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THE σ-CONVEXITY OF ALL BOUNDED CONVEX SETS IN $\mathbb{R}^n$ AND $\mathbb{C}^n$

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Introduction. we call a convex subset U in the n-dimensional real Euclidean space $\mathbb{R}^n$ or the n-dimensional complex Euclidean space $\mathbb{C}^n$ as a σ-convex set if $U = \left\{\sum_{j=1}^{\infty} a_j\lambda^{(j)}; \lambda^{(j)} \in U, a_j \geq 0, \sum_{j=1}^{\infty} a_j = 1\right\}$. In this papaer, we shall show that any bounded convex subsets in $\mathbb{R}^n$ or $\mathbb{C}^n$ are σ-convex. We have not seen the notation of σ-convexity elsewhere. Takemoto and Uchiyama [1] showed that any bounded convex subsets in $\mathbb{C}$ are σ-convex. This result gave a useful role for the arguments of the numerical ranges of operators on Hilbert spaces.

Main Theorem. We also give in this paper the proof that any bounded convex subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$ are σ-convex.

Definition. Let U be a subset of $\mathbb{R}^n$ or $\mathbb{C}^n$. We call U as a σ-convex set if U satisfies the following relation: Let $\{\lambda^{(j)}\}_{j=1}^{\infty}$ be a sequence in U and $\{a_j\}_{j=1}^{\infty}$ a sequence with $a_j \geq 0$ and $\sum_{j=1}^{\infty} a_j = 1$, then the element $\sum_{j=1}^{\infty} a_j\lambda^{(j)}$ is an element of U.

Theorem. Let U be a bounded convex subset of $\mathbb{R}^n$ or $\mathbb{C}^n$, then we have the following relation $U = \left\{\sum_{j=1}^{\infty} a_j\lambda^{(j)}; \lambda^{(j)} \in U, a_j \geq 0, \sum_{j=1}^{\infty} a_j = 1\right\}$.

Proof. We can get the conclusion in the case of the complex Euclidean spaces by using a similar argument in the case of the n-dimensional real Euclidean spaces. So, we shall only show the σ-convexity for any bounded convex subsets in $\mathbb{R}^n$ by using the mathematical induction.

Let $V = \left\{\sum_{j=1}^{\infty} a_j\lambda^{(j)}; \lambda^{(j)} \in U, a_j \geq 0, \sum_{j=1}^{\infty} a_j = 1\right\}$, then V is contained in the the closure $\overline{U}$ of U. If λ is an element of $V - U$, then λ is an element of $\overline{U} - U$. Since U is a convex set and λ is not an element of U, there exists a hyperplane $H = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n; \sum_{k=1}^{n} b_k x_k = c \right\}$ containing λ that U lies in the half-plane of the hyperplane H where $\{x_1, \cdots \cdots, x_n\}$ and c are real numbers. Then we have the relation $\sum_{k=1}^{n} b_k \lambda_k = c$ where $\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \vdots \\ \lambda_n \end{bmatrix}$. Without loss of generality, we can assume that $\lambda = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$ (this induces $c = 0$) and $U \subset \left\{ \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}; \sum_{k=1}^{n} b_k x_k \leq 0 \right\}$.

The case of $n = 1$: The element λ and each $\lambda^{(j)}$ are real numbers in the case of $n = 1$. Since each $\lambda^{(j)}$ is a negative real number, $a_j \geq 0$ ($j = 1, 2, \cdots$) and $\sum_{j=1}^{\infty} a_j = 1$, $\sum_{j=1}^{\infty} a_j \lambda^{(j)} < 0 = \lambda$. This is a contradiction. Thus λ is an element of U.

The case of $n = 2$: By the assumption, the hyperplane H is a line $L; b_1 x + b_2 y = 0$ in the xy-plane $\mathbb{R}^2$. Furthermore, we can assume that the line L is the y-axis $\left\{ \begin{bmatrix} x \\ y \end{bmatrix}; x = 0 \right\}$ and $U \subset \left\{ \begin{bmatrix} x \\ y \end{bmatrix}; x \leq 0 \right\}$. Put $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \lambda = \sum_{j=1}^{\infty} a_j \lambda^{(j)}$ and $\lambda^{(j)} = \begin{bmatrix} \lambda_1^{(j)} \\ \lambda_2^{(j)} \end{bmatrix}$. Since $\lambda_1^{(j)} \leq 0$, $a_j \geq 0$ ($j = 1, 2, \cdots$) and $\sum_{j=1}^{\infty} a_j = 1$, we can show the relation $\lambda_1^{(j)} = 0$ ($j = 1, 2, \cdots$). Thus, each $\lambda^{(j)}$ is on the y-axis. Since all $\lambda^{(j)}$ ($j = 1, 2, \cdots$) are element of U, U is a convex set and λ is not an element of U, each $\lambda^{(j)}$ is on the half part of the y-axis with respect to λ. Thus, we have the relation $\lambda_2^{(j)} > 0$ for every j or $\lambda_2^{(j)} < 0$ for every j. Therefore, we have the following relation:

$$0 = y\text{-coefficient of } \lambda = y\text{-coefficient of } \sum_{j=1}^{\infty} a_j \lambda^{(j)} = \sum_{j=1}^{\infty} a_j \lambda_2^{(j)} \neq 0.$$

This is a contradiction. Therefore λ is an element of U in the case of $n = 2$.

The case of $n = 1, 2, \cdots, k-1$: If U is a bounded convex subset of $\mathbb{R}^n$ for $n \leq k-1$, we assume that $U = \left\{ \sum_{j=1}^{\infty} a_j \lambda^{(j)}; \lambda^{(j)} \in U, a_j \geq 0, \sum_{j=1}^{\infty} a_j = 1 \right\}$.

The case of $n = k$: We can assume that the hyperplane H is the subspace $\left\{ \begin{bmatrix} 0 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} ; x_2, \cdots, x_n \in \mathbb{R} \right\}$. Furthermore, we can assume that $\lambda = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$ and $U \subset \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} ; x_1 \leq 0 \right\}$ without loss of generality.

Put $\begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} = \lambda = \sum_{j=1}^{\infty} a_j \lambda^{(j)}$, $\lambda^{(j)} = \begin{bmatrix} \lambda_1^{(j)} \\ \vdots \\ \vdots \\ \lambda_n^{(j)} \end{bmatrix}$, $a_j \geq 0$ and $\sum_{j=1}^{\infty} a_j = 1$.

Since $\lambda_1^{(j)} \leq 0$, $a_j \geq 0$ $(j = 1, 2 \cdots)$ and $\sum_{j=1}^{\infty} a_j = 1$, we can show the relation $\lambda_1^{(j)} = 0$ $(j = 1, 2, \cdots)$. Thus, let $W = U \cap H$, then W is a bounded convex subset in the subspace H and $\{\lambda^{(j)}\}_{j=1}^{\infty} \subset W$. Since the dimension of H is $k-1$, by the assumption of mathematical induction, λ is an element of W. Therefore λ is an element of U and so we have the complete proof of theorem.

References

[1] H. Takemoto and A. Uchiyama, *A remark of the numerical ranges of operators on Hilbert spaces*, Nihonkai Mathematical Journal, vol. 13 (2002), 1–7.

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