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Harmonic Foliations on a Complete Riemannian Manifold

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Abstract. Let $\mathcal{F}$ be a Riemannian foliation with finite energy on a manifold (M, g_M) with a complete bundle-like metric g_M. Assume that the Ricci curvature is non-negative and the transversal scalar curvature is non-positive. If $\mathcal{F}$ is harmonic, then $\mathcal{F}$ is totally geodesic.

0 Introduction

A foliation $\mathcal{F}$ on a manifold M is *harmonic*, if the canonical projection $\pi : TM \to Q$ of the tangent bundle to the normal bundle $Q = TM/L$ is a harmonic Q-valued 1-form ([2,3]). For this one needs the connection ∇' defined by (3.10) in Q, and a Riemannian metric g_M in M.

A rich variety of harmonic foliations were discussed in [2]. It is well-known that $\mathcal{F}$ is harmonic if and only if all leaves of $\mathcal{F}$ are minimal submanifolds of M ([2]).

On the other hand, if $\mathcal{F}$ is Riemannian, i.e., if there exists a holonomy invariant metric g_Q on Q, there is a unique metric and torsion-free connection ∇ in Q ([2]).

In 1984, F.W.Kamber and Ph.Tondeur([3]) studied the interplay of the harmonicity property with the curvature of the Riemannian metric g_M and the curvature of the connection ∇, which is metric and trosion-free with respect to the holonomy invariant metric g_Q on Q. Namely, let $\mathcal{F}$ be a Riemannian foliation on a closed oriented manifold M. Let g_M be a Riemannian metric on M with non-negative Ricci curvature and assume the

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normal sectional curvature K_∇ of g_Q to be non-positive. If π is a harmonic form, then each leaf is a totally geodesic submanifold of M.

In this paper, we extend several results of Kamber and Tondeur([3]) to the case of complete manifolds.

The paper is organized as follows. In section 1, we review the known facts on a vector bundle. In section 2, we study the cut off functions, which is main tools for our research in complete manifolds. In section 3, we give some results when $\mathcal{F}$ is a Riemannian foliation on a complete manifold (M, g_M) with holonomy invariant metric g_Q (g_M is not assumed to be bundle-like). With respect to ∇, "$\pi : TM \to Q$ is harmonic" does not mean that $\mathcal{F}$ is harmonic, i.e., all leaves of $\mathcal{F}$ are minimal submanifolds of (M, g_M). On the other hand, if g_M is a bundle-like metric and the holonomy invariant metric g_Q is induced from g_M, then the unique metric and torsion-free connection ∇ is given by (3.10), and then "π is harmonic" means that $\mathcal{F}$ is harmonic.

On the other hand, the tension field τ plays an important role in studying a foliation on a Riemannian manifold with bundle-like metric. When a foliation is minimal, i.e., $\tau = 0$, many results are obtained. An apparent weakening of the condition of the vanishing tension field $\tau \in \Gamma Q$ would be to require $\nabla\tau = 0$. But this parallel condition of τ is meaningless because $\nabla\tau = 0$ implies $\tau = 0$ on a compact manifold([2]). In appendix, we prove that the parallel condition $\nabla\tau = 0$ is also meaningless on complete manifolds.

The main tools we use are the Weitzenböck formulas and cut off functions.

1 Preliminaries

We review some basic facts on a vector bundle ([4]). Let $E \to M$ be a smooth Riemannian vector bundle over a Riemannian manifold M, i.e., E is a vector bundle over M and there is a C^∞-assignment of an inner product $< \cdot, \cdot >$ to each fiber E_x of E over $x \in M$. Let $A^r(E)$ be the space of E-valued r-forms over M. We assume a (metric) connection ∇ is given in E, i.e., $\nabla : A^0(E) \to A^1(E)$ is an $\mathbb{R}$ -linear map such that $\nabla(fs) = f\nabla s + sdf$, $f \in A^0(M)$, $s \in \Gamma(E)$ and such that

$$X < s_1, s_2 > = < \nabla_X s_1, s_2 > + < s_1, \nabla_X s_2 > \tag{1.1}$$

for any $X \in TM$ and $s_1,\ s_2 \in A^0(E)$. By the usual algebraic formalism, $\nabla : A^0(E) \to A^1(E)$ can be extended to an anti-derivation

$$d_\nabla : A^r(E) \to A^{r+1}(E)$$

by the following rule: if $\sum_a s_a\eta^a \in A^r(E)$, then

$$d_\nabla(s_a\eta^a) = \nabla s_a \wedge \eta^a + s_a(d\eta^a) \tag{1.2}$$

for $s_a \in \Gamma(E),\ \eta^a \in A^r(M)$. For a Riemannian matric g_M on M, we extend the star operator $* : A^r(M) \to A^{n-r}(M)(n = dimM)$ to $* : A^r(E) \to A^{n-r}(E)$ as follows: If $s \in \Gamma(E)$ and $\eta \in A^r(E)$, then $*(s\eta) = s(*\eta)$. Moreover the operator $d^*_\nabla : A^r(E) \to A^{r-1}(E)$ given by

$$d^*_\nabla\phi = (-1)^{n(r+1)+1} * d_\nabla * \phi, \quad \phi \in A^r(E) \tag{1.3}$$

is the formal adjoint of d_∇ with respect to a suitable inner product induced from $<\ ,\ >$ and g_M. The Laplacian Δ for $A^*(E)$ is given by

$$\Delta = d_\nabla d^*_\nabla + d^*_\nabla d_\nabla. \tag{1.4}$$

Let $e_1, \cdots, e_n$ be an orthonormal basis of T_xM and $E_1, \cdots, E_n$ a local framing of TM in a neighborhood of x, coinciding with $e_1, \cdots, e_n$ at x and satisfying $\nabla^M_{e_\alpha} E_\beta = (\nabla^M_{E_\alpha} E_\beta)_x = 0(\alpha, \beta = 1, \cdots, n)$, where ∇^M denotes the Riemannian connection of (M, g_M). Let ω^α be the dual coframe field of e_α. Then on $A^*(E)$ we have

$$d_\nabla = \sum \omega^\alpha \wedge \tilde{\nabla}_{e_\alpha}, \quad d^*_\nabla = -\sum i(e_\alpha)\tilde{\nabla}_{e_\alpha}, \tag{1.5}$$

where $\tilde{\nabla}_X(s\eta) = (\nabla_X s)\eta + s(\nabla^M_X \eta)$ and $i(X)(s\eta) = s[i(X)\eta]$ for $s \in \Gamma(E),\ \eta \in A^*(M)$. From these, we obtain the following Weitzenböck formula: for any $\phi \in A^1(E)$,

$$\Delta\phi = -\sum \tilde{\nabla}_{e_\alpha}\tilde{\nabla}_{E_\alpha}\phi + S(\phi)_x, \tag{1.6}$$

where $S(\phi)_x(X)$ is defined by

$$S(\phi)_x(X) = \sum\{R^E(e_\alpha, X)\phi(e_\alpha) - \phi(R^M(e_\alpha, X)e_\alpha)\}. \tag{1.7}$$

Here R^E denotes the curvature of the connection ∇ in E and R^M the curvature of the Riemannian connection ∇^M in TM. Formula (1.6) yields then the following "scalar" Weizenböck formula

$$-\frac{1}{2}\Delta^M|\phi|^2 = |\tilde{\nabla}\phi|^2 - < \Delta\phi, \phi > + < S(\phi), \phi >, \tag{1.8}$$

where Δ^M is the ordinary Laplacian d^*d on functions on M and $|\phi|^2 =< \phi, \phi >$ is given by $|\phi|_x^2 = \sum < \phi(e_\alpha), \phi(e_\alpha) >$. The first term on the right hand side of (1.8) is given by

$$|\tilde{\nabla}\phi|_x^2 = \sum < \tilde{\nabla}_{e_\alpha}\phi, \tilde{\nabla}_{e_\alpha}\phi > .$$

Now we define the global scalar product $\ll \cdot, \cdot \gg$ by

$$\ll \phi, \psi \gg = \int_M < \phi, \psi > \quad \text{for } \phi, \psi \in A^*(E). \tag{1.9}$$

Let $A_0^r(E)$ be the subspace of $A^r(E)$ with compact supports and $L_2^r(E)$ the completion of $A_0^r(E)$ with respect to the global scalar product $\ll , \gg$. Then we have

$$\ll d_\nabla\phi, \psi \gg = \ll \phi, d_\nabla^*\psi \gg$$

for any $\phi \in A_0^r(E)$ and $\psi \in A_0^{r+1}(E)$.

2 Cut off functions

Let x_0 be a point of M and fix it. For each point $y \in M$, we denote by $\rho(y)$ the geodesic distance from x_0 to y. Let $B(\ell) = \{y \in M \mid \rho(y) < \ell\}$ for $\ell > 0$. Then there exists a Lipschitz continuous function ω_ℓ on M satisfying the following properties:

$$\begin{aligned}
&0 \le \omega_\ell(y) \le 1 \quad \text{for any } y \in M, \\
&\operatorname{supp} \omega_\ell \subset B(2\ell), \\
&\omega_\ell(y) = 1 \quad \text{for any } y \in B(\ell), \\
&\lim_{\ell \to \infty} \omega_\ell = 1, \\
&|d\omega_\ell| \le \frac{C}{\ell} \quad \text{almost everywhere on } M,
\end{aligned} \tag{2.1}$$

where $C(> 0)$ is a constant independent of ℓ([1]). Then we have

Lemma 2.1 ([1]) *For any $\phi \in A^r(E)$, there exists a positive constant A independent of ℓ such that*

$$||d\omega_\ell \wedge \phi||^2_{B(2\ell)} \leq \frac{A}{\ell^2}||\phi||^2_{B(2\ell)},$$
$$||d\omega_\ell \wedge *\phi||^2_{B(2\ell)} \leq \frac{A}{\ell^2}||\phi||^2_{B(2\ell)},$$

where $||\phi||^2_{B(2\ell)} = \int_{B(2\ell)} < \phi, \phi >$.

Now, we remark that, for $\phi \in L^r_2(E) \cap A^r(E)$, $\omega_\ell\phi$ has compact support and $\omega_\ell\phi \to \phi(\ell \to \infty)$ in the strong sense. From (1.2) and (1.3), we have

$$\begin{aligned} d_\nabla(\omega_\ell^2\phi) &= \omega_\ell^2 d_\nabla\phi + 2\omega_\ell d\omega_\ell \wedge \phi, \\ d^*_\nabla(\omega_\ell^2\phi) &= \omega_\ell^2 d^*_\nabla\phi - *(2\omega_\ell d\omega_\ell \wedge *\phi) \end{aligned} \tag{2.2}$$

for any $\phi \in A^r(E)$. By using the inequality $| < a, b > | \leq \frac{1}{t}|a|^2 + t|b|^2$ for any positive real number t, we have

$$| \ll \omega_\ell d^*_\nabla\phi, *(d\omega_\ell \wedge *\phi) \gg_{B(2\ell)} | \leq \frac{1}{4}||\omega_\ell d^*_\nabla\phi||^2_{B(2\ell)} + 4|| * (d\omega_\ell \wedge *\phi)||^2_{B(2\ell)}.$$

From Lemma 2.1, we have

$$| \ll \omega_\ell d^*_\nabla\phi, *(d\omega_\ell \wedge *\phi) \gg_{B(2\ell)} | \leq \frac{1}{4}||\omega_\ell d^*_\nabla\phi||^2_{B(2\ell)} + \frac{4A}{\ell^2}||\phi||^2_{B(2\ell)}. \tag{2.3}$$

Similarly we have

$$| \ll \omega_\ell\tilde{\nabla}\phi, d\omega_\ell \wedge \phi \gg_{B(2\ell)} | \leq \frac{1}{4}||\omega_\ell\tilde{\nabla}\phi||^2_{B(2\ell)} + \frac{4A}{\ell^2}||\phi||^2_{B(2\ell)}. \tag{2.4}$$

3 Harmonicity of foliations

Let $L \subset TM$ be an integrable subbundle defining a foliation $\mathcal{F}$ and $Q = TM/L$ the normal bundle of $\mathcal{F}$. Since $\mathcal{F}$ is Riemannian, there exist a holonomy invariant metric g_Q on Q and a unique metric and torsion free connection ∇ in Q([2]). A Riemannian metric g_M on M defines a splitting σ of the exact sequence

$$0 \longrightarrow L \longrightarrow TM \underset{\sigma}{\overset{\pi}{\rightleftarrows}} Q \longrightarrow 0, \tag{3.1}$$

where $\sigma(Q)$ is the orthogonal complement $L^\perp$ of L in TM. The induced connection $\tilde{\nabla}$ on Q-valued forms involve ∇ and ∇^M. Let $\{E_\alpha\}_{\alpha=1,\cdots,n}$ be an orthonormal framing with respect to g_M such that $e_i \in L_x$, $i = 1, \cdots, p$ and $e_a \in \sigma Q_x$, $a = p+1, \cdots, n = p+q$ with $\nabla^M_{e_\alpha} E_\beta = 0$. But we neither claim nor require that $(E_i)_y \in L_y$ for $1 \le i \le p$ or $(E_a)_y \in \sigma Q_y$ for $p+1 \le a \le n$ at points $y \neq x$. We do have $(\pi E_i)_x = \pi e_i = 0$. In the case where g_M is a bundle-like metric, the vectors $(\pi E_a)_x = \pi e_a$ form an orthonormal basis of Q_x([3]).

Consider the canonical projection $\pi : TM \to Q$ as a $Q-$valued 1-form, i.e., $\pi \in A^1(Q)$. Then it is well known that $d_\nabla \pi = 0$([2]), since $d_\nabla \pi$ equals the torsion T_∇ given by

$$T_\nabla(X,Y) = \nabla_X \pi(Y) - \nabla_Y \pi(X) - \pi[X,Y]$$

which is zero. Hence we have the following lemma.

Lemma 3.1 *Let $\mathcal{F}$ be a Riemannian foliation with finite energy on a complete Riemannian manifold (M, g_M) with holonomy invariant metric g_Q on Q. If $\Delta\pi \in L^1_2(Q)$, then*

$$\frac{1}{2}\|d^*_\nabla \pi\|^2 \le \limsup \ll \Delta\pi, \omega_\ell^2 \pi \gg \le \frac{3}{2}\|d^*_\nabla \pi\|^2.$$

Proof. We know that $d_\nabla \pi = 0$([2]). Hence from (1.4) and (2.2), we have

$$\begin{aligned} \ll \Delta\pi, \omega_\ell^2 \pi \gg_{B(2\ell)} &= \ll d^*_\nabla \pi, d^*_\nabla(\omega_\ell^2 \pi) \gg_{B(2\ell)} \\ &= \ll \omega_\ell d^*_\nabla \pi, \omega_\ell d^*_\nabla \pi \gg_{B(2\ell)} \\ &\quad -2 \ll \omega_\ell d^*_\nabla \pi, *(d\omega_\ell \wedge *\pi) \gg_{B(2\ell)} . \end{aligned}$$

From (2.3), we get

$$\begin{aligned} \frac{1}{2}\|\omega_\ell d^*_\nabla \pi\|^2_{B(2\ell)} - \frac{8A}{\ell^2}\|\pi\|^2_{B(2\ell)} &\le \ll \Delta\pi, \omega_\ell^2 \pi \gg_{B(2\ell)} \\ &\le \frac{3}{2}\|\omega_\ell d^*_\nabla \pi\|^2_{B(2\ell)} + \frac{8A}{\ell^2}\|\pi\|^2_{B(2\ell)}. \end{aligned}$$

Since $\pi, \Delta\pi \in L^1_2(Q)$, $d^*_\nabla \pi$ is square-integrable. Hence we obtain the inequality by letting $\ell \to \infty$. □

Moreover, we have the following lemma from (1.6).

Lemma 3.2 *Let $\mathcal{F}$ be a Riemannian foliation on (M, g_M) with holonomy invariant metric g_Q on Q (g_M is not assumed to be bundle-like). Then for any $\phi \in A^r(Q)$, we have*

$$\begin{aligned} \ll \Delta\phi, \omega_\ell^2\phi \gg_{B(2\ell)} &= 2 \ll \omega_\ell\tilde{\nabla}\phi, d\omega_\ell \wedge \phi \gg_{B(2\ell)} + \|\omega_\ell\tilde{\nabla}\phi\|^2_{B(2\ell)} \\ &\quad + \ll S(\phi), \omega_\ell^2\phi \gg_{B(2\ell)} . \end{aligned}$$

Proof. From (1.6), we have, at $x \in M$,

$$\begin{aligned} < \Delta\phi, \omega_\ell^2\phi > &= -\sum < \tilde{\nabla}_{E_\alpha}\tilde{\nabla}_{E_\alpha}\phi, \omega_\ell^2\phi > + < S(\phi), \omega_\ell^2\phi > \\ &= -\sum E_\alpha < \tilde{\nabla}_{E_\alpha}\phi, \omega_\ell^2\phi > + \sum < \tilde{\nabla}_{E_\alpha}\phi, \tilde{\nabla}_{E_\alpha}(\omega_\ell^2\phi) > \\ &\quad + < S(\phi), \omega_\ell^2\phi > \\ &= -\sum E_\alpha < \tilde{\nabla}_{E_\alpha}\phi, \omega_\ell^2\phi > + \sum < \tilde{\nabla}_{E_\alpha}\phi, 2\omega_\ell d\omega_\ell(E_\alpha) \wedge \phi > \\ &\quad + |\omega_\ell\tilde{\nabla}\phi|^2 + < S(\phi), \omega_\ell^2\phi > \\ &= -div(\omega_\ell X_\ell) + \sum < \tilde{\nabla}_{E_\alpha}\phi, 2\omega_\ell d\omega_\ell(E_\alpha) \wedge \phi > + |\omega_\ell\tilde{\nabla}\phi|^2 \\ &\quad + < S(\phi), \omega_\ell^2\phi >, \end{aligned}$$

where a vector field X_ℓ satisfies

$$g_M(X_\ell, Y) = < \tilde{\nabla}_Y\phi, \omega_\ell\phi >$$

for any Y. The last line is proved as follows: at $x \in M$,

$$\begin{aligned} div(\omega_\ell X_\ell) &= \sum g_M(\nabla^M_{E_\alpha}(\omega_\ell X_\ell), E_\alpha) \\ &= \sum E_\alpha g_M(\omega_\ell X_\ell, E_\alpha) = \sum E_\alpha < \tilde{\nabla}_{E_\alpha}\phi, \omega_\ell^2\phi > . \end{aligned}$$

By integrating and by the divergence theorem([1]), which is applicable to Lipschitz continuous forms, we obtain our results. □

From (2.4) and Lemma 3.2, we have

$$\begin{aligned} \frac{1}{2}\|\omega_\ell\tilde{\nabla}\phi\|^2_{B(2\ell)} &+ \ll S(\phi), \omega_\ell^2\phi \gg_{B(2\ell)} - \frac{8A}{\ell^2}\|\phi\|^2_{B(2\ell)} \\ &\leq \ll \Delta\phi, \omega_\ell^2\phi \gg_{B(2\ell)} \\ &\leq \frac{3}{2}\|\omega_\ell\tilde{\nabla}\phi\|^2_{B(2\ell)} + \ll S(\phi), \omega_\ell^2\phi \gg_{B(2\ell)} + \frac{8A}{\ell^2}\|\phi\|^2_{B(2\ell)}. \end{aligned}$$

From the first inequality above, we have the following Proposition.

Proposition 3.3 ([1]) *Suppose* $< S(\phi), \phi > \geq -C|\phi|^2$ *for some constant* $C > 0$ *independent of* $x \in M$ *and every* $\phi \in A^r(Q)$. *If* ϕ *and* $\Delta\phi$ *are in* $L_2^r(Q)$, *then* $\tilde{\nabla}\phi$ *is in* L_2.

Lemma 3.4 *Suppose* $< S(\phi), \phi > \geq -C|\phi|^2$ *for some constant* $C > 0$ *independent of* $x \in M$ *and every* $\phi \in A^r(Q)$. *If* ϕ *and* $\Delta\phi$ *are in* $L_2^*(Q)$, *then*

$$\frac{1}{2}||\tilde{\nabla}\phi||^2 + \mathcal{S}(\phi) \leq \text{limsup} \ll \Delta\phi, \omega_\ell^2\phi \gg_{B(2\ell)} \leq \frac{3}{2}||\tilde{\nabla}\phi||^2 + \mathcal{S}(\phi),$$

where $\mathcal{S}(\phi) = \text{limsup} \ll S(\phi), \omega_\ell^2\phi \gg_{B(2\ell)}$.

Hence if the foliation has finite energy (i.e., $||\pi||^2 < \infty$) such that $\Delta\pi \in L_2^1(Q)$, then we have

$$\frac{1}{2}||\tilde{\nabla}\pi||^2 + \mathcal{S}(\pi) \leq \text{limsup} \ll \Delta\pi, \omega_\ell^2\pi \gg_{B(2\ell)} \leq \frac{3}{2}||\tilde{\nabla}\pi||^2 + \mathcal{S}(\pi). \qquad (3.2)$$

From (3.2) and Lemma 3.1, we have the following Proposition.

Proposition 3.5 *Let* $\mathcal{F}$ *be a Riemannian foliation with finite energy on a complete Riemannian manifold. If* $\Delta\pi \in L_2^1(Q)$ *and* $< S(\pi), \pi > \geq -C|\pi|^2$ *for some constant* $C > 0$, *then we have*

$$\frac{1}{2}||\tilde{\nabla}\pi||^2 + \mathcal{S}(\pi) \leq \frac{3}{2}||d_{\nabla}^*\pi||^2,$$
$$\frac{1}{2}||d_{\nabla}^*\pi||^2 \leq \frac{3}{2}||\tilde{\nabla}\pi||^2 + \mathcal{S}(\pi).$$

To analyze the sign of the term $\mathcal{S}(\pi)$, it is convenient to introduce the self-adjoint operator $B_\pi : TM \to TM$ ([2]) by

$$g_M(B_\pi X, Y) = g_Q(\pi(X), \pi(Y)) \quad \text{for } X,\ Y \in TM. \qquad (3.3)$$

Clearly, $KerB_\pi = L$, $ImB_\pi = \sigma Q \cong L^\perp$. We further refine the choice of local framings by requiring that the orthogonal basis $e_1, \cdots, e_n$ of T_xM also diagonalize B_π, i.e.,

$$B_\pi(e_i) = 0 \ (i = 1, \cdots, p); \ B_\pi(e_a) = \lambda_a e_a \ (a = p+1, \cdots, n), \qquad (3.4)$$

where $\lambda_a > 0$, since g_Q is positive definite. Clearly we have

$$g_Q(\pi(e_a), \pi(e_b)) = \lambda_a \delta_{ab}. \tag{3.5}$$

Now, we consider the normal sectional curvature $K^\nabla(e_a, e_b)$ in direction of the normal 2-plane spanned by e_a, e_b defined by

$$K^\nabla(e_a, e_b) = \frac{1}{\lambda_a \lambda_b} g_Q(R^\nabla_{\pi(e_a),\pi(e_b)}\pi(e_b), \pi(e_a)), \tag{3.6}$$

where $R^\nabla_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is the curvautre tensor on Q. Note that since ∇ is a basic connection, $i(X)R^\nabla = 0$ for $X \in \Gamma L([2])$, hence $R^\nabla_{\pi(e_a),-} = R^\nabla_{e_a,-}$. The transversal Ricci operator $\rho^\nabla : Q \to Q$ and the transversal scalar curvature σ^∇ are given respectively by

$$\rho^\nabla(X) = \sum_a R^\nabla_{X,e_a} e_a, \quad \sigma^\nabla = Tr(\rho^\nabla). \tag{3.7}$$

All these geometric quantities should be thought of as the corresponding curvature properties of a Riemannian manifold serving as a model space for $\mathcal{F}$. Furthermore, we have

$$\begin{aligned} g_Q(\pi(\rho^{\nabla^M}(e_a)), \pi(e_a)) =& g_M((B_\pi \circ \rho^{\nabla^M})e_a, e_a) \\ =& g_M(\rho^{\nabla^M}(e_a), B_\pi e_a) \\ =& \lambda_a g_M(\rho^{\nabla^M}(e_a), e_a), \end{aligned} \tag{3.8}$$

where ρ^{∇^M} is the Ricci operator of ∇^M given by $\rho^{\nabla^M}(X) = \sum R^M_{X,e_\alpha} e_\alpha$. From (1.7), we obtain

$$< S(\pi), \pi >_x = -\sum_{a \neq b} \lambda_a \lambda_b K^\nabla(e_a, e_b) + \sum_a \lambda_a g_M(\rho^{\nabla^M}(e_a), e_a). \tag{3.9}$$

Thus non-negative Ricci curvature on M and non-positive normal sectional curvature K^∇ imply $< S(\pi), \pi > \geq 0$. From Proposition 3.5 and (3.9), we obtain the following Theorem.

Theorem 3.6 *Let $\mathcal{F}$ be a Riemannian foliation with finite energy on a complete Riemannian manifold (M, g_M) with holonomy invariant metric g_Q on Q (g_M is not assumed to be bundle-like). Assume that the Ricci curvature ρ^M on M is non-negative and the normal sectional curvature K^∇ of g_Q is non-positive. Then*

$$d^*_\nabla \pi = 0 \text{ if and only if } \tilde{\nabla}\pi = 0 \text{ and } \limsup \ll S(\pi), \omega_\ell^2 \pi \gg = 0.$$

Note that for the normal bundle Q of a foliation on M the connection ∇' on Q defined by a Riemannian metric g_M via (3.10) below need not be metric with respect to g_Q induced by g_M. Thus we say that $\phi \in A^r(M,Q)$ is *harmonic* if $d_\nabla \phi = 0$ and $d_\nabla^* \phi = 0$. In case ∇ is metric, this condition is equivalent to $\Delta\phi = 0$ for ϕ with $\phi, \Delta\phi \in L_2^*(Q)$.

The condition $\tilde{\nabla}\pi = 0$ implies that

$$(\tilde{\nabla}\pi)(X,Y) = (\tilde{\nabla}_X\pi)(Y) = \nabla_X\pi(Y) - \pi(\nabla_X^M Y) = 0.$$

In particular, for $X, Y \in \Gamma L$, $\nabla_X^M Y \in \Gamma L$. This means that each leaf $\mathcal{L}$ is a totally geodesic submanifold of M. Hence we have the following Corollary.

Corollary 3.7 *Let $\mathcal{F}$ be a Riemannian foliation satisfying the conditions in* Theorem 3.6.

(1) *If π is a harmonic form, then each leaf is a totally geodesic submanifold of M.*

(2) *If there exists some point $x \in B(2\ell)$ such that $< S(\pi), \omega_\ell^2\pi >_x \neq 0$, then π is not a harmonic form.*

If the codimension of $\mathcal{F}$ is one, then the normal sectional curvature K_∇ is zero. Hence Corollary 3.7 holds under the assumption that the Ricci curvature of g_M is non-negative.

Now we discuss the bundle-like metric case([2]), i.e., g_Q can be assumed to be induced by g_M as

$$g_Q(s,t) = g_M(\sigma(s), \sigma(t))$$

for any s, $t \in \Gamma Q$. The projection $\pi : TM \to Q$ is then an orthogonal projection. The particular connection ∇' in Q defined by

$$\begin{cases} \nabla'_X s = \pi([X, \sigma(s)]) & \text{for } X \in \Gamma L \\ \nabla'_X s = \pi(\nabla_X^M \sigma(s)) & \text{for } X \in \Gamma L^\perp \end{cases} \tag{3.10}$$

is then the unique metric and torsion-free connection with respect to g_Q. The harmonicity of π, i.e., the condition $d_{\nabla'}^*\pi = 0$(since we already have $d_{\nabla'}\pi = 0$), is then equivalent to the property that all leaves of $\mathcal{F}$ are minimal submanifolds of (M, g_M) ([2]). Noting that $(\tilde{\nabla}_X\pi)(X) = 0$ for any $X \in \Gamma Q$, we see that $\tau = d_{\nabla'}^*\pi$. Then $\mathcal{F}$ is harmonic if and only if $\tau = 0$ (see Appendix or [2]).

The operator $B_\pi : TM \to TM$ defined by (3.3) is the map $\sigma \circ \pi$ and the non-zero eigenvalues λ_a equal 1. Then we have

$$< S\pi, \pi >_x = -\sigma^{\nabla'} + \sum_a < \rho^{\nabla^M}(e_a), e_a >$$

where $\sigma^{\nabla'}$ is the transversal scalar curvature of Q. Hence from Theorem 3.6, we have the following Corollary.

Corollary 3.8 *Let $\mathcal{F}$ be a Riemannian foliation with finite energy on M with a complete bundle-like metric g_M. Assume that the Ricci curvature ρ^M on M is non-negative and the transversal scalar curvature is non-positive. If $\mathcal{F}$ is harmonic, then $\mathcal{F}$ is totally geodesic.*

Appendix

Let $\mathcal{F}$ be a foliation on a Riemannian manifold (M, g_M) with bundle-like metric g_M. The Q-valued symmetric bilinear form $\alpha = -\tilde{\nabla}\pi$ restricted to any leaf $\mathcal{L} \subset M$ of $\mathcal{F}$ is then the second fundamental form of the Riemannian submanifold $\mathcal{L} \subset M$. By [2], the tension $\tau = Tr\alpha$ of $\mathcal{F}$ is evaluated at $x \in M$ by

$$\tau_x = Tr\alpha = \sum_\beta \alpha(e_\beta, e_\beta) = \sum_i \alpha(e_i, e_i) \in Q_x.$$

It is immediate that $\tau = d^*_{\nabla'}\pi$, and $\mathcal{F}$ is harmonic iff $\tau = 0$([2]).

This tension field τ plays an important role in studying a foliated Riemannian manifold. When a foliation is minimal, i.e., $\tau = 0$, many results are similar to those in an ordinary manifold. So an apparent weakening of the condition of the vanishing tension field would be to require $\nabla'\tau = 0$. But the ∇'-parallel condition of τ is meaningless on a compact manifold because $\nabla'\tau = 0$ implies $\tau = 0$([2]). On a complete Riemannian manifold, we obtain the following result which is similar to the one in [2].

Theorem A. *Let $\mathcal{F}$ be a Riemannian foliation with finite energy on M with a complete bundle-like metric g_M. Then we have*

$$\nabla'\tau = 0 \Longrightarrow \tau = 0.$$

Proof. For a 0-form $\tau \in A^0(Q)$, we have by definition $d_{\nabla'}\tau = \nabla'\tau$. Since $d_{\nabla'}\pi = 0$, we have

$$\Delta\pi = d_{\nabla'}d^*_{\nabla'}\pi = d_{\nabla'}\tau = \nabla'\tau.$$

This implies that if $\nabla'\tau = 0$, then $\Delta\pi = 0$. From the first inequality in Lemma 3.4, we obtain $\tau = d^*_{\nabla'}\pi = 0$. □

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