatorname{rank} X(H_v/H'_v)$ if and only if φ_0 is not injective, because $X(G/G')$ is a free abelian group and the restriction homomorphism φ_0 is surjective. Since $\ker \varphi_0 \simeq \ker \pi$, this condition is equivalent to the condition that π is not injective. Therefore, by Proposition 1.2, we obtain (1) $\Longleftrightarrow$ (2).

Note that the condition (2) is equivalent to the condition that there exists a non-trivial character $\chi \in {H_v}^{\perp}$ satisfying $\chi|_{G'} = 1$. In particular, the condition (2) follows from (3). Conversely, if the condition (2) is satisfied, then there exists a non-trivial character $\chi \in {H_v}^{\perp}$ satisfying $\chi|_{G'} = 1$. Therefore, by Lemma 2.1, we obtain the condition (3). □

In general, a rational representation (G, V) of a connected linear algebraic group G on a finite dimensional vector space V is called a prehomogeneous vector space (abbrev. P.V.) if there exists $v \in V$ satisfying $\dim H_v = \dim G - \dim V$, where $H_v = \{g \in G;\ g \cdot v = v\}$ (see [2], § 2.1). Thus Theorem 2.2 can be regarded as a generalization of the criterion by Servedio [7], § 3 (see also [2], Proposition 7.41).

In the same situation as in Theorem 2.2, there exists an F.P. (G,V) such that (G',V) is not an F.P. but a P.V.; i.e., a non-generic G-orbit is decomposed into infinitely many G'-orbits:

Example 2.3. Let (G,V) be the representation, of dimension $d=(1,3,1)$, associated with the A_3-type quiver $\cdot \longleftarrow \cdot \longrightarrow \cdot$; that is, $G=GL(1)\times GL(3)\times GL(1)$ acts on $V=M(1,3)\oplus M(1,3)$ by $g\cdot v=(g_1v_1g_2{}^{-1},\ g_3v_2g_2{}^{-1})$ for $g=(g_1,g_2,g_3)\in G$ and $v=(v_1,v_2)\in V$. Then the torus-restricted subgroup $G'=SL(1)\times GL(3)\times SL(1)$ also acts on V. It is well known that (G,V) is an F.P. (it has exactly five orbits). Moreover we see that (G',V) is a prehomogeneous vector space with a generic point $v=\big((1\,0\,0),\ (0\,0\,1)\big)$; that is, we have $G\cdot v=G'\cdot v$.

On the other hand, let $H_w=\{g\in G;\ g\cdot w=w\}$ be the isotropy subgroup at $w=\big((0\,0\,1),\ (0\,0\,1)\big)\in V$, and put $H'_w=H_w\cap G'$. Then we have $\dim G/G'=2>\dim H_w/H'_w=1$. Hence the orbit $G\cdot w$ is decomposed into infinitely many G'-orbits; that is, (G',V) is *not* an F.P.

In this case, since we can write the orbit as $G\cdot w=U_1\cup U_2\cup U_3$, where

$$U_i=\big\{\big((\alpha x_1\,\alpha x_2\,\alpha x_3),\ (x_1\,x_2\,x_3)\big)\in V;\ \alpha\in\mathbb{K}^\times \text{ and } x_i\neq 0\big\},$$

a relatively G-invariant (and G'-invariant) regular function on the orbit $G\cdot w$ is given by $f(x)=x_i/y_i$ on U_i $(i=1,2,3)$ for $x=\big((x_1\,x_2\,x_3),\ (y_1\,y_2\,y_3)\big)\in V$.

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