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Ricci tensor of C-totally real submanifolds in Sasakian space forms

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Abstract

B.-Y. Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for a submanifold in a Riemannian space form with arbitrary codimension. The Lagrangian version of this inequality was proved by the same author. In this article, we obtain a sharp estimate of the Ricci tensor of a C-totally real submanifold M in a Sasakian space form $\widetilde{M}(c)$, in terms of the main extrinsic invariant, namely the squared mean curvature. If M satisfies the equality case identically, then it is minimal. Moreover, in this case, M is a ruled submanifold.

1. Preliminaries.

A $(2m+1)$-dimensional Riemannian manifold $(\widetilde{M}, g)$ is said to be a *Sasakian manifold* if it admits an endomorphism ϕ of its tangent bundle $T\widetilde{M}$, a vector field ξ and a 1-form η, satisfying:

$$\begin{cases} \phi^2 = -Id + \eta \otimes \xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \ \eta \circ \phi = 0, \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \eta(X) = g(X, \xi), \\ (\widetilde{\nabla}_X \phi)Y = -g(X, Y)\xi + \eta(Y)X, \ \widetilde{\nabla}_X \xi = \phi X, \end{cases}$$

for any vector fields X, Y on $T\widetilde{M}$, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g. A plane section π in $T_p\widetilde{M}$ is called a ϕ-section if it is spanned by X and ϕX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a ϕ-section is called a ϕ-sectional curvature. A Sasakian manifold with constant ϕ-sectional curvature c is said to be a *Sasakian space form* and is denoted by $\widetilde{M}(c)$.

The curvature tensor $\widetilde{R}$ of a Sasakian space form $\widetilde{M}(c)$ is given by [1]

$$(1.1) \qquad \widetilde{R}(X, Y)Z = \frac{c+3}{4}\{g(Y, Z)X - g(X, Z)Y\} +$$

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$$+\frac{c-1}{4}\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi+$$
$$+g(\phi Y,Z)\phi X-g(\phi X,Z)\phi Y-2g(\phi X,Y)\phi Z\},$$

for any tangent vector fields X, Y, Z on $\widetilde{M}(c)$.

As examples of Sasakian space forms we mention $\mathbf{R}^{2m+1}$ and S^{2m+1}, with standard Sasakian structures (see [1],[10]).

Let M be an n-dimensional submanifold of a Sasakian space form $\widetilde{M}(c)$ of constant ϕ-sectional curvature c. We denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM, p \in M$, and ∇ the Riemannian connection of M, respectively. Also, let h be the second fundamental form and R the Riemann curvature tensor of M. Then the equation of Gauss is given by

$$\tilde{R}(X,Y,Z,W)=R(X,Y,Z,W)+ \tag{1.2}$$
$$+g(h(X,W),h(Y,Z))-g(h(X,Z),h(Y,W)),$$

for any vectors X, Y, Z, W tangent to M.

Let $p \in M$ and $\{e_1, ..., e_{2m+1}\}$ an orthonormal basis of the tangent space $T_p\widetilde{M}$, such that $e_1, ..., e_n$ are tangent to M at p. We denote by H the mean curvature vector, that is

$$H(p)=\frac{1}{n}\sum_{i=1}^{n}h(e_i,e_i). \tag{1.3}$$

Also, we set

$$h_{ij}^r=g(h(e_i,e_j),e_r), \quad i,j\in\{1,...,n\}, r\in\{n+1,...,2m+1\}, \tag{1.4}$$

and

$$\|h\|^2=\sum_{i,j=1}^{n}g(h(e_i,e_j),h(e_i,e_j)). \tag{1.5}$$

Recall that for a submanifold M in a Riemannian manifold, the relative null space of M at a point $p \in M$ is defined by

$$\mathcal{N}_p=\{X\in T_pM \,|\, h(X,Y)=0, \text{ for all } Y\in T_pM\}.$$

In the proof of Theorem 2.1, we will use the following result of B.-Y. Chen.

Lemma [2]. *Let $n \geq 2$ and $a_1, ..., a_n, b$ real numbers such that*

$$\left(\sum_{i=1}^{n}a_i\right)^2=(n-1)\left(\sum_{i=1}^{n}a_i^2+b\right)$$

Then $2a_1a_2 \geq b$, *with equality holding if and only if*

$$a_1 + a_2 = a_3 = ... = a_n.$$

2. Ricci tensor and squared mean curvature.

B.Y. Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for submanifolds in real space forms (see [3]). Afterwards, he obtained the Lagrangian version of this relationship (see [4]). First, we prove a similar inequality for an n-dimensional C-totally real submanifold M of a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$ of constant ϕ-sectional curvature c.

A submanifold M normal to ξ in a Sasakian space form $\widetilde{M}(c)$ is said to be a *C-totally real submanifold.*

It follows that ϕ maps any tangent space of M into the normal space, that is, $\phi(T_pM) \subset T_p^{\perp}M$, for every $p \in M$.

Theorem 2.1. *Let M be an n-dimensional C-totally real submanifold in a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$ of constant ϕ-sectional curvature c. Then:*

i) *For each unit vector $X \in T_pM$, we have*

$$\text{Ric}(X) \leq \frac{1}{4}\{n^2 \|H\|^2 + (n-1)(c+3)\}. \tag{2.1}$$

ii) *If $H(p) = 0$, then a unit tangent vector X at p satisfies the equality case of (2.1) if and only if $X \in \mathcal{N}_p$.*

iii) *The equality case of (2.1) holds identically for all unit tangent vectors at p if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Proof. i) Let $X \in T_pM$ be a unit tangent vector X at p. We choose an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{2m+1}\}$ such that $e_1, ..., e_n$ are tangent to M at p, with $e_n = X$, $e_{2m+1} = \xi$ and e_{n+1} parallel to the mean curvature vector $H(p)$ (if $H(p) = 0$, then e_{n+1} can be any unit normal vector orthogonal to ξ).

Then, from the Gauss equation, we have

$$n^2 \|H\|^2 = 2\tau + \|h\|^2 - \frac{1}{4}n(n-1)(c+3), \tag{2.3}$$

where τ denotes the scalar curvature at p, that is,

$$\tau = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j).$$

We put

$$\delta = 2\tau - \frac{n^2}{2}\|H\|^2 - \frac{1}{4}n(n-1)(c+3).$$

Then, from (2.3), we get

$$n^2\|H\|^2 = 2(\delta + \|h\|^2). \tag{2.4}$$

With respect to the above orthonormal basis, (2.4) takes the following form:

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = 2\left\{\delta + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\right\}.$$

If we put $a_1 = h_{11}^{n+1}, a_2 = \sum_{i=2}^{n-1} h_{ii}^{n+1}$ and $a_3 = h_{nn}^{n+1}$, the above equation becomes

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left\{\delta + \sum_{i=1}^{3} a_i^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 - \sum_{2\leq\alpha\neq\beta\leq n-1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1}\right\}.$$

Thus a_1, a_2, a_3 satisfy the Lemma of Chen (for $n = 3$), i.e.

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left(b + \sum_{i=1}^{3} a_i^2\right).$$

Then $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3$.
In the case under consideration, this means

$$\sum_{1\leq\alpha\neq\beta\leq n-1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1} \geq \delta + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2,$$

or equivalently,

$$\frac{n^2}{2}\|H\|^2 + \frac{1}{4}n(n-1)(c+3) \geq \tag{2.5}$$

$$\geq 2\tau - \sum_{1\leq\alpha\neq\beta\leq n-1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1} + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2.$$

Using again the Gauss equation, we have

$$2\tau - \sum_{1\leq\alpha\neq\beta\leq n-1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1} + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 = \tag{2.6}$$

$$= 2S(e_n, e_n) + \frac{1}{4}(n-1)(n-2)(c+3) + 2\sum_{i=1}^{n-1}(h_{in}^{n+1})^2 +$$

$$+\sum_{n+2}^{2m}\left\{(h_{nn}^r)^2+2\sum_{i=1}^{n-1}(h_{in}^r)^2+(\sum_{\alpha=1}^{n-1}h_{\alpha\alpha}^r)^2\right\},$$

where S is the Ricci tensor of M.

Combining (2.5) and (2.6), we obtain

$$\frac{n^2}{2}\|H\|^2+\frac{1}{2}(n-1)(c+3)\geq 2S(e_n,e_n)+2\sum_{i=1}^{n-1}(h_{in}^{n+1})^2+$$

$$+\sum_{r=n+2}^{2m}\left\{\sum_{i=1}^{n-1}(h_{in}^r)^2+(\sum_{\alpha=1}^{n-1}h_{\alpha\alpha}^r)^2\right\},$$

which implies (2.1).

ii) Assume $H(p)=0$. Equality holds in (2.1) if and only if

$$\begin{cases} h_{1n}^r=\ldots=h_{n-1,n}^r=0 \\ h_{nn}^r=\sum_{i=1}^{n-1}h_{ii}^r \end{cases},\quad r\in\{n+1,\ldots,2m\}. \tag{2.7}$$

Then $h_{in}^r=0,\forall i\in\{1,\ldots,n\},r\in\{n+1,\ldots,2m\}$, i.e. $X\in\mathcal{N}_p$.

iii) The equality case of (2.1) holds for all unit tangent vectors at p if and only if

$$\begin{cases} h_{ij}^r=0, i\neq j, r\in\{n+1,\ldots,2m\}, \\ h_{11}^r+\ldots+h_{nn}^r-2h_{ii}^r=0,\quad i\in\{1,\ldots,n\}, r\in\{n+1,\ldots,2m\}. \end{cases} \tag{2.8}$$

We distinguish two cases:

a) $n\neq 2$, then p is a totally geodesic point;

b) $n=2$, it follows that p is a totally umbilical point.

The converse is trivial. □

By polarization, from Theorem 2.1, we derive:

Theorem 2.2. *Let M be an n-dimensional C-totally real submanifold in a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$ of constant ϕ-sectional curvature c. Then the Ricci tensor S satisfies*

$$S\leq\frac{1}{4}\left\{n^2\|H\|^2+(n-1)(c+3)\right\}g. \tag{2.9}$$

The equality case of (2.9) holds identically if and only if either M is a totally geodesic submanifold or $n=2$ and M is a totally umbilical submanifold.

3. Minimality of C-totally real submanifolds.

Let $\widetilde{M}(c)$ be a $(2n+1)$-dimensional Sasakian space form and M an n-dimensional C-totally real submanifold of $\widetilde{M}(c)$.

We denote by $\mathcal{R}$ the maximum Ricci curvature function on M (see [4]), defined by

$$\mathcal{R}(p) = \max\{S(u,u) | u \in T_p^1 M\}, \quad p \in M,$$

where $T_p^1 M = \{u \in T_p M | g(u,u) = 1\}$.

If $n = 3$, $\mathcal{R}$ is the Chen first invariant δ_M defined in [2]. For $n > 3$, $\mathcal{R}$ is the Chen invariant $\delta(n-1)$ (see [5]).

In this section, we derive an inequality for the Chen invariant $\mathcal{R}$ and prove that any C-totally real submanifold which satisfies the equality case, identically, is minimal. This is the Sasakian version of a result ([4]) of B.-Y. Chen for Lagrangian submanifolds in complex space forms.

Theorem 3.1. *Let M be an n-dimensional C-totally real submanifold in a $(2n+1)$-dimensional Sasakian space form $\widetilde{M}(c)$ of constant ϕ-sectional curvature c. Then*

$$\mathcal{R} \leq \frac{1}{4}\{n^2\|H\|^2 + (n-1)(c+3)\}. \tag{3.1}$$

If M satisfies the equality case of (3.1) *identically, then M is a minimal submanifold.*

Proof. The inequality (3.1) is an immediate consequence of the inequality (2.9).

We assume that M is a C-totally real submanifold of $\widetilde{M}(c)$, which satisfies the equality case of (3.1) at a point $p \in M$. We may choose an orthonormal basis $\{e_1, ..., e_n\}$ of T_pM such that $\mathcal{R}(p) = S(e_n, e_n)$. By the proof of Theorem 2.1, it follows that the equations (2.7) hold, where h_{ij}^r are the coefficients of the second fundamental form with respect to the orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{2n+1}\}$, with $e_{n+j} = \phi e_j, j \in \{1, ..., n\}$, $e_{2n+1} = \xi$ and e_{n+1} parallel to the mean curvature vector $H(p)$.

Let A denote the shape operator of M in $\widetilde{M}(c)$. It is easy to prove that

$$A_{\phi X} Y = A_{\phi Y} X, \tag{3.2}$$

for all vector fields X, Y tangent to M (see, for instance, [9]). Then we have $h_{ij}^{n+k} = h_{ik}^{n+j} = h_{jk}^{n+i}$, for any $i, j, k \in \{1, ..., n\}$.

Thus, using the equations (2.7), we find

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h_{ii}^{n+1} e_{n+1} = \frac{2}{n} h_{nn}^{n+1} e_{n+1} = \frac{2}{n} h_{1n}^{2n} e_{n+1} = 0.$$

Therefore M is a minimal submanifold. □

Corollary 3.2. *Let M be an n-dimensional C-totally real submanifold of a $(2n+1)$-dimensional Sasakian space form $\widetilde{M}(c)$. If $\dim \mathcal{N}_p$ is positive constant, then M satisfies the equality case of* (3.1) *identically and is foliated by totally geodesic submanifolds.*

Proof. By the above proof, it follows that M satisfies the equality case of (3.1) at a point $p \in M$ if and only if $\dim \mathcal{N}_p \geq 1$.

Assume that $\dim \mathcal{N}_p$ is positive constant.

We prove that $\mathcal{N}$ is involutive and its leaves are totally geodesic.

Let $Y, Z \in \mathcal{N}$ and $X \in \mathcal{N}^{\perp}$. Codazzi equation implies $g(X, \nabla_Y Z) = 0$. Thus $\nabla_Y Z \in \mathcal{N}$, for all $Y, Z \in \mathcal{N}$. Therefore each leaf of $\mathcal{N}$ is totally geodesic.

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