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SPECTRAL MAPPING THEOREM FOR APPROXIMATE SPECTRA AND ITS APPLICATIONS

FUMIHIKO KIMURA

Abstract. In this paper, we show that the spectral mapping theorem holds for the approximate point (or approximate defect) spectrum of a bounded linear operator on a Banach space. Moreover, we analyze the spectra of an elementary operator by means of those spectral mapping theorems.

1. Introduction

Let $\mathfrak{X}$ be an infinite-dimensional complex Banach space and denote the algebra of all bounded linear operators on $\mathfrak{X}$ by $\mathcal{L}(\mathfrak{X})$. For $T \in \mathcal{L}(\mathfrak{X})$, $\sigma(T)$ stands for the spectrum of T. The usual spectral mapping theorem for T has been generalized to theorems of the form

$$\sigma_i(f(T)) = f(\sigma_i(T)) \tag{1.1}$$

where $\sigma_i(T)$ is a certain subset of $\sigma(T)$. Throughout this paper, $\mathcal{A}(\sigma(T))$ denotes the set of all complex-valued functions f analytic on a neighborhood Ω_f of $\sigma(T)$. For $f \in \mathcal{A}(\sigma(T))$, $f(T)$ is the usual analytic functional calculus of T. The first aim of this paper is to show that the approximate point spectrum σ_π and the approximate defect spectrum σ_δ both satisfy (1.1).

Definition 1.1. For $T \in \mathcal{L}(\mathfrak{X})$, the approximate point spectrum $\sigma_\pi(T)$ and the approximate defect spectrum $\sigma_\delta(T)$ are defined by

$$\begin{aligned}
\sigma_\pi(T) &= \{\lambda \in \mathbb{C} \mid \lambda - T \text{ is not bounded below}\} \\
&= \left\{\lambda \in \mathbb{C} \mid \inf_{\|x\|=1} \|(\lambda - T)x\| = 0\right\}, \\
\sigma_\delta(T) &= \{\lambda \in \mathbb{C} \mid \lambda - T \text{ is not surjective}\}
\end{aligned}$$

where $\mathbb{C}$ denotes complex plane.

It is known that they are both compact subsets of $\sigma(T)$ and $\sigma(T) = \sigma_\pi(T) \cup \sigma_\delta(T)$. Our result is as follows.

Theorem 1.2. *For any $T \in \mathcal{L}(\mathfrak{X})$ and $f \in \mathcal{A}(\sigma(T))$,*

$$\sigma_\pi(f(T)) = f(\sigma_\pi(T)) \tag{1.2}$$

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and

$$\sigma_\delta(f(T)) = f(\sigma_\delta(T)). \tag{1.3}$$

These seem to be very humble spectral mapping theorems, but the author does not know articles in which this result is stated explicitly with its proof. However, concerning this topic, there are some results which are worth noticing. To begin with, note that many mathematicians have shown (1.2) and (1.3) in the case where f is a rational function with no poles in $\sigma(T)$ (for instance, [1, Theorem 2]). Moreover, according to [13, p. 326], (1.2) holds whenever f is not constant on each component of its domain Ω_f. Further, we should introduce the following remarkable result of V. Rakočević. Following [9], we define

$$\sigma_{ab}(T) = \bigcap \left\{ \sigma_\pi(T+K) \mid TK = KT,\ K \in \mathcal{K}(\mathfrak{X}) \right\}$$

and call it Brouder's essential approximate point spectrum of T. ($\mathcal{K}(\mathfrak{X})$ stands for the algebra of all compact operators on $\mathfrak{X}$.) Rakočević [10, Theorem 3.4] showed that (1.1) holds for σ_{ab}. In this context, see also Schmoeger [12].

In this paper, we give a complete proof of Theorem 1.2 by a common technique appearing in [7],[10],[12], and so on. Moreover, in the last section, we analyze the spectra of an elementary operator on $\mathcal{L}(\mathfrak{X})$ by means of Theorem 1.2.

2. Proof of Theorem 1.2

Lemma 2.1. *For a complex number λ, if there exists a sequence of unit vectors $\{x_k\}$ such that $\|(\lambda - T)x_k\| \to 0$, then $\|(f(\lambda) - f(T))x_k\| \to 0$ for any $f \in \mathcal{A}(\sigma(T))$.*

Proof. We cite [5, Lemma 2] for convenience. □

From Lemma 2.1, it follows that the one-way spectral mapping theorem holds for σ_π.

Lemma 2.2. *$\sigma_\pi(p(T)) = p(\sigma_\pi(T))$ holds for every polynomial p.*

Proof. If a polynomial p is constant, $p(z) \equiv \lambda$, then

$$\sigma_\pi(p(T)) = \sigma_\pi(\lambda I) = \sigma(\lambda I) = \{\lambda\} = p(\sigma_\pi(T)).$$

(For concluding the last equality, we mention that $\sigma_\pi(T) \neq \emptyset$ for all T.) Since we have only to show $\sigma_\pi(p(T)) \subseteq p(\sigma_\pi(T))$ when p is not constant. For any fixed complex number λ, let $q(z) = \lambda - p(z)$ and $q(z) = \alpha \prod_{j=1}^{k} (\mu_j - z)$ the factorization of $q(z)$. (α is a nonzero complex number.) Then

$$\lambda - p(T) = q(T) = \alpha \prod_{j=1}^{k} (\mu_j - T).$$

Therefore, if $\mu_j - T$ is bounded below for all $j = 1, \dots, k$, then $\lambda - p(T)$ is also bounded below. Thus, if $\lambda \in \sigma_\pi(p(T))$, then $\mu_j \in \sigma_\pi(T)$ for some j and $\lambda = p(\mu_j) \in p(\sigma_\pi(T))$. □

Let $\mathrm{Rat}(\sigma(T))$ be the set of all rational functions with no poles in $\sigma(T)$.

Lemma 2.3. *$\sigma_\pi(f(T)) = f(\sigma_\pi(T))$ holds for every $f \in \mathrm{Rat}(\sigma(T))$.*

Proof. We have to show $\sigma_\pi(f(T)) \subseteq f(\sigma_\pi(T))$. Since f belongs to $\mathrm{Rat}(\sigma(T))$, $f(z) = p(z)/q(z)$ where p, q are polynomials and q has no zeros in $\sigma(T)$. Let $\lambda \in \sigma_\pi(f(T))$. Then there exists a sequence of unit vectors $\{x_k\}$ such that

$$\|(\lambda - f(T))x_k\| = \|(\lambda - q(T)^{-1}p(T))x_k\| \to 0.$$

Thus $\|(\lambda q(T) - p(T))x_k\| \to 0$ and this means $0 \in \sigma_\pi(\lambda q(T) - p(T)) = \sigma_\pi((\lambda q - p)(T))$. By Lemma 2.2, $\sigma_\pi((\lambda q - p)(T)) = (\lambda q - p)(\sigma_\pi(T))$ and hence there exists a $\mu \in \sigma_\pi(T)$ such that $\lambda q(\mu) - p(\mu) = 0$. $\lambda = p(\mu)/q(\mu) = f(\mu) \in f(\sigma_\pi(T))$. □

In order to show that (1.2) and (1.3) are valid for an arbitrary $f \in \mathcal{A}(\sigma(T))$, we introduce some definitions and lemmas about the continuity of set-valued functions.

Definition 2.4. For a sequence $\{\delta_n\}$ of compact subsets of $\mathbb{C}$, $\limsup_{n\to\infty} \delta_n$ and $\liminf_{n\to\infty} \delta_n$ are defined by

$$\limsup_{n\to\infty} \delta_n = \{\lambda \in \mathbb{C} \mid \liminf_{n\to\infty} d(\lambda, \delta_n) = 0\}$$

and

$$\liminf_{n\to\infty} \delta_n = \{\lambda \in \mathbb{C} \mid \lim_{n\to\infty} d(\lambda, \delta_n) = 0\}.$$

Observe that $\liminf_{n\to\infty} \delta_n \subseteq \limsup_{n\to\infty} \delta_n$ holds in general. If these two sets coincide, then we say that $\{\delta_n\}$ converges and its limit set is given by $\lim_{n\to\infty} \delta_n = \limsup_{n\to\infty} \delta_n = \liminf_{n\to\infty} \delta_n$.

Definition 2.5. Let $\varphi : \mathcal{L}(\mathfrak{X}) \to \{\text{compact subsets of } \mathbb{C}\}$ be a set-valued function. We say that φ is upper semi-continuous at T if $\limsup_{n\to\infty} \varphi(T_n) \subseteq \varphi(T)$ whenever $\|T_n - T\| \to 0$. Also, we say that φ is lower semi-continuous at T if $\varphi(T) \subseteq \liminf_{n\to\infty} \varphi(T_n)$ whenever $\|T_n - T\| \to 0$.

It is well-known that the spectrum σ is upper semi-continuous at every $T \in \mathcal{L}(\mathfrak{X})$. The following two lemmas may also be known, but we will give proofs for the sake of completeness.

Lemma 2.6. *The approximate point spectrum σ_π is upper semi-continuous at every $T \in \mathcal{L}(\mathfrak{X})$.*

Proof. Suppose that $\|T_n - T\| \to 0$ and $\lambda \in \limsup_{n\to\infty} \sigma_\pi(T_n)$. It is to be shown that $\lambda \in \sigma_\pi(T)$. By assumption, we have $\liminf_{n\to\infty} d(\lambda, \sigma_\pi(T_n)) = 0$ and hence there exist a subsequence $\{T_{n_k}\}$ and $\lambda_k \in \sigma_\pi(T_{n_k})$ such that $|\lambda - \lambda_k| \to 0$ as $k \to \infty$. Then

$$\|(\lambda - T) - (\lambda_k - T_{n_k})\| \leq |\lambda - \lambda_k| + \|T_{n_k} - T\| \to 0.$$

Since $\lambda_k - T_{n_k}$ is not bounded below for all k, $\lambda - T$ also fails to be bounded below, that is, $\lambda \in \sigma_\pi(T)$. □

Lemma 2.7. *If $\|T_n - T\| \to 0$ and $T_nT = TT_n$ for all n, then $\lim_{n\to\infty} \sigma_\pi(T_n) = \sigma_\pi(T)$.*

Proof. By Lemma 2.6, it suffices to show that $\sigma_\pi(T) \subseteq \liminf_{n\to\infty} \sigma_\pi(T_n)$. From [3, Theorem 2], it follows that

$$\sigma_\pi(T) \subseteq \sigma_\pi(T_n) + \sigma_\pi(T - T_n).$$

(Here, $M + N$ denotes the set $\{m + n \mid m \in M,\ n \in N\}$ for $M, N \subseteq \mathbb{C}$.) Therefore if $\lambda \in \sigma_\pi(T)$, then $d(\lambda, \sigma_\pi(T_n)) \leq \|T - T_n\| \to 0$ as $n \to \infty$. This means $\lambda \in \liminf_{n\to\infty} \sigma_\pi(T_n)$ □

The next lemma is known as Runge's theorem. See [11, Chapter 13].

Lemma 2.8. *Let S^2 be the Riemann sphere (the one-point compactification of $\mathbb{C}$) and Ω an open subset of $\mathbb{C}$, and suppose that f is a complex-valued analytic function on Ω. If A is a subset of S^2 which has one point in each component of $S^2 \setminus \Omega$, then there exists a sequence of rational functions $\{f_n\}$, with poles only in A, such that f_n converges uniformly to f on compact subsets of Ω.*

Suppose $T \in \mathcal{L}(\mathfrak{X})$ and $f \in \mathcal{A}(\sigma(T))$. Since f is analytic on an open set Ω_f containing $\sigma(T)$, there exists a sequence of rational functions $\{f_n\}$ with no poles in Ω_f and f_n converges uniformly to f on compact subsets of Ω_f. The following lemma guarantees that $f_n(T)$ converges to $f(T)$ in the norm topology of $\mathcal{L}(\mathfrak{X})$.

Lemma 2.9. *Let Ω be an open subset of $\mathbb{C}$ containing $\sigma(T)$. If f_n $(n = 1, 2, \dots)$ and f are analytic on Ω and f_n converges uniformly to f on compact subsets of Ω, then $\|f_n(T) - f(T)\| \to 0$.*

Proof. Choose a Cauchy domain U such as $\sigma(T) \subset U \subset U \cup \Gamma \subset \Omega$, where Γ denotes the boundary of U. By hypothesis, f_n converges uniformly to f on Γ. Thus,

$$\begin{aligned} \|f_n(T) - f(T)\| &= \frac{1}{2\pi i} \cdot \left\| \int_\Gamma [f_n(z) - f(z)](z - T)^{-1} dz \right\| \\ &\leq \frac{1}{2\pi i} \cdot M \cdot l(\Gamma) \cdot \sup_{z \in \Gamma} |f_n(z) - f(z)| \\ &\to 0 \end{aligned}$$

as $n \to \infty$. On the right-hand side of the inequality, $M = \sup_{z \in \Gamma} \|(z - T)^{-1}\|$ and $l(\Gamma)$ stands for the length of Γ. □

Proof of Theorem 1.2. First, we deal with σ_π. It suffices to show that $\sigma_\pi(f(T)) \subseteq f(\sigma_\pi(T))$. By Lemmas 2.8, 2.9, there exists a sequence $\{f_n\} \subset \mathrm{Rat}(\sigma(T))$ such that f_n converges uniformly to f on $\sigma(T)$ and $\|f_n(T) - f(T)\| \to 0$. By Lemmas 2.3, 2.7, now we have only to check

$$\lim_{n \to \infty} f_n(\sigma_\pi(T)) \subseteq f(\sigma_\pi(T)).$$

Let λ be in $\lim_{n\to\infty} f_n(\sigma_\pi(T))$, that is, $\lim_{n\to\infty} d(\lambda, f_n(\sigma_\pi(T))) = 0$. Then there exist a subsequence $\{f_{n_k}\}$ and $\lambda_k \in \sigma_\pi(T)$ such that $|\lambda - f_{n_k}(\lambda_k)| \to 0$ as $k \to \infty$. Since $\sigma_\pi(T)$ is compact, we may assume that there exists a $\mu \in \sigma_\pi(T)$ and $\lim_{n\to\infty} \lambda_k = \mu$. Then it follows that

$$\begin{aligned} |\lambda - f(\mu)| &\leq |\lambda - f_{n_k}(\lambda_k)| + |f_{n_k}(\lambda_k) - f(\lambda_k)| + |f(\lambda_k) - f(\mu)| \\ &\to 0 \end{aligned}$$

as $k \to \infty$. (The second term converges to 0 since f_{n_k} converges uniformly to f on $\sigma(T)$, and the third term converges to 0 since f is continuous at μ.) Therefore we conclude that $\lambda = f(\mu)$ and hence $\lambda \in f(\sigma_\pi(T))$.

From the remark in [3, § 1],

$$\begin{aligned} \sigma_\delta(f(T)) &= \sigma_\pi(f(T)^\dagger) = \sigma_\pi(f(T^\dagger)) \\ &= f(\sigma_\pi(T^\dagger)) = f(\sigma_\delta(T)) \end{aligned}$$

and this completes the proof. ($T^\dagger$ denotes the Banach space adjoint operator of T defined on the topological dual of $\mathfrak{X}$.) □

Remark 2.10. By Theorem 1.2, we can present another proof of the usual spectral mapping theorem. Indeed, for any $T \in \mathcal{L}(\mathfrak{X})$ and $f \in \mathcal{A}(\sigma(T))$,

$$\begin{aligned}\sigma(f(T)) &= \sigma_\pi(f(T)) \cup \sigma_\delta(f(T)) \\ &= f(\sigma_\pi(T)) \cup f(\sigma_\delta(T)) \\ &= f(\sigma_\pi(T) \cup \sigma_\delta(T)) = f(\sigma(T)).\end{aligned}$$

Remark 2.11. In the proofs of Oberai [7, Theorem 3] and Rakočević [10, Theorem 3.4], both argued that there exists a sequence of *polynomials* $\{p_n\}$ converging uniformly to f on a neighborhood of $\sigma(T)$ and thus $p_n(T)$ converges to $f(T)$. But in general, that is an incorrect argument. For example, if $Ue_n = e_{n+1}$ is the bilateral shift operator on a Hilbert space with its orthonormal basis $\{e_n\}_{n=-\infty}^{\infty}$, then $U^{-1} = U^*$ can not be approximated by polynomials of U.

3. Applications for elementary operators

An elementary operator $\Phi_{\mathbf{A},\mathbf{B}}$ on $\mathcal{L}(\mathfrak{X})$ is defined by $\Phi_{\mathbf{A},\mathbf{B}}(X) = A_1XB_1 + \cdots + A_nXB_n$ for all $X \in \mathcal{L}(\mathfrak{X})$, where $\mathbf{A} = (A_1, \ldots, A_n)$ and $\mathbf{B} = (B_1, \ldots, B_n)$ are both n-tuples of mutually commuting operators in $\mathcal{L}(\mathfrak{X})$. $\Phi_{\mathbf{A},\mathbf{B}}$ is a bounded linear operator on $\mathcal{L}(\mathfrak{X})$. In this section, we deal with analytic elementary operators on $\mathcal{L}(\mathfrak{X})$. Let A and B be in $\mathcal{L}(\mathfrak{X})$ and let $f_1, \ldots, f_n$ (resp. $g_1, \ldots, g_n$) be in $\mathcal{A}(\sigma(A))$ (resp. $\mathcal{A}(\sigma(B))$). An analytic elementary operator Ψ on $\mathcal{L}(\mathfrak{X})$ is defined by

$$\Psi(X) = \sum_{j=1}^{n} f_j(A)Xg_j(B) \quad (X \in \mathcal{L}(\mathfrak{X})) \tag{3.1}$$

and we call A and B the generating operators of Ψ.

Theorem LR ([6, Theorem 10]).

$$\sigma(\Psi) = \left\{ \sum_{j=1}^{n} f_j(\alpha)g_j(\beta) \;\middle|\; \alpha \in \sigma(A), \beta \in \sigma(B) \right\}. \tag{3.2}$$

The formula (3.2) claims that the equation

$$f_1(A)Xg_1(B) + \cdots + f_n(A)Xg_n(B) = Y \tag{3.3}$$

has a unique solution X for each Y if and only if the complex-valued function H of two variables of the form $H(z,w) = f_1(z)g_1(w) + \cdots + f_n(z)g_n(w)$ has no zeros on the Cartesian product $\sigma(A) \times \sigma(B)$.

In [5], we obtained the following result for the approximate point and defect spectra of Ψ.

Theorem K ([5, Theorem 1]).

$$\sigma_\pi(\Psi) \supseteq \left\{ \sum_{j=1}^{n} f_j(\alpha)g_j(\beta) \;\middle|\; \alpha \in \sigma_\pi(A), \beta \in \sigma_\delta(B) \right\} \tag{3.4}$$

and

$$\sigma_\delta(\Psi) \supseteq \left\{ \sum_{j=1}^{n} f_j(\alpha)g_j(\beta) \;\middle|\; \alpha \in \sigma_\delta(A), \beta \in \sigma_\pi(B) \right\}. \tag{3.5}$$

In the view of the theory of operator equations, it is instructive to know the condition for that $\sigma(\Psi) = \sigma_\pi(\Psi)$ or $\sigma(\Psi) = \sigma_\delta(\Psi)$ holds. First of this section, we give an application of Theorem LR and Theorem K to this problem. We consider the situation where the generating operators A and B of Ψ are both decomposable operator in the sense of [2, p. 30].

Lemma 3.1. *Let $T \in \mathcal{L}(\mathfrak{X})$ be a decomposable operator. Then*

$$\sigma(T) = \sigma_\pi(T) = \sigma_\delta(T). \tag{3.6}$$

Proof. Since T is decomposable, $\sigma(T) = \sigma_\pi(T)$ holds. (See [2, Chapter 2, Corollary 1.4].) It is known that the decomposability and the 2-decomposability of T is equivalent ([8, Corollary 1]) and Frunză [4] showed that if T is 2-decomposable then the adjoint $T^\dagger$ is also 2-decomposable. Thus, if T is decomposable, then $T^\dagger$ is also a decomposable operator and hence $\sigma_\delta(T) = \sigma_\pi(T^\dagger) = \sigma(T^\dagger) = \sigma(T)$. □

Theorem 3.2. *If the generating operators A and B of Ψ are both decomposable, then*

$$\sigma(\Psi) = \sigma_\pi(\Psi) = \sigma_\delta(\Psi). \tag{3.7}$$

Proof. Direct consequence of Theorem LR, Theorem K, and Lemma 3.1. □

Corollary 3.3. *If A and B are both decomposable, then, for the equation* (3.3), *the following three conditions are mutually equivalent.*

(i) *There exists a unique solution X to* (3.3) *for each Y,*
(ii) *There exists at least one solution X to* (3.3) *for each Y,*
(iii) *There exists a positive number c such that if X_1* (resp. X_2) *is a solution to* (3.3) *for Y_1* (resp. Y_2) *then $\|Y_1 - Y_2\| \geq c\|X_1 - X_2\|$.*

Proof. (i) is equivalent to the condition that $0 \notin \sigma(\Psi)$ where Ψ is the corresponding analytic elementary operator to the equation (3.3). Similarly, (ii) (resp. (iii)) is equivalent to the condition that $0 \notin \sigma_\delta(\Psi)$ (resp. $0 \notin \sigma_\pi(\Psi)$). □

In [5, Remark 2], we presented the following two questions. Can the inclusion (3.4) be replaced by "="? If $\mathfrak{X}$ is a Hilbert space, can the inclusion (3.5) be replaced by "="? For an arbitrary elementary operator $\Phi_{\mathbf{A},\mathbf{B}}$, C. Davis and P. Rosenthal showed the next result.

Theorem DR ([3, Theorem 3]).

$$\sigma_\pi(\Phi_{\mathbf{A},\mathbf{B}}) \subseteq \left\{ \sum_{j=1}^{n} \alpha_j\beta_j \;\middle|\; \alpha_j \in \sigma_\pi(A_j), \beta_j \in \sigma_\delta(B_j) \right\}. \tag{3.8}$$

Moreover, if $\mathfrak{X}$ is a Hilbert space,

$$\sigma_\delta(\Phi_{\mathbf{A},\mathbf{B}}) \subseteq \left\{ \sum_{j=1}^{n} \alpha_j\beta_j \;\middle|\; \alpha_j \in \sigma_\delta(A_j), \beta_j \in \sigma_\pi(B_j) \right\}. \tag{3.9}$$

By means of Theorem DR and Theorem 1.2, we can make a slight progress for getting the condition for the equalities of (3.4) and (3.5).

Theorem 3.4.

$$\sigma_\pi(\Psi) \subseteq \left\{ \sum_{j=1}^{n} f_j(\alpha_j)g_j(\beta_j) \;\middle|\; \alpha_j \in \sigma_\pi(A), \beta_j \in \sigma_\delta(B) \right\}. \tag{3.10}$$

Moreover, if $\mathfrak{X}$ is a Hilbert space,

$$\sigma_\delta(\Psi) \subseteq \left\{ \sum_{j=1}^{n} f_j(\alpha_j) g_j(\beta_j) \;\middle|\; \alpha_j \in \sigma_\delta(A), \beta_j \in \sigma_\pi(B) \right\}. \tag{3.11}$$

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BIBLIOGRAPHY

[1] M. D. Choi and C. Davis, *The spectral mapping theorem for joint approximate point spectrum*, Bull. Amer. Math. Soc., **80** (1974), pp. 317–321.

[2] I. Colojoară and C. Foiaş, *Theory of Generalized Spectral Operators*, Gordon and Breach, New York, 1968.

[3] C. Davis and P. Rosenthal, *Solving linear operator equations*, Can. J. Math., **26** (1974), pp. 1384–1389.

[4] S. Frunză, *A duality theorem for decomposable operators*, Rev. Roumaine Math. Pures Appl., **16** (1971), pp. 1055–1058.

[5] F. Kimura, *Some spectral properties of analytic elementary operators*, Nihonkai Math. J., **13** (2002), pp. 9–16.

[6] G. Lumer and M. Rosenblum, *Linear operator equations*, Proc. Amer. Math. Soc., **10** (1959), pp. 32–41.

[7] K. K. Oberai, *Spectral mapping theorem for essential spectra*, Rev. Roumaine Math. Pures Appl., **25** (1980), pp. 365–373.

[8] M. Radjabalipour, *Equivalence of decomposable and 2-decomposable operators*, Pacific J. Math., **77** (1978), pp. 243–247.

[9] V. Rakočević, *On one subset of M. Schechter's essential spectrum*, Mat. Vesnik, **5** (1981), pp. 389–391.

[10] ———, *Approximate point spectrum and commuting compact perturbations*, Glasgow Math. J., **28** (1986), pp. 193–198.

[11] W. Rudin, *Real and Complex Analysis* (3rd ed.), McGraw-Hill, New York, 1987.

[12] C. Schmoeger, *The spectral mapping theorem for the essential approximate point spectrum*, Colloq. Math., **74** (1997), pp. 167–176.

[13] A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis* (2nd ed.), Robert E. Krieger Publishing Co., Inc., Merbourne, FL, 1986.

MATHEMATICAL INSTITUTE, TOHOKU UNIVERSITY, SENDAI 980-8578, JAPAN
E-mail address: s98m10@math.tohoku.ac.jp

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