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CHARACTERIZATION OF COMPLEX SPACE FORMS IN TERMS OF GEODESICS AND CIRCLES ON THEIR GEODESIC SPHERES

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ABSTRACT. In this paper we pay particular attention to geodesics and circles on geodesic spheres in a given Kähler manifold. We characterize complex space forms in the class of Kähler manifolds from this point of view.

1. Introduction.

In this paper we characterize complex space forms among Kähler manifolds by observing the structure torsion of *geodesics* on their geodesic spheres. A complex n-dimensional complex space form is a Kähler manifold of constant holomorphic sectional curvature c, which is locally congruent to either a complex projective space $\mathbb{C}P^n(c)$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $\mathbb{C}H^n(c)$, according as c is positive, zero or negative. Among real hypersurfaces in a complex space form geodesic spheres have many nice properties. For a unit tangent vector $v \in T_xN$ of a real hypersurface N in a Kähler manifold M, we put $\eta(v) = \langle v, \xi_x \rangle$ and call its structure torsion. Here ξ is the characteristic vector field of N in M which is defined by $\xi = -J\mathcal{N}$ with unit normal vector field $\mathcal{N}$ and complex structure J of M. For a geodesic γ on N which is parameterized by its arc-length, we can define a structure torsion function η_γ by $\eta(\dot\gamma)$. When N is a geodesic sphere in a complex space form, the structure torsion η_γ for an arbitrary geodesic γ is a constant function. Our main result in this paper is Theorem 1 which characterizes complex space forms among Kähler manifolds by this property. We also give a characterization of complex Euclidean spaces by the extrinsic shape of circles on geodesic spheres (Theorem 2).

2. Characterization of complex space forms.

For a Riemannian manifold $(M, \langle\ ,\ \rangle)$ of dimension greater than 2, we denote by $G_x(r)$ a geodesic sphere of radius r centered at $x \in M$, and by $A = A_{x,r}$ the shape operator of $G_x(r)$ in M with respect to the outward unit normal vector field $\mathcal{N}$. We then have the following relationship between the Riemannian connections $\widetilde\nabla$ of M and ∇ of $G_x(r)$:

$$(2.1) \qquad \widetilde\nabla_X Y = \nabla_X Y + \langle A_{x,r}X, Y\rangle \mathcal{N} \quad \text{and} \quad \widetilde\nabla_X \mathcal{N} = -A_{x,r}X.$$

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Our study in this paper is deeply based on the following expansion for the second fundamental form.

Lemma ([CV, Theorem 3.1]). *For nonzero tangent vectors $v, w \in T_xM$ at a point $x \in M$, we choose a unit tangent vector $u \in T_xM$ orthogonal to both v and w. We denote by $v_r, w_r \in T_{\exp_x(ru)}M$ the parallel displacements of v, w along the geodesic segment $\exp_x(su), 0 \leqq s \leqq r$. Then for sufficiently small r we have*

$$\langle A_{m,r}v_r, w_r\rangle = \frac{1}{r}\langle v, w\rangle - \frac{r}{3}\langle R(u,v)w, u\rangle + O(r^2). \tag{2.2}$$

Our characterization of complex space forms is the following:

Theorem 1. *For a Kähler manifold M of complex $n(\geqq 2)$-dimension, the following two conditions are equivalent each other.*

(1) *M is a complex space form.*
(2) *At an arbitrary point $x \in M$, for each geodesic sphere $G_x(r)$ of sufficiently small radius r, every geodesic on $G_x(r)$ has constant structure torsion.*

Proof. Since a geodesic sphere G is a real hypersurface, it admits an almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ induced by the complex structure J of M, which satisfies $\phi^2 = -I + \eta \otimes \xi$. It follows from equalities (2.1) that

$$(\nabla_X\phi)Y = \eta(Y)AX - \langle AX, Y\rangle\xi \quad \text{and} \quad \nabla_X\xi = \phi AX. \tag{2.3}$$

For a geodesic γ on a geodesic sphere G, by the second equality in (2.3) we find that

$$\eta_\gamma' = \nabla_{\dot\gamma}\langle\dot\gamma, \xi\rangle = \langle\dot\gamma, \nabla_{\dot\gamma}\xi\rangle = \langle\dot\gamma, \phi A\dot\gamma\rangle = -\langle A\phi\dot\gamma, \dot\gamma\rangle. \tag{2.4}$$

(1) $\Longrightarrow$ (2). Since it holds $\phi A = A\phi$ for every geodesic sphere G in a complex space form (see for example [NR]), we see that $\eta_\gamma' = \langle\dot\gamma, \phi A\dot\gamma\rangle = -\langle\dot\gamma, \phi A\dot\gamma\rangle$, which shows $\eta_\gamma' = 0$.

(2) $\Longrightarrow$ (1). For a geodesic sphere $G_x(r)$ of radius r, it follows from (2.4) that $\langle v, \phi Av\rangle = 0$ for every tangent vector $v \in TG_x(r)$. In particular, we see $\langle v + w, \phi A(v + w)\rangle = 0$ for arbitrary $v, w \in T_yG_x(r)$ at an arbitrary point $y \in G_x(r)$. This guarantees $\langle(\phi A - A\phi)v, w\rangle = 0$ for arbitrary $v, w \in T_yG_x(r)$, so that $\phi A_{m,r} = A_{m,r}\phi$. This shows that the characteristic vector ξ is a principal curvature vector of $G_x(r)$ in the ambient Kähler manifold M.

Given a unit tangent vector $v \in T_xM$ we take a unit tangent vector $w \in T_xM$ which is orthogonal to both v and Jv and use Lemma by putting $u = Jv$. Since u_r is a normal vector of $G_x(r)$ in M at $y = \exp_x(ru)$ and M is Kähler, the vector $v_r = -Ju_r$ is the characteristic vector of $G_x(r)$ at y. As v_r is a principal curvature vector of $G_x(r)$, the equality (2.2) implies that the curvature tensor R of M satisfies $\langle R(u, Ju)w, u\rangle = 0$. Thus we find that $R(u, Ju)u$ is proportional to Ju for every $u \in T_xM$, so that M is a complex space form (see [T]). □

3. Characterization of a complex Euclidean space.

A smooth curve γ in M parameterized by its arc-length is called a *circle* of curvature $\kappa(\geqq 0)$, if there exists a field of unit vectors Y_s along this curve which satisfies the differential equations $\nabla_{\dot\gamma}\dot\gamma = \kappa Y_s, \nabla_{\dot\gamma}Y_s = -\kappa\dot\gamma$, where κ is a constant and $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ on M. A circle of null curvature is nothing but a circle. More generally, for a smooth curve γ in M parameterized by its arc-length, we usually call the norm $\|\nabla_{\dot\gamma}\dot\gamma\|$ the *first curvature* of γ in the sense of Frenet formula. In their paper[CV], Chen and Vanhecke gave the following characterization of complex space forms.

Proposition. *A complex $n(\geqq 2)$-dimensional Kähler manifold M is a complex space form if and only if at an arbitrary point $x \in M$ for each geodesic sphere $G_x(r)$ of sufficiently small radius r every geodesic on $G_x(r)$ has constant first curvature as a curve in the ambient space M,*

Motivated by this result, we characterize a complex Euclidean space in the class of Kähler manifolds by observing circles of positive curvature on its geodesic spheres.

Theorem 2. *For a complex $n(\geqq 2)$-dimensional Kähler manifold M the following two conditions are equivalent each other.*

(1) *M is locally congruent to a complex Euclidean space.*

(2) *At an arbitrary point $x \in M$, for each geodesic sphere $G_x(r)$ of sufficiently small radius r, there exists $\kappa = \kappa_{x,r} > 0$ satisfying that every circle of curvature κ on $G_x(r)$ has constant first curvature as a curve in the ambient space M.*

Though this theorem can be obtained as a consequence of the result in [M], we here give a complete proof without using the result.

Proof. (1) $\Longrightarrow$ (2). Since the geodesic sphere $G(r)$ of radius r in $\mathbb{C}^n$ is a standard sphere of constant sectional curvature $1/r^2$, it is known that every circle of curvature κ on $G(r)$ can be regard as a circle of curvature $\sqrt{\kappa^2 + (1/r^2)}$ in $\mathbb{C}^n$, so that our assertion is obvious.

(2) $\Longrightarrow$ (1). For an arbitrary orthogonal pair of vectors $u, v \in T_yG_x(r)$ at a point $y \in G_x(r)$ we take a circle $\gamma = \gamma(s), s \in I$ of curvature of κ on a geodesic sphere $G_x(r)$ with initial condition that $\gamma(0) = x, \dot\gamma(0) = u$ and $Y_0(= (1/\kappa)\nabla_{\dot\gamma}\dot\gamma(0)) = v$. It follows from the first equality in (2.1) that

$$\widetilde{\nabla}_{\dot\gamma(s)}\dot\gamma(s) = \kappa Y_s + \langle A_{x,r}\dot\gamma, \dot\gamma\rangle\mathcal{N}, \quad s \in I.$$

By the assumption, the first curvature $\kappa_1 = \|\widetilde{\nabla}_{\dot\gamma}\dot\gamma\|$ of the curve γ in the ambient Kähler manifold M is constant, so that this equality implies that $\langle A_{m,r}\dot\gamma, \dot\gamma\rangle$ is constant on I. Hence we have

$$\begin{aligned} 0 = \frac{d}{ds}\langle A_{m,r}\dot\gamma, \dot\gamma\rangle &= \langle(\nabla_{\dot\gamma}A_{m,r})\dot\gamma, \dot\gamma\rangle + 2\langle A_{m,r}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle \\ &= \langle(\nabla_{\dot\gamma}A_{m,r})\dot\gamma, \dot\gamma\rangle + 2\kappa\langle A_{m,r}\dot\gamma, Y_s\rangle. \end{aligned}$$

Evaluating this equation at $s = 0$, we get

$$\langle(\nabla_u A_{m,r})u, u\rangle + 2\kappa\langle A_{m,r}u, v\rangle = 0. \tag{3.1}$$

On the other hand, for another circle $\rho = \rho(s)$ of the same curvature κ on $G_x(r)$ with initial condition that $\rho(0) = x$, $\dot\rho(0) = u$ and $\nabla_{\dot\rho}\dot\rho(0) = -\kappa v$, we see that

$$\langle(\nabla_u A_{m,r})u, u\rangle - 2\kappa\langle A_{m,r}u, v\rangle = 0 \tag{3.2}$$

which corresponds to equation (3.1). Thus, from (3.1) and (3.2) we obtain $\langle A_{m,r}u, v\rangle = 0$ for an arbitrary orthonormal pair of vectors u, v at each point y of $G_x(r)$, so that our geodesic sphere $G_x(r)$ is totally umbilic in M. This, together with equation (2.2), yields that the curvature tensor R of the ambient Kähler manifold M satisfies $\langle R(u, v)w, u\rangle = 0$ for any orthonormal vectors $u, v, w \in T_xM$. By virtue of Cartan's result we find from this property that M is of constant sectional curvature. Therefore we can conclude that our Kähler manifold M of complex dimension $n(\geqq 2)$ is nothing but a complex Euclidean space. $\square$

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