Nihonkai Math. J.
Vol.13(2002), 167-177

THE CONDITIONS THAT THE TOEPLITZ OPERATOR IS NORMAL OR ANALYTIC

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Abstract. P. R. Halmos [6; Problem 5] asked whether every subnormal Toeplitz operators on H^2 was either analytic or normal. A negative example was given by C. C. Cowen and J. J. Long [5; Theorem]. In this paper, we shall give the conditions that the Toeplitz operator T_φ is normal or analytic and show, as their applications, the following results: (1) If T_φ is hyponormal with $\mathcal{N}_{T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*} = \{f \in H^2 \ : \ (T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)f = o\}$ as its invariant subspace and if $\mathcal{N}_{H_\varphi} \cup \mathcal{N}_{H_{\bar\varphi}} \neq \{o\}$, then T_φ is normal or analytic ([1; Theorem]) and (2) Every quasi-normal Toeplitz operator is only normal or a scalar multiple of an isometry ([2; Theorem]).

1. Preliminaries. A bounded measurable function $\varphi \in L^\infty$ on the circle induces the multiplication operator on L^2 called the **Laurent operator** L_φ given by $L_\varphi f = \varphi f$ for $f \in L^2$. And the Laurent operator induces in a natural way twin operators on H^2 called **Toeplitz operator** T_φ given by $T_\varphi f = PL_\varphi f$ for $f \in H^2$, where P is the orthogonal projection from L^2 onto H^2 and **Hankel operator** H_φ given by $H_\varphi f = J(I-P)L_\varphi f$ for $f \in H^2$, where J is the unitary operator on L^2 defined by $J(z^{-n}) = z^{n-1}$, $n = 0, \pm1, \pm2, \cdots$. The following results are well known.

Proposition 1. ([3; Theorem IV]) If $\mathcal{M}$ is a non-zero invariant subspace of T_z, then there exists an isometric Toeplitz operator T_g uniquely, up to a unimodular constant, such that $\mathcal{M} = T_g H^2$.

Mathematics Subject Classification 2000 : 47B35
Keywords : Toeplitz operators

Proposition 2. ([4; Theorems 6 and 7]) $A \in \mathcal{B}(H^2)$ is a Toeplitz operator if and only if $T_z{}^*AT_z = A$. And, in particular, $A \in \mathcal{B}(H^2)$ is an **analytic** Toeplitz operator (i.e., $A = T_\varphi$ for some $\varphi \in H^\infty$) if and only if $T_zA = AT_z$.

Proposition 3. ([4; Theorem 8]) $T_\varphi T_\psi$ is a Toeplitz operator if and only if $\bar{\varphi}$ or $\psi \in H^\infty$, where the bar denotes the complex conjugate. And, in this case, $T_\varphi T_\psi = T_{\varphi\psi}$. In particular, T_φ is an analytic Toeplitz operator or a co-analytic Toeplitz operator if and only if $T_\varphi{}^2$ is a Toeplitz operator.

Proposition 4. ([7; Theorem 7, Corollary 6]) If φ is a non-constant function in L^∞, then $\sigma_p(T_\varphi) \cap \overline{\sigma_p(T_\varphi{}^*)} = \emptyset$ where $\sigma_p(T_\varphi)$ denotes the point spectrum of T_φ. And, as a special case, for a non-constant function φ in L^∞, if T_φ is **hyponormal** (i.e., $T_\varphi{}^*T_\varphi \geq T_\varphi T_\varphi{}^*$), then $\sigma_p(T_\varphi) = \emptyset$.

Proposition 5. If φ and ψ are in H^∞, then $T_\varphi H^2 \subseteq T_\psi H^2$ if and only if there exists a $g \in H^\infty$ uniquely such that $T_\varphi = T_\psi T_g = T_{\psi g}$. And then $\varphi = \psi g$. Particularly, if φ and ψ are inner, then g is also inner.

Proposition 6. ([7; Theorem 5]) For a T_φ such as $\|T_\varphi\| = 1$, if $\{f \in H^2 \ : \ \|T_\varphi{}^n f\|_2 = \|f\|_2,\ n = 0, 1, 2, \cdots\} \neq \{o\}$, then T_φ is an isometry.

Proposition 7. ([8; Theorems 3 and 1, Corollary 2]) $H_\psi{}^*H_\varphi = T_{\bar{\psi}\varphi} - T_{\bar{\psi}}T_\varphi$ and, in particular, we have $H_{\bar{\varphi}}{}^*H_{\bar{\varphi}} - H_\varphi{}^*H_\varphi = T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*$. For any $\psi \in H^\infty$, $H_\varphi T_\psi = H_{\varphi\psi}$ and $T_\psi{}^*H_\varphi = H_\varphi T_{\psi^*} = H_{\varphi\psi^*}$ where $\psi^*(z) = \overline{\psi(\bar{z})}$.

Proposition 8. ([8; Theorem 2]) The following assertions are equivalent.

(1) $\mathcal{N}_{H_\varphi} \stackrel{\text{def}}{=} \{f \in H^2 \ : \ H_\varphi f = o\} \neq \{o\}$

(2) $[H_\varphi H^2]^{\sim L^2} \neq H^2$

(3) $\varphi = \bar{g}h$ for some inner function g and $h \in H^\infty$ such that g and h have no common non-constant inner factor and that $\mathcal{N}_{H_\varphi} = T_g H^2$.

Proposition 9. ([8; Corollary 3]) $H_\varphi H_\psi = O$ if and only if $H_\varphi = O$ or $H_\psi = O$. In particular, there is no non-zero nilpotent Hankel operator.

2. Main results. A function in L^∞ is said to be of **bounded type** or **in the Nevanlina class** if it can be written as the quotient of two functions in H^∞. The Nevanlina class function is characterized as follows.

Proposition 10. ([1; Lemma 3]) $\varphi \in L^\infty$ is of bounded type if and only if $\mathcal{N}_{H_\varphi} \neq \{o\}$.

Proof. ($\leftarrow$) It is clear by Proposition 8.

($\rightarrow$) Let $\varphi = \frac{u}{f}$ for some u and f in H^∞. Then $H_\varphi f = J(I-P)\varphi f = J(I-P)u = o$ and $\mathcal{N}_{H_\varphi} \neq \{o\}$. □

Lemma 1. ([1; Lemma 8]) If $\mathcal{N}_{H_\varphi} \neq \{o\}$, then $\vee\{\mathcal{N}_{H_\varphi},\ T_\varphi \mathcal{N}_{H_\varphi}\} = H^2$.

Proof. If $\mathcal{N}_{H_\varphi} \neq \{o\}$, then, by Proposition 8, $\varphi = \bar{g}h$ for some inner function g and $h \in H^\infty$ such that g and h have no commom non-constant inner factor and that $\mathcal{N}_{H_\varphi} = T_g H^2$. Then $T_\varphi \mathcal{N}_{H_\varphi} = T_g{}^* T_h T_g H^2 = T_h H^2$ and $\{o\} \neq \vee\{\mathcal{N}_{H_\varphi},\ T_\varphi \mathcal{N}_{H_\varphi}\} = \vee\{T_g H^2,\ T_h H^2\}$ is invariant under T_z and hence, by Proposition 1, there exists an inner function q uniquely, up to a unimodular constant, such that $\vee\{T_g H^2,\ T_h H^2\} = T_q H^2$. Hence $T_g H^2 \cup T_h H^2 \subseteq T_q H^2$ and, by Proposition 5, there exist u and v in H^∞ such that $g = qu$ and $h = qv$. Since g and h have no common non-constant inner factor, q is constant and we have the conclusion. □

Lemma 2. ([1; Lemma 10]) For inner functions g and q, if $T_g H^2 \subseteq \mathcal{N}_{H_{\bar\varphi}{}^* H_{\bar\varphi} - H_\varphi{}^* H_\varphi}$ and if $T_q H^2 \subseteq \mathcal{N}_{H_\varphi}$, then either $T_g H^2$ or $T_q H^2$ is contained in $\mathcal{N}_{H_{\bar\varphi}}$.

Proof. For any u and v in H^2, we have, by Proposition 7,

$$\begin{aligned} 0 &= \langle T_q u,\ (H_{\bar\varphi}{}^* H_{\bar\varphi} - H_\varphi{}^* H_\varphi) T_g v\rangle = \langle H_{\bar\varphi} T_q u,\ H_{\bar\varphi} T_g v\rangle - \langle H_\varphi T_q u,\ H_\varphi T_g v\rangle \\ &= \langle H_{\bar\varphi} T_q u,\ H_{\bar\varphi} T_g v\rangle = \langle H_{\bar\varphi q} u,\ H_{\bar\varphi g} v\rangle = \langle H_{\bar\varphi g}{}^* H_{\bar\varphi q} u,\ v\rangle \end{aligned}$$

and $H_{\bar\varphi g}{}^* H_{\bar\varphi q} = O$ and hence, by Proposition 9, $H_{\bar\varphi g} = O$ or $H_{\bar\varphi q} = O$. Therefore $T_g H^2 \subseteq \mathcal{N}_{H_{\bar\varphi}}$ or $T_q H^2 \subseteq \mathcal{N}_{H_{\bar\varphi}}$. □

Lemma 3. For any $\varphi \notin H^\infty$, T_φ has no such type of invariant subspace as $T_g H^2$ for some non-constant inner function g.

Proof. If $T_\varphi T_g H^2 \subseteq T_g H^2$ for some non-constant inner function g, then there exists a $C \in \mathcal{B}(H^2)$ such that $T_{\varphi g} = T_g C$ because $T_\varphi T_g = T_{\varphi g}$ by Proposition 3. Since g is inner, $C = T_g{}^* T_{\varphi g} = T_{\bar g} T_{\varphi g} = T_{\bar g \varphi g} = T_\varphi$ and $T_{\varphi g} = T_g T_\varphi$ and hence $\varphi \in H^\infty$ by Proposition 3 because $\bar g \notin H^\infty$. □

Theorem 1. If $\{o\} \neq \mathcal{N}_{H_{\bar{\varphi}}} \subseteq \mathcal{N}_{H_\varphi} \cap \mathcal{N}_{H_{\bar{\varphi}}{}^* H_{\bar{\varphi}} - H_\varphi{}^* H_\varphi}$ and if $\mathcal{N}_{H_{\bar{\varphi}}{}^* H_{\bar{\varphi}} - H_\varphi{}^* H_\varphi}$ is invariant under T_φ, then T_φ is normal or analytic.

Proof. Since $\{o\} \neq \mathcal{N}_{H_{\bar{\varphi}}} \subseteq \mathcal{N}_{H_\varphi}$, we have, by Proposition 8, $\varphi = \bar{g}h$, $\bar{\varphi} = \bar{q}k$ for some inner functions g and q and some $h,\ k \in H^\infty$ such that each pair $(g,\ h)$ and $(q,\ k)$ has no common non-constant inner factor and that $\mathcal{N}_{H_\varphi} = T_g H^2$, $\mathcal{N}_{H_{\bar{\varphi}}} = T_q H^2$. And then $\mathcal{N}_{H_{\bar{\varphi}}} \subseteq \mathcal{N}_{H_\varphi}$ implies that

$$T_q H^2 \subseteq T_g H^2 \tag{1}$$

and, by Proposition 5, there exists an inner function u uniquely, up to a unimodular constant, such that

$$q = gu. \tag{2}$$

Since $T_q H^2 = T_u T_g H^2$ by (2) and since, by Proposition 3, $T_\varphi T_q H^2 = T_{\bar{g}h} T_{gu} H^2 = T_h T_u H^2 = T_u T_{\bar{g}h} T_g H^2 = T_u T_\varphi T_g H^2$,

$$\vee\{T_q H^2,\ T_\varphi T_q H^2\} = T_u[\vee\{T_g H^2,\ T_\varphi T_g H^2\}] = T_u H^2 \tag{3}$$

by Lemma 1 and

$$T_u H^2 \subseteq \mathcal{N}_{H_{\bar{\varphi}}{}^* H_{\bar{\varphi}} - H_\varphi{}^* H_\varphi} \tag{4}$$

because $\vee\{T_q H^2,\ T_\varphi T_q H^2\} \subseteq \mathcal{N}_{H_{\bar{\varphi}}{}^* H_{\bar{\varphi}} - H_\varphi{}^* H_\varphi}$ by the assumption and hence, by Lemma 2, either $T_u H^2$ or $T_g H^2$ is contained in $\mathcal{N}_{H_{\bar{\varphi}}}$.

If $T_u H^2 \subseteq \mathcal{N}_{H_{\bar{\varphi}}} = T_q H^2$, then $T_\varphi T_q H^2 \subseteq T_q H^2$ by (3) and $\varphi \in H^\infty$ by Lemma 3.

If $T_g H^2 \subseteq \mathcal{N}_{H_{\bar{\varphi}}} = T_q H^2$, then $T_g H^2 = T_q H^2$ by (1) and u in (2) is a constant inner function and hence $\mathcal{N}_{H_{\bar{\varphi}}{}^* H_{\bar{\varphi}} - H_\varphi{}^* H_\varphi} = H^2$ by (4). Therefore T_φ is normal because $H_{\bar{\varphi}}{}^* H_{\bar{\varphi}} - H_\varphi{}^* H_\varphi = T_\varphi{}^* T_\varphi - T_\varphi T_\varphi{}^*$ by Proposition 7. □

Since, for any $f \in H^2$, $\|H_{\bar{\varphi}} f\|_2{}^2 = \|H_\varphi f\|_2{}^2 + \langle (H_{\bar{\varphi}}{}^* H_{\bar{\varphi}} - H_\varphi{}^* H_\varphi) f,\ f\rangle$, any two intersection of the following three sets $\mathcal{N}_{H_\varphi}$, $\mathcal{N}_{H_{\bar{\varphi}}}$ and $\mathcal{N}_{H_{\bar{\varphi}}{}^* H_{\bar{\varphi}} - H_\varphi{}^* H_\varphi}$ is contained in the rest set. Hence the condition $\mathcal{N}_{H_{\bar{\varphi}}} \subseteq \mathcal{N}_{H_\varphi} \cap \mathcal{N}_{H_{\bar{\varphi}}{}^* H_{\bar{\varphi}} - H_\varphi{}^* H_\varphi}$ in Theorem 1 is equivalent to $\mathcal{N}_{H_{\bar{\varphi}}} = \mathcal{N}_{H_\varphi} \cap \mathcal{N}_{H_{\bar{\varphi}}{}^* H_{\bar{\varphi}} - H_\varphi{}^* H_\varphi}$. And if T_φ is hyponormal, then $\langle (H_{\bar{\varphi}}{}^* H_{\bar{\varphi}} - H_\varphi{}^* H_\varphi) f,\ f\rangle = \|(T_\varphi{}^* T_\varphi - T_\varphi T_\varphi{}^*)^{\frac{1}{2}} f\|_2{}^2$ by Proposition 7 and we have easily the following.

Lemma 4. ([1; Lemma 2]) If T_φ is hyponormal, then $\mathcal{N}_{H_{\bar\varphi}} = \mathcal{N}_{H_\varphi} \cap \mathcal{N}_{H_{\bar\varphi}{}^* H_{\bar\varphi} - H_\varphi{}^* H_\varphi}$.

Lemma 5. For any $\varphi \in L^\infty$ such as $\mathcal{N}_{H_\varphi} = \{o\}$ and for any inner function g, $\mathcal{N}_{H_{\varphi\bar g}} = \{o\}$ and $\mathcal{N}_{H_{\varphi g}} = \{o\}$.

Proof. By Proposition 7, $H_\varphi H^2 = H_{\varphi\bar g g} H^2 = H_{\varphi \bar g} T_g H^2 \subseteq H_{\varphi\bar g} H^2$ and $T_{g^*}{}^*[H_\varphi H^2]^{\sim L^2} \subseteq [T_{g^*}{}^* H_\varphi H^2]^{\sim L^2} = [H_{\varphi g} H^2]^{\sim L^2}$ and hence we have the conclusion by Proposition 8 because g^* is also inner. □

If T_φ is hyponormal, then, by Lemma 4, $\mathcal{N}_{H_{\bar\varphi}} \subseteq \mathcal{N}_{H_\varphi}$. Moreover, if $\varphi \notin H^\infty$, then we have the following.

Lemma 6. ([1; Lemma 6]) If T_φ is hyponormal and if $\varphi \notin H^\infty$, then $\mathcal{N}_{H_{\bar\varphi}} \neq \{o\} \rightleftharpoons \mathcal{N}_{H_\varphi} \neq \{o\}$.

Proof. By the above inclusion, we may show that $\mathcal{N}_{H_\varphi} \neq \{o\}$ implies $\mathcal{N}_{H_{\bar\varphi}} \neq \{o\}$. Then, by Proposition 8, $\varphi = \bar g h$ for some inner function g and some $h \in H^\infty$ such that g and h have no common non-constant inner factor and that $\mathcal{N}_{H_\varphi} = T_g H^2$. Furthermore, since $\varphi \notin H^\infty$, g is not constant. Therefore there is a non-zero vector $u \in H^2$ such that $\langle u,\ T_g H^2\rangle = 0$. Let $\mathcal{M} = H_{\bar\varphi} T_g H^2$ and let $y = H_{\bar\varphi} u$.

If $\mathcal{N}_{H_{\bar\varphi}} = \{o\}$, then $\mathcal{N}_{H_{\bar\varphi g}} = \{o\}$ by Lemma 5 and $\mathcal{M} = H_{\bar\varphi g} H^2$ is dense in H^2 by Proposition 8. And $H_\varphi u \neq o$ because u is orthogonal to $T_g H^2 = \mathcal{N}_{H_\varphi}$.

Now we need the following :

Claim. If $\mathcal{M}$ is a dense linear manifold of a non-zero Hilbert space $\mathcal{H}$ and if $y \in \mathcal{H}$, then $(0,\ \infty) \subseteq \{\|y + x\| \ :\ o \neq x \in \mathcal{M}\}$.

(In fact, for $\epsilon > 0$, find $o \neq x \in \mathcal{M}$ such that $\|y + x\| \leq \epsilon$. The function $\alpha\ :\ [1,\ \infty) \to \mathbb{R}$ defined by $\alpha(t) = \|y + tx\|$ is continuous and $\lim_{t\to\infty} \alpha(t) = \infty$. It follows that $[\epsilon,\ \infty) \subseteq \alpha([1,\ \infty)) \subseteq \{\|y + x\|\ :\ o \neq x \in \mathcal{M}\}$.)

It follows from Claim that there is a non-zero $u_1 \in H^2$ such that $\|H_{\bar\varphi}(u + T_g u_1)\|_2 = \|y + H_{\bar\varphi} T_g u_1\|_2 = \|H_\varphi u\|_2$. Let $v_1 = u + T_g u_1$. Since $\mathcal{N}_{H_\varphi} = T_g H^2$, $\|H_\varphi v_1\|_2 = \|H_\varphi u\|_2 = \|H_{\bar\varphi}(u + T_g u_1)\|_2 = \|H_{\bar\varphi} v_1\|_2$ and $v_1 \in \mathcal{N}_{H_{\bar\varphi}{}^* H_{\bar\varphi} - H_\varphi{}^* H_\varphi}$. Since $o \neq u_1 \in H^2$, there exists a positive integer n such that $\langle u_1,\ z^{n-1}\rangle \neq 0$. Let $\mathcal{M}_1 = H_{\bar\varphi} T_{gz^n} H^2$. Then $\mathcal{M}_1$ is dense in H^2 by

Proposition 8 because $\mathcal{N}_{H_{\bar{\varphi}gz^n}} = \{o\}$ by Lemma 5 and, by Claim, there is a non-zero $u_2 \in H^2$ such that $\|H_{\bar{\varphi}}(u + T_{gz^n}u_2)\|_2 = \|y + H_{\bar{\varphi}}T_{gz^n}u_2\|_2 = \|H_{\varphi}u\|_2$. Let $v_2 = u + T_{gz^n}u_2$. Then, by the same reason as above, $\|H_{\varphi}v_2\|_2 = \|H_{\bar{\varphi}}v_2\|_2$ and $v_2 \in \mathcal{N}_{H_{\bar{\varphi}}^*H_{\bar{\varphi}}-H_{\varphi}^*H_{\varphi}}$. Thus $v_1 - v_2$ belongs to both $T_gH^2 = \mathcal{N}_{H_{\varphi}}$ and $\mathcal{N}_{H_{\bar{\varphi}}^*H_{\bar{\varphi}}-H_{\varphi}^*H_{\varphi}}$ and is non-zero by the following reason.

If $o = v_1 - v_2 = T_g(u_1 - T_z{}^n u_2)$, then $u_1 = T_z{}^n u_2$ and $\langle u_1,\ z^{n-1}\rangle = \langle T_z{}^n u_2,\ z^{n-1}\rangle = \langle u_2,\ T_z{}^*1\rangle = 0$ which is a contradiction.

Therefore $o \neq v_1 - v_2 \in \mathcal{N}_{H_{\bar{\varphi}}}$ by Lemma 4. This contradicts the assumption that $\mathcal{N}_{H_{\bar{\varphi}}} = \{o\}$. □

Corollary 1. ([1; Theorem]) If T_φ is hyponormal with $\mathcal{N}_{T_\varphi^*T_\varphi - T_\varphi T_\varphi^*}$ as its invariant subspace and if $\mathcal{N}_{H_\varphi} \cup \mathcal{N}_{H_{\bar{\varphi}}} \neq \{o\}$, then T_φ is normal or analytic.

Proof. By Lemma 4, $\mathcal{N}_{H_{\bar{\varphi}}} = \mathcal{N}_{H_\varphi} \cap \mathcal{N}_{H_{\bar{\varphi}}^*H_{\bar{\varphi}}-H_{\varphi}^*H_{\varphi}}$. Moreover, in the case where $\varphi \notin H^\infty$, $\mathcal{N}_{H_{\bar{\varphi}}} \neq \{o\}$ by Lemma 6. Since, by Proposition 7, $H_{\bar{\varphi}}{}^*H_{\bar{\varphi}} - H_\varphi{}^*H_\varphi = T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*$, the conclusion follows from Theorem 1. □

It is clear that every **subnormal operator** A on $\mathcal{H}$ (i.e., A has a normal extension N on $\mathcal{K} \supseteq \mathcal{H}$) has $\mathcal{N}_{A^*A-AA^*}$ as its invariant subspace. In fact, let Q be the projection from $\mathcal{K}$ on $\mathcal{H}$. Then, for each $x \in \mathcal{N}_{A^*A-AA^*}$, $\|Nx\| = \|Ax\| = \|A^*x\| = \|QN^*x\| \leq \|N^*x\| = \|Nx\|$ and $QN^*x = N^*x$ and hence $\|A^*Ax\| = \|AA^*x\| = \|NQN^*x\| = \|NN^*x\| = \|N^*Nx\| = \|N^2x\| = \|A^2x\|$. This implies that $\mathcal{N}_{A^*A-AA^*}$ is invariant under A and we have the following.

Corollary 2. ([1; Corollary A]) If T_φ is subnormal and if $\mathcal{N}_{H_\varphi} \cup \mathcal{N}_{H_{\bar{\varphi}}} \neq \{o\}$, then T_φ is normal or analytic.

Lemma 7. A is **quasi-normal** (i.e., A commutes with A^*A) if and only if A is hyponormal and $(A^*A)^2 = A^{*2}A^2$.

Proof. If A is hyponormal and if $(A^*A)^2 = A^{*2}A^2$, then $A^*(A^*A - AA^*)A = O$ and, by the hyponormality, $(A^*A - AA^*)^{\frac{1}{2}}A = O$ and hence $(A^*A - AA^*)A = O$. Therefore A commutes with A^*A. The converse assertion is clear. □

Lemma 8. For $\varphi \in H^\infty$, if $(T_\varphi{}^*T_\varphi)^2 = T_\varphi{}^{*2}T_\varphi{}^2$, then φ is a scalar multiple of an inner function.

Proof. By Proposition 3 and by the assumption,
$T_{\bar{\varphi}\varphi}{}^2 = (T_\varphi{}^*T_\varphi)^2 = T_\varphi{}^{*2}T_\varphi{}^2 = T_{\bar{\varphi}^2}T_{\varphi^2} = T_{\bar{\varphi}^2\varphi^2} = T_{|\varphi|^4}$ and $\bar{\varphi}\varphi \in H^\infty$ and hence $|\varphi|$ is constant. Therefore φ is a scalar multiple of an inner function. □

For $\varphi \in L^\infty$, let $X_\varphi = T_\varphi T_z - T_z T_\varphi$ and let $Y_\varphi = T_z{}^*T_\varphi{}^*T_\varphi T_z - T_\varphi{}^*T_\varphi$.

Then $X_\varphi = O \rightleftharpoons \varphi \in H^\infty$ by Proposition 2,

$Y_\varphi = O \rightleftharpoons T_\varphi{}^*T_\varphi$ is a Toeplitz operator by Proposition 2

$\rightleftharpoons \varphi \in H^\infty$ by Proposition 3,

and $Y_\varphi = T_z{}^*T_\varphi{}^*(T_zT_\varphi + X_\varphi) - T_\varphi{}^*T_\varphi = T_z{}^*T_\varphi{}^*X_\varphi$.

Since $Y_\varphi = T_z{}^*T_\varphi{}^*(I - T_zT_z{}^*)T_\varphi T_z$ and $(I - T_zT_z{}^*)H^2 = \vee\{1\}$, Y_φ is an at most rank one positive operator and $Y_\varphi T_z{}^*T_\varphi{}^*1 = \|Y_\varphi\|T_z{}^*T_\varphi{}^*1$. And since, for any $f \in H^2$, $\|X_\varphi f\|_2{}^2 = \|(I - T_zT_z{}^*)T_\varphi T_z f\|_2{}^2 = \langle Y_\varphi f,\ f\rangle = \|Y_\varphi{}^{\frac{1}{2}} f\|_2{}^2$, we have $\mathcal{N}_{X_\varphi} = \mathcal{N}_{Y_\varphi}$ and $X_\varphi{}^*H^2 = Y_\varphi H^2 = \vee\{T_z{}^*T_\varphi{}^*1\}$ and hence

$$\begin{aligned} H^2 &= \{f \in H^2 :\ Y_\varphi f = o\} \oplus \{f \in H^2 :\ Y_\varphi f = \|Y_\varphi\| f\} \\ &= \mathcal{N}_{X_\varphi} \oplus \vee\{T_z{}^*T_\varphi{}^*1\} \end{aligned} \qquad (\sharp)$$

and also we have $X_\varphi H^2 \subseteq \mathcal{N}_{T_z{}^*} = \vee\{1\}$.

Lemma 9. If $\{o\} \neq \mathcal{N}_{T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*} \neq H^2$, then $Y_\varphi - Y_{\bar{\varphi}} \neq O$ and $(Y_\varphi - Y_{\bar{\varphi}})H^2 = \vee\{T_z{}^*T_\varphi{}^*1,\ T_z{}^*T_\varphi 1\}$.

Proof. If $Y_\varphi - Y_{\bar{\varphi}} = O$, then $T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*$ is a Hermitian Toeplitz operator by Proposition 2 because $Y_\varphi - Y_{\bar{\varphi}} = T_z{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)T_z - (T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)$. Let $T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^* = T_\psi$. Then the assumption $\{o\} \neq \mathcal{N}_{T_\psi} \neq H^2$ implies $\psi \neq o$ and $0 \in \sigma_p(T_\psi)$. This contradicts Proposition 4. And since, for any $f \in H^2$,

$$\begin{aligned} (Y_\varphi - Y_{\bar{\varphi}})f = \langle f,\ T_z{}^*T_\varphi{}^*1\rangle &\left(\frac{\|Y_\varphi\|}{\|T_z{}^*T_\varphi{}^*1\|_2{}^2}\right) T_z{}^*T_\varphi{}^*1 \\ - \langle f,\ T_z{}^*T_\varphi 1\rangle &\left(\frac{\|Y_\varphi\|}{\|T_z{}^*T_\varphi 1\|_2{}^2}\right) T_z{}^*T_\varphi 1, \end{aligned}$$

we have $(Y_\varphi - Y_{\bar\varphi})H^2 = \vee\{T_z{}^*T_\varphi{}^*1,\ T_z{}^*T_\varphi 1\}$ because it is clear in the case where $T_z{}^*T_\varphi{}^*1$ and $T_z{}^*T_\varphi 1$ are linearly dependent and, in the other case, we can select $f \in H^2$ such as $\langle f,\ T_z{}^*T_\varphi{}^*1\rangle = 0 \neq \langle f,\ T_z{}^*T_\varphi 1\rangle$ and also $\langle f,\ T_z{}^*T_\varphi{}^*1\rangle \neq 0 = \langle f,\ T_z{}^*T_\varphi 1\rangle$. □

Theorem 2. If T_φ satisfies the following conditions :
(i) $(T_\varphi{}^*T_\varphi)^2 = T_\varphi{}^{*2}T_\varphi{}^2$, (ii) $\{o\} \neq \mathcal{N}_{T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*}$, (iii) Every eigen-space of $T_\varphi{}^*T_\varphi$ is invariant under $T_\varphi{}^*$ and (iv) $T_\varphi{}^*T_z{}^*T_\varphi{}^*1$ and $T_\varphi{}^*T_z{}^*T_\varphi 1$ are linearly dependent, then T_φ is normal or a scalar multiple of an isometry.

Proof. By Lemma 8, we have only to prove that there is no non-normal, non-analytic Toeplitz operator which satisfies the above conditions (i), (ii), (iii) and (iv). Let T_φ be non-normal and non-analytic. Since $T_\varphi{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)T_\varphi = O$ by the condition (i),

$$\begin{aligned}T_\varphi{}^*(Y_\varphi - Y_{\bar\varphi})T_\varphi &= T_\varphi{}^*T_z{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)T_zT_\varphi \\ &= (T_z{}^*T_\varphi{}^* - X_\varphi{}^*)(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)(T_\varphi T_z - X_\varphi) \\ &= -T_z{}^*T_\varphi{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)X_\varphi \\ &\qquad - X_\varphi{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)(T_\varphi T_z - X_\varphi) \qquad (1)\end{aligned}$$

and $T_z{}^*T_\varphi{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)X_\varphi H^2 \subseteq X_\varphi{}^*H^2 + T_\varphi{}^*(Y_\varphi - Y_{\bar\varphi})H^2$ and hence, by Lemma 9,

$$\begin{aligned}&T_z{}^*T_\varphi{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)1 \\ =&\alpha T_z{}^*T_\varphi{}^*1 + \beta T_\varphi{}^*T_z{}^*T_\varphi{}^*1 + \gamma T_\varphi{}^*T_z{}^*T_\varphi 1 \quad \text{for some} \quad \alpha, \beta,\ \gamma \in \mathbb{C} \qquad (2)\end{aligned}$$

because the conditions of Lemma 9 are satisfied by the condition (ii) and by the non-normality of T_φ. And since

$$\begin{aligned}&T_z{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)1 = (T_\varphi{}^*T_z{}^* + X_\varphi{}^*)T_\varphi 1 - (T_\varphi T_z{}^* + X_{\bar\varphi}{}^*)T_\varphi{}^*1 \\ =&T_\varphi{}^*T_z{}^*T_\varphi 1 + aT_z{}^*T_\varphi{}^*1 - T_\varphi T_z{}^*T_\varphi{}^*1 + bT_z{}^*T_\varphi 1 \quad \text{for some} \quad a,\ b \in \mathbb{C},\end{aligned}$$

$$\begin{aligned}&T_z{}^*T_\varphi{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)1 = (T_\varphi{}^*T_z{}^* + X_\varphi{}^*)(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)1 \\ =&T_\varphi{}^*T_z{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)1 + X_\varphi{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)1 \\ =&T_\varphi{}^*T_z{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)1 + cT_z{}^*T_\varphi{}^*1 \quad \text{for some} \quad c \in \mathbb{C} \\ =&T_\varphi{}^*(T_\varphi{}^*T_z{}^*T_\varphi 1 + aT_z{}^*T_\varphi{}^*1 - T_\varphi T_z{}^*T_\varphi{}^*1 + bT_z{}^*T_\varphi 1) + cT_z{}^*T_\varphi{}^*1 \\ =&T_\varphi{}^{*2}T_z{}^*T_\varphi 1 + aT_\varphi{}^*T_z{}^*T_\varphi{}^*1 - T_\varphi{}^*T_\varphi T_z{}^*T_\varphi{}^*1 + bT_\varphi{}^*T_z{}^*T_\varphi 1 + cT_z{}^*T_\varphi{}^*1\end{aligned}$$

and, by (2),

$$T_\varphi{}^*T_\varphi T_z{}^*T_\varphi{}^*1 = T_\varphi{}^{*2}T_z{}^*T_\varphi 1 + (c-\alpha)T_z{}^*T_\varphi{}^*1 + (a-\beta)T_\varphi{}^*T_z{}^*T_\varphi{}^*1 + (b-\gamma)T_\varphi{}^*T_z{}^*T_\varphi 1. \qquad (3)$$

Since, by (1),

$$T_\varphi{}^*(Y_\varphi - Y_{\bar\varphi})T_\varphi H^2 \subseteq T_z{}^*T_\varphi{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)X_\varphi H^2 + X_\varphi{}^*H^2 \qquad (4)$$

and since $T_\varphi{}^*(Y_\varphi - Y_{\bar\varphi})T_\varphi H^2 \subseteq T_\varphi{}^*(Y_\varphi - Y_{\bar\varphi})H^2$,

$$T_\varphi{}^*T_z{}^*T_\varphi{}^*1 = \lambda_1 T_z{}^*T_\varphi{}^*1$$
$$\text{and} \quad T_\varphi{}^*T_z{}^*T_\varphi 1 = \lambda_2 T_z{}^*T_\varphi{}^*1 \quad \text{for some} \quad \lambda_1,\ \lambda_2 \in \mathbb{C} \qquad (5)$$

by the condition (iv), Lemma 9 and (2) and hence, by (3),

$$T_\varphi{}^*T_\varphi(T_z{}^*T_\varphi{}^*1) = \{\lambda_2\lambda_1 + (c-\alpha) + (a-\beta)\lambda_1 + (b-\gamma)\lambda_2\}T_z{}^*T_\varphi{}^*1. \qquad (6)$$

Let $r = \lambda_2\lambda_1 + (c-\alpha) + (a-\beta)\lambda_1 + (b-\gamma)\lambda_2$ and let $\mathcal{M} = \{f \in H^2 \ :\ T_\varphi{}^*T_\varphi f = rf\}$. Since, for any $f \in \mathcal{M}$,

$$\begin{aligned}(T_\varphi{}^*T_\varphi - rI)T_z{}^*f =& T_\varphi{}^*(T_z{}^*T_\varphi - X_{\bar\varphi}{}^*)f - rT_z{}^*f \\ =& (T_z{}^*T_\varphi{}^* - X_\varphi{}^*)T_\varphi f - T_\varphi{}^*X_{\bar\varphi}{}^*f - rT_z{}^*f \\ =& - X_\varphi{}^*T_\varphi f - T_\varphi{}^*X_{\bar\varphi}{}^*f \\ =& - a_1T_z{}^*T_\varphi{}^*1 - T_\varphi{}^*(b_1T_z{}^*T_\varphi 1) \quad \text{for some} \quad a_1,\ b_1 \in \mathbb{C} \\ =& - (a_1 + b_1\lambda_2)T_z{}^*T_\varphi{}^*1 \quad \text{by (5)}\end{aligned}$$

and since $T_z{}^*T_\varphi{}^*1 \in \mathcal{M}$ by (6), $(T_\varphi{}^*T_\varphi - rI)^2T_z{}^*f = o$ and $(T_\varphi{}^*T_\varphi - rI)T_z{}^*f = o$ because $\|(T_\varphi{}^*T_\varphi - rI)T_z{}^*f\|_2{}^2 = \langle (T_\varphi{}^*T_\varphi - rI)^2T_z{}^*f,\ T_z{}^*f\rangle = 0$ and hence $\mathcal{M}$ is invariant under $T_z{}^*$. Since T_φ is non-analytic by the assumption, $T_z{}^*T_\varphi{}^*1 \neq o$ by (♯) and by Proposition 2 and $\mathcal{M} \neq H^2$ by Proposition 3 and hence $\mathcal{M}$ is non-trivial. Therefore $\mathcal{M}^\perp = T_gH^2$ for some non-constant inner function g by Proposition 1. Since $\mathcal{M}$ is invariant under $T_\varphi{}^*$ by the condition (iii), T_gH^2 is invariant under T_φ and $\varphi \in H^\infty$ by Lemma 3. This contradicts the assumption that T_φ is non-analytic. □

Corollary 3. ([2; Theorem]) Every quasi-normal Toeplitz operator is only normal or a scalar multiple of an isometry.

Proof. It is clear that every quasi-normal T_φ satisfies the conditions (i), (ii) and (iii). And, by Theorem 2, we have only to show that quasi-normal T_φ satisfies the condition (iv).

If $T_\varphi{}^*T_z{}^*T_\varphi{}^*1$ and $T_\varphi{}^*T_z{}^*T_\varphi 1$ are linearly independent, then $(Y_\varphi - Y_{\bar{\varphi}})T_\varphi H^2 = \vee\{T_z{}^*T_\varphi{}^*1,\ T_z{}^*T_\varphi 1\}$ because, for any $f \in H^2$,

$$\begin{aligned}(Y_\varphi - Y_{\bar{\varphi}})T_\varphi f &= \langle T_\varphi f,\ T_z{}^*T_\varphi{}^*1\rangle \left(\frac{\|Y_\varphi\|}{\|T_z{}^*T_\varphi{}^*1\|_2{}^2}\right) T_z{}^*T_\varphi{}^*1 \\ &\quad - \langle T_\varphi f,\ T_z{}^*T_\varphi 1\rangle \left(\frac{\|Y_{\bar{\varphi}}\|}{\|T_z{}^*T_\varphi 1\|_2{}^2}\right) T_z{}^*T_\varphi 1 \\ &= \langle f,\ T_\varphi{}^*T_z{}^*T_\varphi{}^*1\rangle \left(\frac{\|Y_\varphi\|}{\|T_z{}^*T_\varphi{}^*1\|_2{}^2}\right) T_z{}^*T_\varphi{}^*1 \\ &\quad - \langle f,\ T_\varphi{}^*T_z{}^*T_\varphi 1\rangle \left(\frac{\|Y_{\bar{\varphi}}\|}{\|T_z{}^*T_\varphi 1\|_2{}^2}\right) T_z{}^*T_\varphi 1.\end{aligned}$$

And since $T_\varphi{}^*(Y_\varphi - Y_{\bar{\varphi}})T_\varphi H^2 \subseteq X_\varphi{}^*H^2$ by (4) in the proof of Theorem 2 because $T_\varphi{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*) = O$ by the quasi-normality of T_φ,

$$\begin{aligned}&T_\varphi{}^*T_z{}^*T_\varphi{}^*1 = \lambda_1 T_z{}^*T_\varphi{}^*1 \\ \text{and}\quad &T_\varphi{}^*T_z{}^*T_\varphi 1 = \lambda_2 T_z{}^*T_\varphi{}^*1 \quad \text{for some}\quad \lambda_1,\ \lambda_2 \in \mathbb{C}\end{aligned}$$

and this contradicts the assumption that $T_\varphi{}^*T_z{}^*T_\varphi{}^*1$ and $T_\varphi{}^*T_z{}^*T_\varphi 1$ are linearly independent. □

Theorem 3. If T_φ is **paranormal** (i.e., $\|T_\varphi f\|_2{}^2 \leq \|T_\varphi{}^2 f\|_2\ \|f\|_2$ for all $f \in H^2$) and if $\varphi = \bar{q}g$ for some inner functions q and g, then T_φ is an isometry.

Proof. By the assumption, $\|T_\varphi\| = 1$. Since $\|T_\varphi q\|_2 = \|P\bar{q}gq\|_2 = \|Pg\|_2 = \|g\|_2 = 1 = \|q\|_2$,

$$\mathcal{M} = \{f \in H^2 : \|T_\varphi f\|_2 = \|f\|_2\} \neq \{o\}.$$

And, by the paranormality, we have $T_\varphi \mathcal{M} \subseteq \mathcal{M}$. In fact, if $f \in \mathcal{M}$, then

$$\|f\|_2{}^2 \geq \|f\|_2\ \|T_\varphi{}^2 f\|_2 \geq \|T_\varphi f\|_2{}^2 = \|f\|_2\ \|T_\varphi f\|_2 = \|f\|_2{}^2$$

and $\|T_\varphi{}^2 f\|_2 = \|T_\varphi f\|_2$ and hence $T_\varphi f \in \mathcal{M}$. Therefore

$$\mathcal{M} = \{f \in H^2 \ : \ \|T_\varphi{}^n f\|_2 = \|f\|_2, \ n = 0, 1, 2, \cdots\} \neq \{o\}$$

and T_φ is an isometry by Proposition 6. □

Acknowledgement The author would like to express his gratitude to the referee for some helpful comments and suggestions.

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Received April 16, 2002 Revised September 17, 2002