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Extremals for families of plane quasiconformal mappings

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Abstract

Let $\mathscr{F}(K)$ be the family of K-quasiconformal mappings from the Riemann sphere $\mathbb{C}^{\#}$ onto $\mathbb{C}^{\#}$, which preserve reals, and moreover, which have three fixed points, $-1, 0$, and ∞. For real t let $\lambda(K,t)$ and $\nu(K,t)$ be the supremum and the infimum, respectively, of the values $f(t)$ for f ranging over the family $\mathscr{F}(K)$. Among others we shall express $X(K,t)$ for $X = \lambda, \nu$, in terms of extremals for various families of K-quasiconformal self-mappings of $\mathbb{C}^{\#}$.

1 Introduction

Let $\mathscr{Q} = \mathscr{Q}(K)$ be the family of all the K-quasiconformal mappings from the Riemann sphere $\mathbb{C}^{\#} = \{|z| \leqslant +\infty\}$ onto $\mathbb{C}^{\#}$. Three families with the inclusion formulae $\mathscr{F} \subset \mathscr{G} \subset \mathscr{H}$ are then defined by $\mathscr{H} = \mathscr{H}(K) = \{ f \in \mathscr{Q};\ f(0) = 0, f(\infty) = \infty\}$; $\mathscr{G} = \mathscr{G}(K) = \{ f \in \mathscr{H};\ f(-1) = -1\}$; $\mathscr{F} = \mathscr{F}(K) = \{ f \in \mathscr{G};\ f(\mathbb{R}) = \mathbb{R}\}$, where $\mathbb{R}$ is the set of all the real numbers, so that $\mathbb{C} = \mathbb{R}^2$ is the complex plane.

In [KY] we studied

$$\lambda(K,t) = \sup_{f \in \mathscr{F}(K)} f(t) \qquad \text{and} \qquad \nu(K,t) = \inf_{f \in \mathscr{F}(K)} f(t)$$

for $t \in \mathbb{R}$ in detail. Since $\mathscr{F}(K)$ is normal, $\lambda(K,t)$ and $\nu(K,t)$ are the maximum and the minimum, respectively. In particular, $\nu(K,t) \leqslant t \leqslant \lambda(K,t)$ for all $t \in \mathbb{R}$ and trivially, $X(K,t) = t$ for $X = \lambda, \nu$, and $t = -1, 0, \infty$; moreover, $X(1,t) \equiv t$. Furthermore, $\nu(K,t) > 0$ for all $t > 0$. For a fixed $K \geqslant 1$ the function $X(K,t)$ is increasing for $t \in \mathbb{R}$. For fixed $t > 0$ the functions $\lambda(K,t)$ and $\nu(K,t)$ are increasing and decreasing functions of $K \geqslant 1$,

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respectively. We shall sometimes write $X(t) = X(K,t)$ for $X = \lambda, \nu$, and $t \in \mathbb{R}$ whenever $K \geqslant 1$ is fixed.

Set

$$p(f,t) = \max_{|z|=t} |f(z)| \qquad \text{and} \qquad q(f,t) = \min_{|z|=t} |f(z)|$$

for f complex-valued and continuous on the circle $\{|z| = t\}$, $t > 0$.

We begin with $\mathscr{G}$, the family of the members of $\mathscr{Q}$ with the fixed points, $-1, 0$, and ∞.

Theorem 1. *For* $t > 0$,

$$\lambda(K,t) = \max_{f \in \mathscr{G}(K)} p(f,t) \quad \text{and} \quad \nu(K,t) = \min_{f \in \mathscr{G}(K)} q(f,t). \tag{1.1}$$

S. Agard [A, p. 10, (3.1)] claimed that $\lambda(K,t) = \sup_{f \in \mathscr{G}(K)} p(f,t)$ for $t \geqslant 1$. More precisely he wrote $P_2(a,K) = \sup_{f \in \mathscr{G}^*(K)} p(f,a)$ for $a \geqslant 1$, where $\mathscr{G}^*(K)$ is the family of mappings $-f(-z)$ for $f \in \mathscr{G}(K)$.

Our next theorem is concerned with $\mathscr{H}$.

Theorem 2. *For* $r > 0$ *and* $t > 0$,

$$\lambda(K,t) = \max_{f \in \mathscr{H}(K)} \frac{p(f,tr)}{q(f,r)} \quad \text{and} \quad \nu(K,t) = \min_{f \in \mathscr{H}(K)} \frac{q(f,tr)}{p(f,r)}. \tag{1.2}$$

The λ-part in (1.2) for $t = 1$ is given in [LVV, Theorem 1]. Theorem 2 has two corollaries which will be described in Section 3.

Let $\mathscr{Q}(K,D)$ be the family of all the K-quasiconformal mappings from a domain $D \subset \mathbb{C}^{\#}$ into $\mathbb{C}^{\#}$. One can therefore regard $\mathscr{Q}(K) = \mathscr{Q}(K, \mathbb{C}^{\#}) \subset \mathscr{Q}(K,D)$. Let f be a homeomorphism from D into $\mathbb{C}^{\#}$. For $t > 0$ and $D \ni z \neq \infty \neq f(z)$, set

$$\Delta_t^+(f,z) = \limsup_{r \to +0} \frac{\max_{|\zeta|=tr} |f(\zeta+z) - f(z)|}{\min_{|\zeta|=r} |f(\zeta+z) - f(z)|},$$

$$\Delta_t^-(f,z) = \liminf_{r \to +0} \frac{\min_{|\zeta|=tr} |f(\zeta+z) - f(z)|}{\max_{|\zeta|=r} |f(\zeta+z) - f(z)|},$$

so that $0 \leqslant \Delta_t^-(f,z) \leqslant \Delta_t^+(f,z) \leqslant +\infty$. Let Y stand for Δ_t^+ or Δ_t^-. In case $D \ni z \neq \infty = f(z)$, define $Y(f,z) = Y(1/f,z)$. In case $\infty \in D$, set $g(\zeta) = f(1/\zeta)$, $\zeta \in D$, and define $Y(f,\infty) = Y(g,0)$.

Our third result considers the family $\mathscr{Q}(K,D)$ the most restricted of which is $\mathscr{Q}(K)$.

Theorem 3. *For a domain $D \subset \mathbb{C}^\#$ and $t > 0$,*

$$(1.3)\quad \lambda(K,t) = \sup_{f\in\mathscr{Q}(K,D)} \Delta_t^+(f,z) \quad \textit{and} \quad \nu(K,t) = \inf_{f\in\mathscr{Q}(K,D)} \Delta_t^-(f,z) \quad \textit{for all} \quad z \in D.$$

Hence the supremum and the infimum in (1.3) are independent of the particular choice of a pair z, D with $z \in D$. Theorem 3, actually, follows from Theorem 4 which will be described later in Section 4. See also the Remark at the end of Section 4.

How about the results in Theorems 1–3 in case $t < 0$? We have only to remember the formulae $\lambda(t) = -1 - \nu(-1-t)$ and $\nu(t) = -1 - \lambda(-1-t)$ for $t \in \mathbb{R}$, and further $X(t) = -1/\{1 + X(-1-1/t)\}$ for $X = \lambda, \nu$ and for $t \in \mathbb{R}\setminus\{0\}$; see [KY, Theorem 3.1].

For example, following are the consequences of (1.1).

For $t < -1$,

$$\lambda(K,t) = -1 - \min_{f\in\mathscr{G}(K)} q(f,-1-t) \quad \text{and} \quad \nu(K,t) = -1 - \max_{f\in\mathscr{G}(K)} p(f,-1-t);$$

and for $-1 < t < 0$,

$$\lambda(K,t) = -1/\left(1 + \max_{f\in\mathscr{G}(K)} p(f,-1-1/t)\right) \quad \text{and} \quad \nu(K,t) = -1/\left(\min_{f\in\mathscr{G}(K)} p(f,-1-1/t)\right).$$

2 Proof of Theorem 1

The hyperbolic distance $\sigma(z,w)$ of z and w in $\mathbb{C}^* = \mathbb{C}\setminus\{-1,0\}$ is given by the line integral

$$(2.1)\qquad \sigma(z,w) = \int_z^w P(\zeta)|d\zeta|$$

along a geodesic from z to w, where the hyperbolic density P satisfies the differential equation $\Delta \log P = 4P^2$ in $\mathbb{C}^*$.

Supposing $t > 0$ we first prove $\lambda(t) \geqslant p(f,t)$ and $q(f,t) \geqslant \nu(t)$ for $f \in \mathscr{G}$. We begin with the λ-part. Since $t \leqslant \lambda(t)$ it suffices to prove that $|f(\zeta)| \leqslant \lambda(t)$ for $\zeta \in \{|\zeta| = t\}$

and for $f \in \mathscr{G}$ under the condition that $t < |f(\zeta)|$. Recall the Teichmüller theorem [LVV, p. 6] which says, in our terms, that

$$\sigma(w, g(w)) \leqslant \log \sqrt{K} \tag{2.2}$$

for all $w \in \mathbb{C}^*$ and all $g \in \mathscr{G}$; see [KY, Section 4] also. On the other hand,

$$\int_{\nu(t)}^{t} P(x)dx = \int_{t}^{\lambda(t)} P(x)dx = \log \sqrt{K}, \quad x \in \mathbb{R}; \tag{2.3}$$

see [KY, Section 4]. Consequently, (2.2) for $w = \zeta$ and $g = f$ yields that

$$\int_{t}^{\lambda(t)} P(x)dx \geqslant \sigma(\zeta, f(\zeta)). \tag{2.4}$$

Since $P(|w|) \leqslant P(w)$, $w \in \mathbb{C}^*$, by [LVV, p. 6, Lemma], where $\rho(-w) = P(w)$, it follows that

$$\sigma(\zeta, f(\zeta)) = \int_{\zeta}^{f(\zeta)} P(w)|dw| \geqslant \int_{t}^{|f(\zeta)|} P(x)dx, \tag{2.5}$$

which, combined with (2.4), shows that $\lambda(t) \geqslant |f(\zeta)|$. Hence $\lambda(t) \geq p(f, t)$.

To prove the inequality $\nu(t) \leqslant q(f, t)$ for $f \in \mathscr{G}$, we may suppose that $|f(\zeta)| < t$ for $|\zeta| = t$, so that we have, this time,

$$\int_{|f(\zeta)|}^{t} P(x)dx \leqslant \int_{f(\zeta)}^{\zeta} P(w)|dw| = \sigma(\zeta, f(\zeta)) \leqslant \int_{\nu(t)}^{t} P(x)dx.$$

Hence $|f(\zeta)| \geqslant \nu(t)$.

For $t \in \mathbb{R}$ we have $f_1 \in \mathscr{F}$ and $f_2 \in \mathscr{F}$ such that $f_1(t) = \nu(t)$ and $f_2(t) = \lambda(t)$; see [KY, Theorem 1.1]. Hence, $\nu(t) \geqslant q(f_1, t)$ and $\lambda(t) \leqslant p(f_2, t)$ for $t > 0$, so that $\nu(t) = q(f_1, t)$ and $\lambda(t) = p(f_2, t)$. These equalities complete the proof of (1.1).

3 Proof of Theorem 2 and asymptotic behavior

For the proof of $p(f, tr)/q(f, r) \leqslant \lambda(t)$ for $f \in \mathscr{H}$ we choose a and b such that

$$|a| = tr \quad \text{with} \quad |f(a)| = p(f, tr); \; |b| = r \quad \text{with} \quad |f(b)| = q(f, r). \tag{3.1}$$

Set $g(z) = -f(-bz)/f(b)$ for $z \in \mathbb{C}^\#$. Then $g \in \mathscr{G}$ and $|g(-a/b)| = p(f, tr)/q(f, r)$. Since $|g(-a/b)| \leqslant \lambda(t)$ by Theorem 1 with $|-a/b| = t$, the requested inequality follows.

For the proof of $\nu(t) \leqslant q(f, tr)/p(f, r)$ we replace the pair p, q with the pair q, p in (3.1), and apply the ν-part in (1.1).

To prove the maximality for λ, we choose $f \in \mathscr{F}$ with $f(t) = \lambda(t)$ and set $g(z) = -f(-z/r)$, so that $g \in \mathscr{H}$ and $q(g,r) \leqslant 1$ by $g(r) = 1$. Since $p(g,tr) = p(f,t) \geqslant f(t) = \lambda(t)$, it follows that $p(g,tr)/q(g,r) \geqslant \lambda(t)$, whence $p(g,tr)/q(g,r) = \lambda(t)$.

The proof of the minimality for ν is now obvious.

Corollary 1 to Theorem 2 *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal with $f(\mathbb{C}) = \mathbb{C}$. Then for $z \in \mathbb{C}$, $w \in \mathbb{C} \setminus \{z\}$ and $\zeta \in \mathbb{C} \setminus \{z\}$,*

$$\nu\left(K, \left|\frac{\zeta - z}{w - z}\right|\right) |f(w) - f(z)| \leqslant |f(\zeta) - f(z)| \leqslant \lambda\left(K, \left|\frac{\zeta - z}{w - z}\right|\right) |f(w) - f(z)|. \tag{3.2}$$

Proof. The function $g(\eta) = f(\eta + z) - f(z)$ of $\eta \in \mathbb{C}$ is K-quasiconformal with $g(\mathbb{C}) = \mathbb{C}$, so that, by defining $g(\infty) = \infty$, one observes that $g \in \mathscr{H}$. Set $\gamma = (\zeta - z)/(w - z)$. Then, for $\eta_o = \gamma(w - z) = \zeta - z$, one has $|\eta_o| = tr$ where $t = |\gamma| > 0$ and $r = |w - z| > 0$. The following are consequences of (1.2).

$$|f(\zeta) - f(z)| = |g(\eta_o)| \leqslant p(g, tr)$$

$$\leqslant \lambda(K,t) q(g,r) \leqslant \lambda(K,t)|g(w - z)| = \lambda(K,t)|f(w) - f(z)|;$$

$$|f(\zeta) - f(z)| = |g(\eta_o)| \geqslant q(g, tr)$$

$$\geqslant \nu(K,t)p(g,r) \geqslant \nu(K,t)|g(w - z)| = \nu(K,t)|f(w) - f(z)|.$$

As an application of (3.2) let $0 < a \leqslant b < +\infty$ and let

$$a|w - z| \leqslant |\zeta - z| \leqslant b|w - z|.$$

Then

$$\nu(K,a)|f(w) - f(z)| \leqslant |f(\zeta) - f(z)| \leqslant \lambda(K,b)|f(w) - f(z)|. \tag{3.3}$$

Actually this is trivial in case $z = w$, so that $\zeta = z = w$. In case $z \neq w$, one observes that $\zeta \neq z$, so that (3.3) follows from (3.2).

Remark. The λ-part in (3.2) for $|\zeta - z|/|w - z| \geqslant 1$ is observed in [A]. The case $K = 1$ in (3.2) is trivial because f is linear, $f(z) = az + b$, $a \neq 0$.

It follows from the local $1/K$-Hölder-continuity in terms of the spherical distance on $\mathbb{C}^{\#}$ of $f \in \mathscr{Q}(K)$ described in [LV, p. 71] that if f is a K-quasiconformal mapping from $\mathbb{C}$ onto $\mathbb{C}$, then

$$\limsup_{w \to z} \frac{|f(w) - f(z)|}{|w - z|^{1/K}} < +\infty \tag{3.4}$$

for $z \in \mathbb{C}$ and

$$\limsup_{w \to \infty} \frac{|w|^{1/K}}{|f(w)|} < +\infty.$$

Since we may replace f with its inverse in the last, it follows that

$$\limsup_{w \to \infty} \frac{|f(w)|}{|w|^{K}} < +\infty. \tag{3.5}$$

For later use we need two examples f_1 and f_2 of K-quasiconformal mappings from $\mathbb{C}$ onto $\mathbb{C}$. Set $f_1(w) = K\operatorname{Re} w + i \operatorname{Im} w$ for $K > 1$. Then, since $|f_1(w)|/|w|^{1/K} \geqslant |w|^{1-1/K}$ it follows that

$$\lim_{w \to \infty} \frac{|f_1(w)|}{|w|^{1/K}} = +\infty. \tag{3.6}$$

Next, set $f_2(w) = (w - z)|w - z|^{1/K - 1}$ for $K > 1$. Then

$$\lim_{w \to z} \frac{|f_2(w)|}{|w - z|^{K}} = +\infty. \tag{3.7}$$

Corollary 2 to Theorem 2. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal with $f(\mathbb{C}) = \mathbb{C}$. Then for $z \in \mathbb{C}$,*

$$\limsup_{w \to z} \frac{|f(w) - f(z)|}{|w - z|^{K}} \leqslant 16^{K-1} \liminf_{w \to z} \frac{|f(w) - f(z)|}{|w - z|^{K}}; \tag{3.8}$$

$$\limsup_{w \to z} \frac{|f(w) - f(z)|}{|w - z|^{1/K}} \leqslant 16^{1-1/K} \liminf_{w \to z} \frac{|f(w) - f(z)|}{|w - z|^{1/K}}. \tag{3.9}$$

Furthermore,

$$\limsup_{w \to \infty} \frac{|f(w)|}{|w|^{K}} \leqslant 16^{K-1} \liminf_{w \to \infty} \frac{|f(w)|}{|w|^{K}}; \tag{3.10}$$

$$\limsup_{w\to\infty} \frac{|f(w)|}{|w|^{1/K}} \leqslant 16^{1-1/K} \liminf_{w\to\infty} \frac{|f(w)|}{|w|^{1/K}}. \tag{3.11}$$

The (inferior) limit in (3.8) may be $+\infty$ as (3.7) shows, whereas the (inferior) limit in (3.11) may be $+\infty$; see (3.6).

Proof of (3.8)–(3.11). Inequalities (3.8) and (3.10) follow from (3.9) and (3.11), respectively. For the proof just consider the inverse of f which is K-quasiconformal again.

For the proof of (3.9) we recall [KY, (6.24)] to obtain

$$\lim_{t\to+\infty} t^{-1/K}\lambda(K,t) = 16^{1-1/K}. \tag{3.12}$$

Set $t = |\zeta - z|/|w - z|$ in the λ-part in (3.2). Then

$$t^{-1/K}|f(\zeta) - f(z)| \leqslant t^{-1/K}\lambda(K,t)|f(w) - f(z)|,$$

whence

$$\frac{|f(\zeta) - f(z)|}{|\zeta - z|^{1/K}} \leqslant t^{-1/K}\lambda(K,t) \cdot \frac{|f(w) - f(z)|}{|w - z|^{1/K}}. \tag{3.13}$$

Let $\zeta \to z$, so that $t \to 0$. Then

$$\limsup_{\zeta\to z} \frac{|f(\zeta) - f(z)|}{|\zeta - z|^{1/K}} \leqslant 16^{1-1/K} \frac{|f(w) - f(z)|}{|w - z|^{1/K}}. \tag{3.14}$$

Hence (3.9) follows from (3.14).

To prove (3.11) we set $g(\zeta) = 1/\big(f(1/\zeta) - f(0)\big)$ for $\zeta \neq 0$ and $g(0) = 0$. Then g is K-quasiconformal from $\mathbb{C}$ onto $\mathbb{C}$, so that, one may apply (3.9) to g and to $z = 0$. Then

$$\limsup_{w\to\infty} \frac{|w|^{1/K}}{|f(w)|} = \limsup_{w\to 0} \frac{|g(w)|}{|w|^{1/K}} \leqslant \tag{3.15}$$

$$16^{1-1/K} \liminf_{w\to 0} \frac{|g(w)|}{|w|^{1/K}} = 16^{1-1/K} \liminf_{w\to\infty} \frac{|w|^{1/K}}{|f(w)|}.$$

Hence (3.11) follows on taking the reciprocal in the first and in the last in (3.15).

4 Theorem 4 and Proof of Theorem 3

A Jordan domain $Q = Q(a,b,c,d) \subset \mathbb{C}^{\#}$ with four distinct points, $a, b, c,$ and d on its boundary curve in the positive order can be mapped by a conformal maping, or a univalent and meromorphic function, ϕ, which is said to be canonical, onto the the interior of the rectangle with the vertices $\phi(a) = 0$, $\phi(b) = M(Q) > 0$, $\phi(c) = M(Q) + i$, and $\phi(d) = i$, where the homeomorphic extension of ϕ to the closure $\overline{Q}$ of Q is again denoted by ϕ. Then $M(Q)$ is uniquely determined by Q. Let f be a sense-preserving homeomorphism from a domain $D \subset \mathbb{C}^{\#}$ into $\mathbb{C}^{\#}$. We then denote $f(Q) = f(Q)(f(a), f(b), f(c), f(d))$ for $Q = Q(a,b,c,d)$ with $\overline{Q} \subset D$. Let $U(z) \subset D$ be an open disk of center $z \in D$ and set

$$\omega(f, U(z)) = \sup_{\overline{Q} \subset U(z)} \frac{M(f(Q))}{M(Q)}.$$

We shall be concerned with

$$\omega(f, z) = \inf_{U(z) \subset D} \omega(f, U(z)).$$

Then $0 \leqslant \omega(f,z) \leqslant +\infty$. Note that $\omega(f,z)$ is a 'local' quantity and does not depend on D as far as f is defined near z. More precisely, let $U_r(z) = \{w; |w - z| < r\} \subset D$. Then $\omega(f,z) = \lim_{r \to +0} \omega(f, U_r(z))$. If $f \in \mathcal{Q}(K, D)$, then $\omega(f,z) \leqslant K$ at each $z \in D$. Set $\Omega(f,z) = \max(\omega(f,z), 1)$.

Theorem 4. *For a domain $D \subset \mathbb{C}^{\#}$, $t > 0$, $z \in D$, and for $f \in \mathcal{Q}(K, D)$, one has*

$$\Delta_t^+(f,z) \leqslant \lambda(\Omega(f,z), t) \quad \text{and} \quad \Delta_t^-(f,z) \geqslant \nu(\Omega(f,z), t). \tag{4.1}$$

The inequalities in (4.1) *are sharp: Given $t > 0$, $z \in D$, and $\varepsilon > 0$, there exist f_λ and f_ν of $\mathcal{Q}(K) \subset \mathcal{Q}(K, D)$ such that $\Omega(f_\lambda, z) = \omega(f_\lambda, z) = \Omega(f_\nu, z) = \omega(f_\nu, z) = K$ and furthermore,*

$$\Delta_t^+(f_\lambda, z) > \lambda(K, t) - \varepsilon \quad \text{and} \quad \Delta_t^-(f_\nu, z) < \nu(K, t) + \varepsilon. \tag{4.2}$$

The λ-part in the case $t = 1$ is a generalization of [LVV, Theorem 2] in which $D = \mathbb{C}^{\#}$. Our Theorem 3 is now an immediate consequence of Theorem 4 with $\lambda(\Omega(f,z), t) \leqslant \lambda(K, t)$ and $\nu(\Omega(f,z), t) \geqslant \nu(K, t)$ for $f \in \mathcal{Q}(K, D)$.

Proof of Theorem 4. First we recall $\lambda(t) = 1/\nu(1/t)$ and $\nu(t) = 1/\lambda(1/t)$ for $t \in \mathbb{R} \setminus \{0\}$; see [KY, Theorem 3.1]. The ν-part in (4.1) immediately follows from the λ-part in (4.1). In fact.

$$\Delta_t^-(f,z) = 1/\Delta_{1/t}^+(f,z) \geqslant 1/\lambda(\Omega(f,z),1/t) = \nu(\Omega(f,z),t).$$

To prove the λ-part in (4.1) we may suppose that $z \neq \infty \neq f(z)$ for a fixed $f \in \mathscr{Q}(K,D)$. For $\varepsilon > 0$ we have $U \equiv \{\zeta;\, |\zeta - z| < \rho\} \subset D$ such that $\omega(f,U) < \omega(f,z) + \varepsilon \leqslant \Omega(f,z) + \varepsilon \equiv K'$. Set $\phi(\zeta) = \rho\zeta + z$, $\zeta \in \mathbb{C}^\#$, and choose a conformal mapping ψ from $f(U)$ onto the disk $\delta \equiv \{\zeta; |\zeta| < 1\}$ so that $\psi(f(z)) = 0$. By reflexion the composed mapping $\psi \circ f \circ \phi$ from δ onto δ, which is K'-quasiconformal, can be extended K'-quasiconformally to the whole $\mathbb{C}^\#$ in the standard manner [L, p. 16], so that the resulting function f^* is in $\mathscr{H}(K')$. It then follows from Theorem 2 that

$$\Delta_t^+(\psi \circ f \circ \phi, 0) = \Delta_t^+(f^*, 0) \leqslant \lambda(K',t).$$

On the other hand, setting $\beta = |\psi'(f(z))|$, one observes that

$$\Delta_t^+(\psi \circ f \circ \phi, 0) = \limsup_{r \to +0} \frac{\max_{|\zeta|=t\rho r} |\psi \circ f(\zeta + z) - \psi \circ f(z)|}{\min_{|\zeta|=\rho r} |\psi \circ f(\zeta + z) - \psi \circ f(z)|}$$

$$= \Delta_t^+(\psi \circ f, z) = \limsup_{r \to +0} \frac{\beta \max_{|\zeta|=tr} |f(\zeta + z) - f(z)|}{\beta \min_{|\zeta|=r} |f(\zeta + z) - f(z)|} = \Delta_t^+(f,z).$$

Hence $\Delta_t^+(f,z) \leqslant \lambda(K',t)$. Since $\varepsilon > 0$ is arbitrary, and since the function $\lambda(K'',t)$ of $K'' \geqslant 1$ is continuous, we have the λ-part in (4.1).

To prove the λ -part in (4.2) we shall find a sequence $\{r_k\}_{k=1}^\infty$, with $0 < r_k \searrow 0$, and a function $\Phi \in \mathscr{G}(K)$ such that $\omega(\Phi, 0) = K$ and

$$\frac{p(\Phi, tr_k)}{q(\Phi, r_k)} > \lambda(t) - \varepsilon/2 \quad \text{for} \quad k = 1, 2, \dots \tag{4.3}$$

Once (4.3) is established, it follows that $\Delta_t^+(\Phi, 0) > \lambda(t) - \varepsilon$, so that we have only to set $f_\lambda(\zeta) = \Phi(\zeta - z)$, $\zeta \in \mathbb{C}^\#$.

There exists $F \in \mathscr{F}(K)$ with $F(t) = \lambda(t)$. Let us recall the detailed construction of F described in the proof of [KY, Theorem 1.1]. The upper half-plane $H = H(0, x, \infty, -1)$ for $x > 0$ admits a canonical mapping ϕ_x with $\phi_x(x) = M(x) = (2/\pi)\mu(1/\sqrt{1+x})$, where $\mu(r)$ is the modulus (= module in) [L, p. 11] of the Grötzsch ring domain $\delta \setminus [0, r]$, $0 < r < 1$. Set $K = M(\lambda(t))/M(t)$, so that $K \geqslant 1$ by $\lambda(t) \geqslant t$. Then the reflection F of $\phi_{\lambda(t)}^{-1} \circ \Psi \circ \phi_t$

with respect to the real axis is the requested, where $\Psi(\zeta) = K\,\mathrm{Re}\,\zeta + i\,\mathrm{Im}\,\zeta$ satisfies $\omega(\Psi,\zeta) = K$ for all $\zeta \in \mathbb{C}$. Hence $\omega(F,0) = K$ and $F \in \mathscr{F}(K)$. Since $KM(t) = M(\lambda(t))$ we have $F(t) = \lambda(t)$.

The image $F(A_n)$ of the ring domain $A_n = \{\zeta; 1/n < |\zeta| < n\}$ for a natural number $n \geqslant \max(1+t, 1+1/t)$ can be mapped onto a ring domain $B_n = \{\zeta; a_n < |\zeta| < b_n\}$, $0 < a_n < 1 < b_n$, by a conformal mapping h_n with $h_n(-1) = -1$. Actually, let M_n be the modulus (= module in) [L. p. 10] of $F(A_n)$ so that we have a conformal mapping τ from $F(A_n)$ onto the ring domain $\{\zeta;\ 1 < |\zeta| < e^{M_n}\}$ with $\tau(-1) < 0$. Set $a_n = -1/\tau(-1)$ and $b_n = -e^{M_n}/\tau(-1)$. Then $h_n = -\tau/\tau(-1)$ is the requested.

For each fixed $n \geqslant \max(1+t, 1+1/t)$ the K-quasiconformal mapping $h_n \circ F : A_n \to B_n$ can then be extended to the whole $\mathbb{C}^{\#}$ by repetition of the reflections, so that the resulting function g_n is in $\mathscr{G}$. Hence $g_n(-1) = h_n \circ F(-1) = -1$ and $g_n(t) = h_n \circ F(t) = h_n(\lambda(t))$. By reflecting $2k$ times internally, every point $\zeta \in A_n$ is mapped to $n^{-4k}\zeta$, so that

$$g_n(n^{-4k}\zeta) = (a_n/b_n)^{2k} g_n(\zeta). \tag{4.4}$$

Hence, for $k = 1, 2, \ldots$,

$$q(g_n, n^{-4k}) \leqslant (a_n/b_n)^{2k}|g_n(-1)| = (a_n/b_n)^{2k}$$

and

$$p(g_n, tn^{-4k}) \geqslant (a_n/b_n)^{2k}|g_n(t)|,$$

so that

$$\frac{p(g_n, tn^{-4k})}{q(g_n, n^{-4k})} \geqslant |g_n(t)| = |h_n(\lambda(t))|. \tag{4.5}$$

Since $\mathscr{G}$ is normal in $\mathbb{C}^{\#}$ in terms of the spherical distance by [L, p. 14, Theorem 2.1] we have a subsequence of $\{g_n\}$, which we denote again by $\{g_n\}$ for simplicity, and which converges to $g \in \mathscr{G}$ in the Euclidean distance on each open disk (of finite radius) in $\mathbb{C}$; see [L, p. 15, Theorem 2.3].

Since $h_n = g_n \circ F^{-1}$ maps $F(A_n)$ conformally onto B_n, it is conformal for some n onwards in every open disk in $\mathbb{C} \setminus \{0\}$. Consequently, the limiting function h of $\{h_n\}$ is a conformal mapping from $\mathbb{C} \setminus \{0\}$ onto $\mathbb{C} \setminus \{0\}$. We can then extend h to $\mathbb{C}^{\#}$ by setting $h(0) = 0$ and $h(\infty) = \infty$. Since $h(\zeta) = \zeta$ for $\zeta \in \{-1, 0, \infty\}$, h must be the identity.

For $\varepsilon > 0$ we have an N such that $|h_N(\lambda(t))| > \lambda(t) - \varepsilon/2$. Hence (4.3) follows from (4.5) on setting $\Phi = g_N$ and $r_k = N^{-4k}$, $k = 1, 2, \ldots$. Apparently, $\omega(g_N, 0) = \omega(F, 0) = K$.

For the ν-part in (4.2) let $0 < \varepsilon' < 1/\nu(t) - 1/(\nu(t) + \varepsilon)$ and let f_λ be the function for the λ-part, this time, for ε' and $1/t$ instead of ε and t. Then $f_\nu = f_\lambda$ is the requested because $\Delta_t^-(f_\lambda, z) = 1/\Delta_{1/t}^+(f_\lambda, z)$ and $\nu(t) = 1/\lambda(1/t)$.

Remark. Let us consider the meaning of $Y(f, z)$ for $Y = \Delta^+, \Delta^-$, and for a homeomorphism f from $D \subset \mathbb{C}^\#$ into $\mathbb{C}$, which is differentiable at $z \in D \setminus \{\infty\}$, and which satisfies $|\partial f(z)| > |\overline{\partial} f(z)|$. Then,

$$\Delta^+(f, z) = \frac{t(|\partial f(z)| + |\overline{\partial} f(z)|)}{|\partial f(z)| - |\overline{\partial} f(z)|} = tD_f(z),$$

$$\Delta^-(f, z) = \frac{t(|\partial f(z)| - |\overline{\partial} f(z)|)}{|\partial f(z)| + |\overline{\partial} f(z)|} = t/D_f(z),$$

where $\partial f = (f_x - i f_y)/2$, $\overline{\partial} f = (f_x + i f_y)/2$, and $D_f = (|\partial f| + |\overline{\partial} f|)/(|\partial f| - |\overline{\partial} f|)$ is the dilatation quotient [L, p. 19].

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