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On totally real submanifolds of a complex projective space*

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Abstract

Montiel, Ros and Urbano [3] showed a complete characterization of compact totally real minimal submanifold M of $CP^n(c)$ with Ricci curvature S of M satisfying $S \geq \frac{3(n-2)}{16}c$. The purpose of this paper is to answer Ogiue's conjecture which the above result remains true under the weaker condition of the scalar curvature ρ of M satisfying $\rho \geq \frac{3n(n-2)}{16}c$.

1 Introduction.

Let $CP^n(c)$ be an n-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $c(>0)$ and let M be an n-dimensional compact totally real minimal submanifold isometrically immersed in $CP^n(c)$. Let h be the second fundamental form of M in $CP^n(c)$.
Recently, Montiel, Ros and Urbano [3] proved the following: Let M be an n-dimensional compact totally real minimal submanifold isometrically immersed in $CP^n(c)$. Then the Ricci curvature S of M satisfies

$$S \geq \frac{3(n-2)}{16}c$$

if and only if one of the following conditions holds: a) $S = \frac{n-1}{4}c$ and M is totally geodesic, b) $S = 0, n = 2$ and M is a finite Riemannian covering of a flat torus minimally embedded in $CP^2(c)$ with parallel second fundamental form, c) $S = \frac{3(n-2)}{16}c$, $n > 2$ and M is an embedded submanifold congruent to the standard embedding of: $SU(3)/SO(3), n = 5; SU(6)/Sp(3), n = 14; SU(3), n = 8$; or $E_6/F_4, n = 26$.
Ogiue [5] conjectured the following: Under the weaker assumption of $\rho \geq \frac{3n(n-2)}{16}c$, the above result remains true, where ρ is the scalar curvature of M. With respect to this conjecture the author [4] showed: Let M be an n-dimensional

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compact totally real minimal submanifold isometrically immersed in $CP^n(c)$. Then

$$\rho \geq \frac{3n(n-2)}{16}c$$

if and only if one of the following conditions holds. A) $\rho = \frac{n(n-1)}{4}c$ and M is totally geodesic, B) $\rho = 0, n = 2$ and M is a finite Riemannian covering of the unique flat torus minimally embedded in $CP^2(c)$ with parallel second fundamental form, C) $\rho = \frac{3n(n-2)}{16}c, n > 2$ and M is an embedded submanifold congruent to the standard embedding of: $SU(3)/SO(3), n = 5; SU(6)/Sp(3), n = 14; SU(3), n = 8$ or $E_6/F_4, n = 26.$, D) $\rho = \frac{n(n-1)}{4}c - 4\lambda^2 \geq \frac{3n(n-2)}{16}c, n \neq 2$ and the second fundamental form of M takes the following form :

$$h(e_1, e_1) = \lambda J e_1, h(e_2, e_2) = -\lambda J e_1, h(e_1, e_2) = -\lambda J e_2,$$

$$h(e_1, e_j) = 0, h(e_2, e_j) = 0, h(e_j, e_k) = 0, j, k = 3, \cdots, n$$

for some non-constant function λ with respect to some suitable orthonormal local frame field $\{e_1, e_2, \cdots, e_n\}$, and if ρ is constant, then the case (D) cannot occur, where J is the complex structure.
The purpose of this paper is to show that (D) cannot occur without the assumption of constant scalar curvature, i.e., to prove Ogiue's conjecture.
Theorem Let M be an n-dimensional compact totally real minimal submanifold isometrically immersed in $CP^n(c)$. Then

$$\rho \geq \frac{3n(n-2)}{16}c$$

if and only if one of the following conditions holds:
A) $\rho = \frac{n(n-1)}{4}c$ and M is totally geodesic,
B) $\rho = 0, n = 2$ and M is a finite Riemannian covering of the unique flat torus minimally embedded in $CP^2(c)$ with parallel second fundamental form,
C) $\rho = \frac{3n(n-2)}{16}c, n > 2$ and M is an embedded submanifold congruent to the standard embedding of: $SU(3)/SO(3), n = 5; SU(6)/Sp(3), n = 14; SU(3), n = 8$ or $E_6/F_4, n = 26.$

2 Preliminaries.

Let M be an n-dimensional Riemannian manifold which is isometrically immersed in a Kaehler manifold $\overline{M}$ of dimension $2(n+p)$. We denote by $<\ ,\ >$ the metric of $\overline{M}$ as well as the one induced on M. We call M an totally real submanifold of $\overline{M}$ if M admits an isometric immersion into $\overline{M}$ such that for all x, $J(T_xM) \subset \nu_x$, where J, T_xM and ν_x denote the complex structure of $\overline{M}$, the tangent space of M at x and the normal space at x, respectively. Let

h be the second fundamental form of the immersion and A_ξ the Weingarten endomorphism associated a normal vector ξ. Let denote by ∇' the covariant differentiation with respect to the connection in (tangent bundle) $\oplus$ (normal bundle) for the second fundamental form h. Then the second fundamental form h of the immersion satisfies a differential equation (See [1] and [2]):

Proposition *Let M be an n-dimensional totally real minimal submanifold immersed in $CP^n(c)$. Then we have*

$$
\begin{aligned}
(1) \qquad \frac{1}{2}\Delta|h|^2 &= \|\nabla' h\|^2 + \sum_{\alpha,\beta} \mathrm{trace}(A_\alpha A_\beta - A_\beta A_\alpha)^2 \\
&\quad - \sum_{\alpha,\beta} (\mathrm{trace} A_\alpha A_\beta)^2 + \frac{n+1}{4} c|h|^2 \\
&= \|\nabla' h\|^2 + \sum_{\alpha,\beta} \mathrm{trace}(A_\alpha A_\beta - A_\beta A_\alpha)^2 \\
&\quad - 2\mathrm{trace}(\sum_\alpha A_\alpha^2)^2 + \frac{n+1}{4} c|h|^2
\end{aligned}
$$

3 Proof of Theorem.

At first, we prepare the following lemma:

Lemma ([2]): *Let A and B be symmetric $(n \times n)$-matrices. Then* $\mathrm{trace}(AB - BA)^2 \geq -2\mathrm{trace}A^2 \mathrm{trace}B^2$, *and the equality holds for nonzero matrices A and B if and only if A and B can be transformed simultaneously by an orthogonal matrix into scalar multiples of $\overline{A}$ and $\overline{B}$ respectively, where*

$$
\overline{A} = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & 0 \\ & 0 & 0 \end{pmatrix}; \quad \overline{B} = \begin{pmatrix} 1 & 0 & \\ 0 & -1 & 0 \\ & 0 & 0 \end{pmatrix}.
$$

Moreover, if A_1, A_2 and A_3 are $(n \times n)$-symmetric matirices and if $\mathrm{trace}(A_\alpha A_\beta - A_\beta A_\alpha)^2 = -2\mathrm{trace}A^2 \mathrm{trace}B^2$, $1 \leq \alpha, \beta \leq 3$, *then at least one of the matrices A_α must be zero.*

Next, we consider the case of the second fundamental form h of M taking the following form:

$$
(2) \qquad h(e_1, e_1) = \lambda J e_1, h(e_2, e_2) = -\lambda J e_1, h(e_1, e_2) = -\lambda J e_2, \\ \text{otherwise zero for some function} \lambda.
$$

From Introduction we know that we have only to prove that only the case of $n = 2$ holds if h satisfies (2) . Assume that the second funddamental form h of

M takes (2) . From Proposition we have

$$\begin{aligned}\frac{1}{2}|h|^2 &= \|\nabla' h\|^2 - 24\lambda^4 + \frac{n+1}{4}c|h|^2 \\ &= \|\nabla' h\| - \frac{3}{2}|h|^4 + \frac{n+1}{4}c|h|^2\end{aligned}$$

for the function λ. We put $S_\alpha = \mathrm{trace} A_\alpha^2$ and denote $\sum_\alpha S_\alpha$ and $\sum_{\alpha<\beta} S_\alpha S_\beta$ by $n\sigma_1$ and $\frac{n(n-1)}{2}\sigma_2$, respectively. Then from (2) and Lemma we have

$$\sum_{\alpha,\beta} \mathrm{trace}(A_\alpha A_\beta - A_\beta A_\alpha)^2 = -2\sum_{\alpha\neq\beta} \mathrm{trace}A_\alpha^2 \mathrm{trace}A_\beta^2.$$

Since the absolute values of the scalar multiples of Lemma are equal, we get

$$\sigma_1^2 = \sigma_2.$$

Thus we get

$$\begin{aligned}\frac{1}{2}|h|^2 &= \|\nabla' h\|^2 + \sum_{\alpha,\beta} \mathrm{trace}(A_\alpha A_\beta - A_\beta A_\alpha)^2 - \sum_\alpha S_\alpha^2 + \frac{n+1}{4}c|h|^2 \\ &= \|\nabla' h\|^2 - 2\sum_{\alpha\neq\beta} \mathrm{trace}A_\alpha^2 \mathrm{trace}A_\beta^2 - \sum_\alpha S_\alpha^2 + \frac{n+1}{4}c|h|^2 \\ &= \|\nabla' h\|^2 - 2\sum_{\alpha\neq\beta} S_\alpha S_\beta - \sum_\alpha S_\alpha^2 + \frac{n+1}{4}c|h|^2 \\ &= \|\nabla' h\|^2 - (\sum_\alpha S_\alpha)^2 - 2\sum_{\alpha\neq\beta} S_\alpha S_\beta + \frac{n+1}{4}c|h|^2 \\ &= \|\nabla' h\|^2 - n^2\sigma_1^2 - n(n-1)\sigma_2 + \frac{n+1}{4}c|h|^2 \\ &= \|\nabla' h\|^2 - (2n^2 - n)\sigma_1^2 + n(n-1)(\sigma_1^2 - \sigma_2) + \frac{n+1}{4}c|h|^2 \\ &= \|\nabla' h\|^2 - (2n^2 - n)\sigma_1^2 + \frac{n+1}{4}c|h|^2 \\ &= \|\nabla' h\|^2 - (2 - \frac{1}{n})|h|^4 + \frac{n+1}{4}c|h|^2\end{aligned}$$

Hence we have

$$2 - \frac{1}{n} = \frac{3}{2}.$$

Thus $n = 2$.

References

[1] CHENCHEN, B. Y. AND OGIUE, K., *On totally real submanfolds*, Tans. Amer. Math. Soc. **193**, pp. 257-268 (1974)

[2] CHERN, S. S., DOCARMO, M. P. AND KOBAYASHI, S., *Minimal submanifolds of a sphere with second fundamental form of constant length*, Functional Analysis and Related Fields (Proc. Conf. for M. Stone, Univ. Chicago Chicago III. 1968) Springer New York , pp. 59-75 (1970).

[3] MONTIEL, S., ROS, A. AND URBANO, F., *Curvature pinching and eigenvalue rigidity for minimal submanifolds*, Math. Z. **191**, pp. 537-548 (1986).

[4] MATSUYAMA, Y., *Curvature pinching for totally real submanifolds of a complex projective space*, J. Math. Soc. Japan **52**, pp. 51-64 (2000).

[5] OGIUE, K, *Recent topics of differential geometry*, Mathematics: the puplication of the Mathematical Society of Japan **39** pp. 305-319 (1987).

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