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# Growth of transcendental entire solution of some $q$-difference equation

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**Abstract**

We consider a linear $q$-difference equation $qzf(qz)+(1-Az)f(z)=1$, with $q=e^{2\pi i\beta}, \beta\in(0,1)\setminus\mathbb{Q}$ and $A=e^{2\pi i\alpha}, \alpha\in(0,1)$. The equation is known to admit a transcendental entire solution $f(z)$ for suitably chosen $\beta$ and $\alpha$. We will show here that $f(z)$ is of positive order for some $\beta$, contrary to $q$-difference equations with $|q|\neq 0,1$.

*Keywords and phrases:* linear $q$-difference equation, growth of entire function.

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## 1 Introduction

We consider here a $q$-difference equation

$$b_p(z)f(q^pz)+\cdots+b_0(z)f(z)=\mathfrak{b}(z),\quad b_j(z),\ \mathfrak{b}(z)\in\mathbb{C}[z], \tag{1.1}$$

with $b_j(z)=\sum_{k=0}^{B_j}b_k^{(j)}z^k$ $(b_{B_j}^{(j)}\neq 0), 0\leq j\leq p$.

When $|q|\neq 0,1$, a transcendental entire solution $f(z)$ of (1.1) are of order 0. In fact, when $0<|q|<1$, it satisfies

$$\log M(r,f)=\frac{\sigma}{-2\log|q|}(\log r)^2(1+o(1)),\quad r\to\infty,$$

in which $\sigma$ is a slope of the Newton diagram for (1.1) [1].

When $|q|=1$, that is $q=e^{2\pi i\lambda}$, there is no such regularity. For example, when $q=-1, \lambda=1/2$, the equation $f(-z)-f(z)=0$ has solutions of behaviors of several type. We ask here what can be said for the case that

$$q=e^{2\pi i\beta},\quad \beta\in(0,1)\setminus\mathbb{Q}. \tag{1.2}$$

Driver et al. [3] showed that there exist $(q,A)$, with $q$ in (1.2) and $A, |A|=1$, such that the equation

$$qzf(qz)+(1-Az)f(z)=1 \tag{1.3}$$

has a transcendental entire solution. We will show here the following theorem, contrary to the case $|q|\neq 0,1$ :

**Theorem 1.1** *The solution $f(z)$ of (1.3) is of positive order, supposed $\beta$ in (1.2) is suitably chosen, as shown at the end of the proof.*

## 2 Proof of Theorem 1.1

We denote by $\{a\}$ the fractional part of $a$.

The solution $f(z)$ of (1.3) is written as

$$f(z)=\sum_{n=0}^{\infty}\alpha_n z^n, \quad \alpha_n=\prod_{k=1}^{n}(A-q^k), \tag{2.1}$$

in which we write $A=e^{2\pi i\alpha}, q=e^{2\pi i\beta}$. Then

$$|A-q^k|=2|\sin[\pi(\alpha-k\beta)]|, \quad |\alpha_n|=2^n\prod_{k=1}^{n}|\sin[\pi(\alpha-k\beta)]|. \tag{2.2}$$

Write

$$\beta=\frac{1|}{|a_1}+\frac{1|}{|a_2}+\frac{1|}{|a_3}+\cdots$$

and define the $k$-th convergent $p_k/q_k$ by $p_0=0, p_1=1, q_0=1, q_1=a_1$, and

$$p_k=a_k p_{k-1}+p_{k-2}, \quad q_k=a_k q_{k-1}+q_{k-2}, \quad k\geq 2.$$

Put

$$s_j=\sum_{k=0}^{j}q_k.$$

By [3] p.476,

$$\{s_j\beta\}=\beta+\sum_{k=1}^{j}(q_k\beta-p_k)\in(0,\beta).$$

By [3] p.474,

$$(-1)^k(q_k\beta-p_k)=|q_k\beta-p_k|\leq 1/q_{k+1}, \tag{2.3}$$

and $|q_k\beta-p_k|\downarrow 0$. We put

$$\alpha=\beta+\sum_{k=1}^{\infty}(q_k\beta-p_k)=\lim_{j\to\infty}\{s_j\beta\}. \tag{2.4}$$

We have by [4] p.24 Theorem 13,

$$|q_k\beta-p_k|>1/(q_{k+1}+q_k). \tag{2.5}$$

Further, we have by [3] p.475, for $1\leq j\leq q_k-1$, with $k$ even,

$$\begin{gathered}\{j\beta\}=\{j(\beta-p_k/q_k)\}+\{jp_k/q_k\}\\ \{\{jp_k/q_k\};\ 1\leq j\leq q_k-1\}=\{j/q_k;\ 1\leq j\leq q_k-1\},\end{gathered} \tag{2.6}$$

since $(p_k, q_k) = 1$. We assume that, following [3] pp.474, 471,

$$\lim_{j\to\infty} (\log q_{j+1})/q_j = \infty, \quad \text{hence} \quad \lim_{j\to\infty} (\log q_{j+1})/s_j = \infty. \tag{2.7}$$

Take $n = s_j - 1$ with $j$ even. For $1 \le k \le n$,

$$\alpha - k\beta = (\alpha - s_j\beta) + (s_j - k)\beta = \sum_{\ell=j+1}^{\infty} (q_\ell\beta - p_\ell) - \sum_{\ell=0}^{j} p_\ell + q_j\beta + (s_{j-1} - k)\beta.$$

By (2.3) and (2.5),

$$\left|\sum_{\ell=j+1}^{\infty} (q_\ell\beta - p_\ell)\right| \le |q_{j+1}\beta - p_{j+1}| \le \frac{1}{q_{j+2}},$$

$$1/(q_{j+1} + q_j) \le \{q_j\beta\} = q_j\beta - p_j \le 1/q_{j+1},$$

further, by (2.3) and (2.6), for $1 \le k \le s_{j-1} - 1$, noting that $s_{j-1} - k < q_j - 1$ by (2.7),

$$\{(s_{j-1} - k)\beta\} \ge \{(s_{j-1} - k)p_j/q_j\} - |(s_{j-1} - k)(\beta - p_j/q_j)| \ge 1/q_j - 1/q_{j+1}.$$

Thus for $k, 1 \le k \le s_{j-1} - 1$,

$$\{|\alpha - k\beta|\} \ge 1/q_j - 2/q_{j+1} - 1/q_{j+2} > 1/(2q_j). \tag{2.8}$$

For $k = s_{j-1}$, by (2.5)

$$\{|\alpha - k\beta|\} > 1/(q_{j+1} + q_j) - 1/q_{j+2} > 1/(2q_{j+1}). \tag{2.9}$$

For $k, s_{j-1} < k < s_j$, we get $1 \le s_j - k < s_j - s_{j-1} = q_j$, and

$$\{(s_j - k)\beta\} = \{(s_j - k)(\beta - p_j/q_j)\} + \{(s_j - k)p_j/q_j\} \ge -1/q_{j+1} + 1/q_j,$$

hence

$$\{|\alpha - k\beta|\} \ge 1/q_j - 1/q_{j+1} - 1/q_{j+2} > 1/(2q_j). \tag{2.10}$$

Noting $\sin x > (2/\pi)x, 0 < x < \pi/2$, we obtain by (2.2), (2.8), (2.9), (2.10),

$$|\alpha_n| \ge 2^n(2/\pi)^n(\pi/(2q_j))^{n-1}(\pi/(2q_{j+1})) = 2^n q_j^{-n+1} q_{j+1}^{-1}.$$

Noting that $\log n = \log(s_j - 1) = \log q_j + \log(1 + \frac{s_{j-1}-1}{q_j})$, we get, writing $n_j = s_j - 1$,

$$\liminf_{j\to\infty} \frac{\log(1/|\alpha_{n_j}|)}{n_j \log n_j} \le \lim_{j\to\infty} \frac{(n_j - 1)}{n_j}\frac{\log q_j}{\log n_j} + \liminf_{j\to\infty} \frac{\log q_{j+1}}{(s_j - 1)\log(s_j - 1)}$$
$$= 1 + \liminf_{j\to\infty} \frac{\log q_{j+1}}{(s_j - 1)\log(s_j - 1)}.$$

Suppose $q_j$ are chosen so that, together with (2.7), the following holds:

$$\liminf_{j\to\infty} \frac{\log q_{j+1}}{(s_j - 1)\log(s_j - 1)} = \liminf_{j\to\infty} \frac{\log q_{j+1}}{q_j \log q_j} = \gamma < \infty, \tag{2.11}$$

then we have, by [2] p.9, that the order $\rho(f)$ of $f(z)$ satisfies

$$\rho(f) \geq 1/(1+\gamma).$$

The condition (2.11) can be satisfied by assigning the partial quotients $a_j$ in the continued fraction suitably.

## References


[1] W. Bergweiler, K. Ishizaki, and N. Yanagihara: *Growth of meromorphic solutions of some functional equations* I. Aeq. Math. 63 (2002), 140–151.

[2] R.P. Boas: *Entire Functions.* Academic Press, New York 1954.

[3] K.A. Driver, D.S. Lubinsky, G. Petruska, and P. Sarnak: *Irregular distribution of* $\{n\beta\}, n = 1, 2, 3, \ldots$ , *quadrature of singular integrands, and curious basic hypergeometric series.* Indag. Math., N.S., 2 (4) (1991), 469–481.

[4] A.Ya. Khinchin: *Continued Fractions.* 3rd Ed. Moskwa 1961. (in Russian)



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