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CHARACTERIZATIONS OF SOME PSEUDO-EINSTEIN RULED REAL HYPERSURFACES IN COMPLEX SPACE FORMS IN TERMS OF RICCI TENSOR

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ABSTRACT. In this paper we define a new notion of *pseudo-Einstein ruled* real hypersurfaces, which are foliated by the leaves of pseudo-Einstein complex hypersurfaces in complex space forms $M_n(c)$, $c\neq 0$. Also we want to give a new characterization of this kind of *pseudo-Einstein ruled* real hypersurfaces in terms of *weakly η-parallel Ricci tensor* and the certain commutative condition defined on the orthogonal distribution T_0 in $M_n(c)$.

1. Introduction

Let us denote by $M_n(c)$ a complex $n(\geq 2)$-dimensional Kaehler manifold of constant holomorphic sectional curvature c, which is said to be a *complex space form.* Then a complete and simply connected complex space form is isometric to a complex projective space $P_n(\mathbb{C})$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n(\mathbb{C})$, according as $c>0$, $c=0$ or $c<0$. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ is denoted by (ϕ,ξ,η,g).

Until now several kinds of real hypersurfaces have been investigated by many differential geometers from different view points ([2],[4],[6],[9],[12] and [13]). Among them in a complex projective space $P_n(\mathbb{C})$ [5] Cecil-Ryan and [9] Kimura proved that they are realized as the tubes of constant radius over Kaehler submanifolds if the structure vector field ξ is principal. Also Berndt [2] showed recently that all real hypersurfaces with constant principal curvatures of a complex hyperbolic space $H_n(\mathbb{C})$ are realized as the tubes of constant radius over certain submanifolds when the structure vector field ξ is principal. Nowadays in $H_n(\mathbb{C})$ they are said to be of type A_0, A_1, A_2, and B.

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When the structure vector field ξ is not principal, Kimura [9] and Ahn, Lee and the second author [1] have constructed an example of ruled real hypersurfaces foliated by totally geodesic leaves, which are integrable submanifolds of the distribution T_0 defined by the subspace $T_0(x) = \{X \in T_x M : X \perp \xi\}$, $x \in M$, along the direction of ξ and *Einstein* complex hypersurfaces in $P_n(\mathbb{C})$ and $H_n(\mathbb{C})$ respectively. The expression of the Weingarten map is given by

$$A\xi = \alpha\xi + \beta U, \ AU = \beta\xi \text{ and } AX = 0,$$

where we have defined a unit vector U orthogonal to ξ in such a way that $\beta U = A\xi - \alpha\xi$ and β denotes the length of a vector field $A\xi - \alpha\xi$ and $\beta(x) \neq 0$ for any point x in M, and for any X in the distribution T_0 and orthogonal to ξ. Recently, several characterizations of such kind of ruled real hypersurfaces have been studied by the papers ([1],[6],[9],[11] and [15]).

Now as a general extension of this fact we introduce a new kind of ruled real hypersurfaces in $M_n(c)$ foliated by pseudo-Einstein leaves, which are integrable submanifolds of the distribution T_0 defined by the subspace $\{X \in T_x M : X \perp \xi\}$, along the direction of ξ and *pseudo-Einstein* complex hypersurfaces in $M_n(c)$. Then such kind of ruled real hypersurfaces are said to be *pseudo-Einstein*, because its Ricci tensor of the integral submanifold $M(t)$ of the distribution T_0 is given by

$$S^t = (\frac{n}{2}c - \mu)I + (\mu - \lambda)\{U \otimes U^* + \phi U \otimes (\phi U)^*\}.$$

Moreover, its expression of the Weingarten map is given by

$$AU = \beta\xi + \gamma U + \delta\phi U \text{ and } \quad A\phi U = \delta U - \gamma\phi U.$$

In Lemma 3.1 we know that the function λ mentioned above is given by $\lambda = 2(\gamma^2 + \delta^2)$.

When $\lambda = \mu$, ruled real hypersurfaces foliated by such kind of leaves are said to be *Einstein*. In particular, $\lambda = \mu = 0$, this kind of Einstein ruled real hypersurfaces are congruent to ruled real hypersurfaces in $M_n(c)$ foliated by totally geodesic Einstein leaves $M_{n-1}(c)$, which are said to be *totally geodesic* ruled real hypersurfaces in the sense of Kimura [9] for $c > 0$ and Ahn, Lee and the second author [1] for $c < 0$. In such a situation the function γ and δ both vanish identically.

From this point of view Kimura and Maeda [11] proved the following

Theorem A. *Let M be a real hypersurface of $P_n\mathbb{C}, n \geqq 3$. Then the second fundamental form is η-parallel and the holomorphic distribution T_0 is integrable if and only if M is locally a ruled real hypersurface.*

Moreover, Ahn, Lee and the second author [1] proved the following

Theorem B. *Let M be a real hypersurface of $M_n(c), c \neq 0, n \geqq 3$. Assume that ξ is not principal. Then it satisfies*

$$g((A\phi - \phi A)X, Y) = 0$$

for any vector fields X and Y in T_0 and the second fundamental form is η-parallel if and only if M is locally a ruled real hypersurface.

Even though the second fundamental form for ruled real hypersurfaces in above is *η-parallel*, but its Ricci tensor is not necessarilly *η-parallel.* In the previous paper [17] the second author introduced the new notion of ***pseudo-Einstein*** ruled real hypersurfaces in $M_n(c)$, $c\neq0$. The ruled real hypersurfaces of $M_n(c)$ defined by Kimura [9] and Ahn, Lee and the second author [1] respectively for $c > 0$ and $c < 0$ are Einstein ruled ones with zero Ricci curvatures. The purpose of this paper is to generalize such a notion of Einstein ruled ones into pseudo-Einstein ones.

We consider a distribution T'' defined by a subspace

$$T''(x) = \{X \in T_0(x) : g(X, U)(x) = 0\}$$

of the tangent subspace $T_0(x)$. Let T' be a distribution defined by a subspace

$$T'(x) = \{X \in T''(x) : g(X, \phi U_{(x)}) = 0\}$$

of the tangent subspace T''.

Now let us consider much more generalized condition than that of *η-paralle* Ricci tensor. The Ricci tensor S of the real hypersurface M of $M_n(c)$ is said to be ***weakly η-parallel***, if it satisfies

(I) $$g((\nabla_X S)Y, Z) = 0, \ X \in T_0, \ Y, Z \in T'.$$

Of course in section 3 it can be verified that the Ricci tensor of pseudo-Einstein ruled real hypersurfaces in $M_n(c)$ is weakly *η-parallel.*

Let us consider another geometric condition defined on the distribution T_0 in such a way that

(II) $$g((A\phi + \phi A)X, Y) = 0$$

for any $X, Y \in T_0$, which gives an integrability of the distribution T_0. Then the purpose of this paper is to give a new characterization of pseudo-Einstein ruled real hypersurface in terms of the Ricci tensor as follows:

Theorem 1. *Let M be a real hypersurface of $M_n(c), c \neq 0$ and $n \geqq 3$. If it satisfies* (I) *and* (II) *and its mean curvature is non-constant along the distribution T', then M is locally congruent to a pseudo-Einstein ruled real hypersurface.*

Moreover, as an application of Theorem 1 we obtain more specified result if we assume that the mean curvature of M is non-vanishing. That is, we have the following

Theorem 2. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$ and $n \geqq 3$ and its mean curvature is non-vanishing. If it satisfies* (I) *and* (II) *and its mean curvature is non-constant along the distribution T', then M is locally congruent to a ruled real hypersurface.*

In section 3 we will recall fundamental properties of pseudo-Einstein ruled real hypersurfaces of $M_n(c)$, $c\neq 0$, and will show that its Ricci tensor is *weakly η-parallel.* In section 4 we shall prove the main theorem.

Kimura and Maeda ([11]) have constructed a ruled real hypersurface M in complex projective space $P_n(\mathbb{C})$ which was foliated by totally geodesic submanifolds $P_{n-1}(\mathbb{C})$. In such a case its mean curvature H is given by $H = \frac{\alpha}{n}$, where the function α is denoted by $\alpha = g(A\xi, \xi)$. In general its mean curvature H is non-vanishing. After finishing the proof of Lemma 4.4, by using Theorem 1 we will prove Theorem 2 which gives another new characterization of such kind of ruled real hypersurfaces in complex space form $M_n(c)$.

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2. Preliminaries

First of all, we recall fundamental properties of real hypersurfaces in a complex space form. Let M be a real hypersurface in a complex n-dimensional complex space form $(M_n(c), \bar{g})$ of constant holomorphic sectional curvature c, and let C be a unit normal vector field on a neighborhood in M. We denote by J the almost complex structure of $M_n(c)$. For a local vector field X on the neighborhood in M, the images of X and C under the linear transformation J can be represented as

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, where η and ξ denote a 1-form and a vector field on the neighborhood in M, respectively. Then it is seen that $g(\xi, X) = \eta(X)$, where g denotes the Riemannian metric tensor on M induced from the metric tensor $\bar{g}$ on $M_n(c)$. The set of tensors (ϕ, ξ, η, g) is called an *almost contact metric structure* on M. They satisfy the following properties :

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1$$

for any vector field X, where I denotes the identity transformation. Furthermore the covariant derivatives of the structure tensors are given by

$$\nabla_X \xi = \phi AX, \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{2.1}$$

for any vector fields X and Y on M, where ∇ is the Riemannian connection on M and A denotes the shape operator of M in the direction of C.

Since the ambient space is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are obtained by:

$$\begin{aligned} R(X,Y)Z = & \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y \\ & + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ & + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.2}$$

$$\begin{aligned} (\nabla_X A)Y - & (\nabla_Y A)X \\ & = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \end{aligned} \tag{2.3}$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X. The second fundamental form is said to be *η-parallel* if the shape operator A satisfies $g((\nabla_X A)Y, Z) = 0$ for any vector fields X, Y and Z in T_0.

On the other hand, the Ricci tensor S is given by

$$S = \frac{c}{4}\{(2n+1)I - 3\eta \otimes \xi\} + hA - A^2, \tag{2.4}$$

where I denotes the identity transformation and h is the trace of A.

Next we assume the condition that

$$g((A\phi + \phi A)X, Y) = 0 \tag{II}$$

for any vector fields X and Y in T_0.

On the other hand, by (2.1) we know

$$\nabla_X Y = (\nabla_X Y)_0 - g(Y, \phi AX)\xi,$$

where $(\nabla_X Y)_0$ denotes the T_0 component of the vector field $\nabla_X Y$. Then by differentiating the condition (II) and using also (2.1) and the above formula, we have

$$\begin{aligned} & g((\nabla_X A)Y, \phi Z) - g((\nabla_X A)Z, \phi Y) \\ & = \beta\Big[g(Y,U)g(AX,Z) - g(Z,U)g(AX,Y) \\ & \quad + g(Y,\phi U)g(\phi AX, Z) - g(Z, \phi U)g(\phi AX, Y)\Big] \end{aligned} \tag{2.5}$$

for any vector fields X, Y and Z in T_0.

Now we here calculate the covariant derivative of the Ricci tensor S. By (2.4) we get for any X, Y in T_0

$$\begin{aligned}(\nabla_X S)Y = & -\frac{3}{4}cg(\phi AX, Y)\xi + dh(X)AY \\ & + h(\nabla_X A)Y - (\nabla_X A)AY - A(\nabla_X A)Y,\end{aligned}$$

from which it turns out to be

$$\begin{aligned}g((\nabla_X S)Y, Z) = & dh(X)g(AY, Z) + hg((\nabla_X A)Y, Z) \\ & - g((\nabla_X A)Y, AZ) - g((\nabla_X A)Z, AY)\end{aligned} \tag{2.6}$$

for any vector fields X, Y and Z in T_0. Replacing Z by ϕZ in this equation, we get

$$\begin{aligned}g((\nabla_X S)Y, \phi Z) = & dh(X)g(AY, \phi Z) + hg((\nabla_X A)Y, \phi Z) \\ & - g((\nabla_X A)Y, A\phi Z) - g((\nabla_X A)\phi Z, AY).\end{aligned}$$

For any vector field V we denote by V_0 the component in the distribution T_0. Since we see $A\phi Z = -\phi AZ - \beta g(Z, \phi U)\xi$ by (II), we have

$$\begin{aligned}& g((\nabla_X A)Y, A\phi Z) + g((\nabla_X A)\phi Y, AZ) \\ & \quad = -g((\nabla_X A)Y, \phi AZ) - \beta g(Z, \phi U)g((\nabla_X A)Y, \xi) \\ & \qquad + g((\nabla_X A)(AZ)_0, \phi Y) + \beta g(Z, U)g((\nabla_X A)\xi, \phi Y),\end{aligned} \tag{2.7}$$

where we denote by $(AZ)_0$ the T_0-component of the vector field AZ. By replacing Z by $(AZ)_0$ in (2.5), the above equation (2.7) is reformed as

$$\begin{aligned}& g((\nabla_X A)Y, A\phi Z) + g((\nabla_X A)\phi Y, AZ) \\ & = -\beta\Big[g(Y, U)g(AX, AZ) - g(AZ, U)g(AX, Y) \\ & \quad + g(Y, \phi U)g(\phi AX, AZ) - g(AZ, \phi U)g(\phi AX, Y) \\ & \quad - \beta^2 g(X, U)g(Y, U)g(Z, U)\Big] \\ & \quad + \beta g(Z, U)g((\nabla_X A)\phi Y, \xi) - \beta g(Z, \phi U)g((\nabla_X A)Y, \xi),\end{aligned} \tag{2.8}$$

where we have used the formula that

$$AZ = (AZ)_0 + \beta g(Z, U)\xi.$$

Now by (II), (2.5), (2.6) and (2.8) we can assert an important formula which will be used to prove our results:

$$
\begin{aligned}
&g((\nabla_X S)Y,\phi Z)+g((\nabla_X S)Z,\phi Y)\\
&=2dh(X)g(AX,\phi Z)+2hg((\nabla_X A)Y,\phi Z)\\
&\quad-\beta h\Big[g(Y,U)g(AX,Z)-g(Z,U)g(AX,Y)\\
&\qquad+g(Y,\phi U)g(\phi AX,Z)-g(Z,\phi U)g(\phi AX,Y)\Big]\\
&\quad+\beta\Big[g(Y,U)g(AX,AZ)-g(AZ,U)g(AX,Y)\\
&\qquad+g(Y,\phi U)g(\phi AX,AZ)-g(AZ,\phi U)g(\phi AX,Y)\\
&\qquad-\beta^2 g(X,U)g(Y,U)g(Z,U)\Big]\\
&\quad-\beta g(Z,U)g(\nabla_X A)\phi Y,\xi)+\beta g(Z,\phi U)g((\nabla_X A)Y,\xi)\\
&\quad+\beta\Big[g(Z,U)g(AX,AY)-g(AY,U)g(AX,Z)\\
&\qquad+g(Z,\phi U)g(\phi AX,AY)-g(AY,\phi U)g(\phi AX,Z)\\
&\qquad-\beta^2 g(X,U)g(Y,U)g(Z,U)\Big]\\
&\quad-\beta g(Y,U)g((\nabla_X A)\phi Z,\xi)+\beta g(Y,\phi U)g((\nabla_X A)Z,\xi).
\end{aligned}
\tag{2.9}
$$

3. Pseudo-Einstein ruled real hypersurface

This section is concerned with necessary properties about ruled real hypersurfaces. First of all, we define a ruled real hypersurface M of $M_n(c)$,$c\neq 0$. Let $\mathfrak{D}$ be a J-invariant integrable $(2n-2)$-dimensional distribution defined on $M_n(c)$ whose integral manifolds are holomorphic planes spanned by unit normals C and JC and let $\gamma : I\rightarrow M_n(c)$ be an integral curve for the vector JC. For any $t(\in I)$ let $M_{n-1}^{(t)}(c)$ be a totally geodesic complex hypersurface through the point $\gamma(t)$ of $M_n(c)$ which is orthogonal to a holomorphic plane spanned by $\gamma'(t)$ and $J\gamma'(t)$. Set $M=\{x\in M_{n-1}^{(t)}(c) : t\in I\}$. Then the construction of M asserts that M is a real hypersurface of $M_n(c)$, which is called a ruled real hypersurface. This means that there exists a ruled real hypersurfaces of $M_n(c)$ with the given distribution $\mathfrak{D}$. We denote by A_C and A_{JC} the shape operator of any integral submanifold $M(t)$ of $\mathfrak{D}$ in $M_n(c)$ in the direction of C and JC.

Under this construction the ruled real hypersurface M of $M_n(c)$,$c\neq 0$,has some fundamental properties. Let M be a ruled real hypersurface with the given distribution $\mathfrak{D}$ of $M_n(c)$, $c\neq 0$ and let A be the its shape operator of the ruled real hypersurface M in $M_n(c)$.

Now let us put $\xi=-JC$ and $A\xi=\alpha\xi+\beta U$, where U is a unit vector orthogonal to ξ and α and β $(\beta\neq 0)$ denote certain differentiable functions defined on M. For any unit vector field V along $\mathfrak{D}$, let V^* be the corresponding 1-form defined by $V^*(V) = g(V, V) = 1$. If they satisfy

$$A_C^2 + A_{JC}^2 = \lambda I + \mu(V \otimes V^* + JV \otimes (JV)^*)$$

for a certain vector field V, where λ and μ are smooth function on M, then the ruled real hypersurface M with the given distribution $\mathfrak{D}$ of $M_n(c)$ is said to be *pseudo-Einstein* and if $\lambda = \mu = 0$, then it is said to be *totally geodesic* and it is the ruled real hypersurface M in the sense of Kimura [9]. If the ruled real hypersurface M is pseudo-Einstein, Einstein or totally geodesic, then it can be easily seen that any integral submanifold of $\mathfrak{D}$ is *pseudo-Einstein, Einstein* or *totally geodesic*, respectively, because $\mathfrak{D}$ is J-invariant.

Since $T_0 (= \mathfrak{D})$ is integrable, we see

(II)
$$g((A\phi + \phi A)X, Y) = 0$$

for any vector fields X and Y in T_0.

On the other hand, $M(t)$ is a submanifold of codimension 2 and ξ and C are orthonormal normal vector fields on its leaf in $M_n(c)$. So we have

(3.1)
$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + g(AX, Y)C \\ &= \nabla_X^t Y + g(A_C X, Y)C + g(A_\xi X, Y)\xi, \end{aligned}$$

where $\bar{\nabla}$ and ∇^t are the covariant derivatives in the ambient space $M_n(c)$ and in the submanifold $M(t)$, respectively and moreover A_C and A_ξ are the shape operators in the direction of C and ξ, respectively. Then we have

$$g(\bar{\nabla}_X Y, \xi) = g(\nabla_X Y, \xi) = -g(\bar{\nabla}_X \xi, Y) = g(A_\xi X, Y),$$

for any $X, Y \in T_0$, from which it implies that

(3.2)
$$A_\xi X = -\phi AX, \quad X \in T_0.$$

On the other hand, by (3.1) we have

$$g(AX, Y) = g(A_C X, Y), \quad X, Y \in T_0$$

and therefore

(3.3)
$$A_C X = AX - \beta g(X, U)\xi, \quad X \in T_0.$$

By (II) we have

(3.4)
$$A\phi X = -\phi AX - \beta g(X, \phi U)\xi, \quad X \in T_0.$$

It is easily seen that the traces of these shape operators are both equal to zero. Since the complex submanifold $M(t)$ of real codimension 2 in $M_n(c)$ is pseudo-Einstein, whose dimension is equal to $2n-2(\geq 2)$, its Ricci tensor S^t is given by

$$S^t = (\frac{n}{2}c - \mu)I + (\mu - \lambda)(U \otimes U^* + \phi U \otimes (\phi U)^*)$$

and we have

$$\text{(3.5)} \qquad \begin{cases} (A_\xi^2 + A_C^2)U = \lambda U \\ (A_\xi^2 + A_C^2)\phi U = \lambda \phi U \\ (A_\xi^2 + A_C^2)X = \mu X, \qquad X \in \mathfrak{D}, X \perp U, X \perp \phi U, \end{cases}$$

where λ and μ are smooth functions on $M(t)$. By the direct calculation of the left hand side of the above relation and using the properties (3.2) and (3.3) we have

Lemma 3.1. (See [16] and [17]) Let M be a proper pseudo-Einstein ruled real hypersurfaces in $M_n(c)$, $c \neq 0, n \geq 3$. Then we have

$$\text{(3.6)} \qquad \begin{cases} AU = \quad \beta\xi + \gamma U + \delta \phi U, \\ A\phi U = \quad \delta U - \gamma \phi U, \quad \lambda = 2(\gamma^2 + \delta^2). \end{cases}$$

In particular, if it is totally geodesic, we have $\gamma = \delta = 0$.

On the other hand, we give the following for any X orthogonal to ξ, U and ϕU.

$$\text{(3.7)} \qquad A^2 X = \beta\epsilon\xi + \frac{\mu}{2}X,$$

because (3.2), (3.3), the third formula of (3.5) and the condition (II) imply that

$$\begin{aligned} \mu X = & - A_\xi \phi A X + A_C\{AX - \beta g(X, U)\xi\} \\ = & 2(A^2 X - \beta g(AX, U)\xi). \end{aligned}$$

When the function μ in (3.5) vanishes, then (3.7) implies that

$$\|AX\|^2 = 0$$

for any $X \in T'$, where $T' = L(\xi, U, \phi U)^\perp$. Then the derivative of the Ricci tensor of pseudo-Einstein ruled real hypersurfaces in $M_n(c)$ satisfies

$$\text{(I)} \qquad g((\nabla_X S)Y, Z) = 0,$$

for any $Y, Z \in T'$ and $X \in T_0$, because for such $X \in T_0$ and $Y, Z \in T'$ we know that $AY = AZ = 0$, and naturally $g((\nabla_X A)Y, Z) = 0$ in (2.6). Of course, in totally geodesic Einstein ruled real hypersurfaces of $M_n(c)$ we know that $AX = 0$ for any X orthogonal to ξ, U and ϕU. In such a case the function μ also vanishes identically.

Now we introduce some examples of pseudo-Einstein ruled real hypersurfaces in complex projective space $P_n(\mathbb{C})$ which were given in [17].

Example 1. Let M be a ruled real hypersurface in $P_n(\mathbb{C})$ foliated by complex hyperplane $P_{n-1}(\mathbb{C})$. Then the expression (3.1) implies that

$$A_\xi X = 0 \text{ and } A_C X = 0$$

for any $X \in \mathfrak{D}$, where $\mathfrak{D}$ denotes the distribution of $P_{n-1}(\mathbb{C})$. This implies $A_\xi^2 + A_C^2 = 0$ on the distribution $\mathfrak{D}$. Then its Ricci tensor is given by $S^t = \frac{nc}{2} I$. So we know that M is a totally geodesic Einstein ruled real hypersurface in $P_n(\mathbb{C})$.

Example 2. Let M be a real hypersurface in $P_n(\mathbb{C})$ foliated by complex quadric Q^{n-1}. Then it can be easily seen in [10] and [17] that the shape operator A_C defined on the distribution of the complex quadric Q^{n-1} satisfies

$$A_C^2 = \lambda^2 I.$$

Moreover, we know that $A_\xi X = -\phi A X$ for $X \in \mathfrak{D}$. Then we know

$$\begin{aligned} A_\xi^2 X =& \phi A \phi A X \\ =& \phi A \phi A_C X \\ =& -\phi^2 A A_C X \\ =& -\phi^2 \{A_C^2 X + \beta g(A_C X, U)\xi\} \\ =& -\phi^2 \{\lambda^2 X\} \\ =& \lambda^2 X, \end{aligned}$$

where in the third equality we have used the integrability of the distribution $\mathfrak{D}$. So it follows that $(A_\xi^2 + A_C^2)X = 2\lambda^2 X$ for any $X \in \mathfrak{D}$. Then the Ricci tensor S^t is given by $S^t = \{\frac{n}{2}c - 2\lambda\}I$. From this we conclude that M is not totally geodesic *Einstein* ruled real hypersurface.

Example 3. Let Γ be a complex curve in $P_n(\mathbb{C})$. Now let us consider

$$\phi_{\frac{\pi}{2}}(\Gamma) = \cup_{x \in \Gamma} \{exp_x \frac{\pi}{2} v | v \text{ is a unit normal vector of } \Gamma \text{ at } x\}.$$

Then $\phi_{\frac{\pi}{2}}(\Gamma)$ is an $(n-1)$-dimensional complex hypersurface in $P_n(\mathbb{C})$ (See [9], [10]), which is a submanifold of real codimension 2 in $P_n(\mathbb{C})$. Moreover, it is a *pseudo-Einstein* complex hypersurface in $P_n(\mathbb{C})$. Then we construct a real hypersurface M in $P_n(\mathbb{C})$ foliated by such kind of leaves along the integral curve of the normal vector field $\xi = -JC$.

For this, we consider a regular curve $\gamma : I \rightarrow M_n(c)$. Then we can construct a ruled real hypersurface M foliated by pseudo-Einstein complex hypersurfaces in such a way that

$$\begin{aligned} M =& \cup_t \gamma(t) \times \phi_{\frac{\pi}{2}}(\Gamma) \\ =& \cup_t \phi_{\frac{\pi}{2}}^{(t)}(\Gamma). \end{aligned}$$

Moreover, let us take a structure vector ξ such that $\xi(\gamma(t)) = \gamma'(t)$ orthogonal to the tangent space of $\phi_{\frac{\pi}{2}}(\Gamma)$ at $\gamma(t)$. The vector $\xi(\gamma(t))$ can be smoothly extended to any point in $\phi_{\frac{\pi}{2}}^{(t)}(\Gamma)$ by parallel displacement P in such a way that $P\xi((\gamma(t)) \perp T_x \phi_{\frac{\pi}{2}}^{(t)}(\Gamma)$ for any x in $\phi_{\frac{\pi}{2}}^{(t)}(\Gamma)$. Then in this case we call such a real hypersurface in $P_n(\mathbb{C})$ *pseudo-Einstein ruled* real hypersurface. Now let us show that its leaves are pseudo-Einstein complex hypersufaces in $P_n(\mathbb{C})$.

In fact, if we consider the principal curvatures of the shape operator A_C defined on the distribution of $\phi_{\frac{\pi}{2}}(\Gamma)$, it is given by

$$\begin{aligned} &cot(\frac{\pi}{2} + \theta) \text{ with multiplicity } 1, \\ &cot(\frac{\pi}{2} - \theta) \text{ with multiplicity } 1, \\ &0 \text{ with multiplicity } 2n - 4. \end{aligned}$$

Then from this expression of the shape operator A_C we can put

$$A_C U = cot(\frac{\pi}{2} + \theta)U, \ A_C \phi U = cot(\frac{\pi}{2} - \theta)\phi U, \text{and } A_C X = 0$$

for a certain vector field $U \in \mathfrak{D}$ and any vector field $X \in \mathfrak{D}$ orthogonal to U and ϕU, where $\mathfrak{D}$ denotes the distribution of $\phi_{\frac{\pi}{2}}(\Gamma)$ orthogonal to the structure vector ξ. Then it can be easily seen that

$$\begin{aligned} A_C^2 U &= cot^2(\frac{\pi}{2} + \theta)U = \frac{\lambda}{2}U, \\ A_C^2 \phi U &= cot^2(\frac{\pi}{2} - \theta)\phi U = \frac{\lambda}{2}\phi U, \\ A_C^2 X &= 0 \end{aligned}$$

for any X orthogonal to U and ϕU. Also if we apply the same method as in Example 2, the shape operator A_ξ can be calculated. So naturally it follows that

$$\begin{aligned} (A_\xi^2 + A_C^2)U &= \lambda U, \\ (A_\xi^2 + A_C^2)\phi U &= \lambda \phi U, \\ (A_\xi^2 + A_C^2)X &= 0 \end{aligned}$$

for any X orthogonal to U and ϕU. Accordingly, we have our assertion.

4. Proof of the Main Theorems

In this section we shall consider a characterization of certain kind of pseudo-Einstein ruled real hypersurfaces in terms of the Ricci tensor S. Let M be a real hypersurface of $M_n(c), c \neq 0$.

Let us first assume that the structure vector ξ is not principal. We denote by M_0 an open subset in M consisting of points x at which $\beta \neq 0$. By the assumption the subset M_0 is not empty. So, we can put $A\xi = \alpha\xi + \beta U$ on M_0, where U is a unit vector field in the holomorphic distribution T_0 and α and β are smooth functions on M.

Let $L(\xi, U)$ or $L(\xi, U, \phi U)$ be a distribution spanned by ξ, U or $\xi, U, \phi U$, respectively. We denote by T'' or T' an orthogonal complement of the distribution $L(\xi, U)$ or $L(\xi, U, \phi U)$, respectively. we also assume the following condition:

$$\text{(I)} \qquad g((\nabla_X S)Y, Z) = 0, \quad X \in T_0, \ Y, Z \in T',$$

$$\text{(II)} \qquad g((A\phi + \phi A)X, Y) = 0, \quad X, Y \in T_0.$$

Now let M_0 be an open subset of M consisting of points x at which $\beta(x) \neq 0$. Since ξ is not principal, M_0 can not be empty. By the assumption (II) it turns out to be

$$(A\phi + \phi A)X = -\beta g(X, \phi U)\xi, \quad X \in T_0. \tag{4.1}$$

Differentiating covariantly (II) with X in T_0 and using (II) and the second equation of (2.1), we get directly

$$\begin{aligned} &g((\nabla_X A)Y, \phi Z) - g((\nabla_X A)Z, \phi Y) \\ &= \beta[-g(AX, Y)g(Z, U) + g(AX, Z)g(Y, U) \\ &\quad - g(\phi AX, Y)g(Z, \phi U) + g(\phi AX, Z)g(Y, \phi U)] \end{aligned} \tag{4.2}$$

for any vector fields X, Y and Z in T_0.

Lemma 4.1. *Let M be a real hypersurface of $M_n(c), c \neq 0, n \geqq 3$ satisfying the conditions* (I) *and* (II). *Then it satisfies*

$$AX \equiv 0 \ (mod\ \xi, U, \phi U) \tag{4.3}$$

on the open subset M_0 for any vector field X in T'.

Proof. By (2.9) and the assumption (I) we have for any vector fields X in T_0 and Y,Z in $T' = \{X \in T''(x) : g(X, \phi U_{(x)}) = 0\}$

$$\begin{aligned} &2\{dh(X)g(AY, \phi Z) + hg((\nabla_X A)Y, \phi Z)\} \\ &\quad - \beta\Big[g(AY, U)g(AX, Z) + g(AZ, U)g(AX, Y) \\ &\quad + g(AY, \phi U)g(\phi AX, Z) + g(AZ, \phi U)g(\phi AX, Y)\Big] \\ &= 0. \end{aligned} \tag{4.4}$$

Hence we have

$$2\{dh(X)g(AY,\phi Z)-dh(Y)g(AX,\phi Z)\}$$
$$+\beta\big[\{g(AX,U)g(AY,Z)+g(AX,\phi U)g(\phi AY,Z)\}$$
$$-\{g(AY,U)g(AX,Z)+g(AY,\phi U)g(\phi AX,Z)\}\big]$$
$$=0$$

for any vector fields X,Y and Z in T'. Accordingly, we have

$$\begin{aligned}&\{2dh(Y)-\beta g(AY,\phi U)\}\phi AX-\{2dh(X)-\beta g(AX,\phi U)\}\phi AY\\&\quad+\beta\{g(AX,U)AY-g(AY,U)AX\}\equiv 0 \pmod{\xi,U,\phi U}\end{aligned}\tag{4.5}$$

for any vector fields X,Y and Z in T'. Under the assumption that the mean curvature H is not constant along the distribution T' we know that $dh(Y)\neq 0$ for any $Y\in T'$. From such a situation we do not have a case that all of coefficients are vanishing simultaneously. When one or two of the coefficients of ϕAX and ϕAY are vanishing, we are able to assert our result by the same method given below.

Now we consider both of two coefficients of ϕAX and ϕAY are non-vanishing. Then we can eliminate one of them as follows. In order to do this, let us replace X with ϕX in the above equation and taking account of the condition (II), we obtain again the linear combination of the vectors $AX, AY, \phi AX$ and ϕAY. From these two equations the vector ϕAY can be eliminated like the following equation for any $X,Y\in T'$

$$\begin{aligned}&\big[\{2dh(\phi X)+\beta g(AX,U)\}\{2dh(Y)-\beta g(AY,\phi U)\}\\&\qquad-\beta g(AY,U)\{2dh(X)-\beta g(AX,\phi U)\}\big]\phi AX\\&+\big[\beta g(AX,U)\{2dh(\phi X)+\beta g(AX,U)\}\\&\qquad-\beta g(AX,\phi U)\{2dh(X)-\beta g(AX,\phi U)\}\big]AY\\&-\big[\beta g(AY,U)\{2dh(\phi X)+\beta g(AX,U)\}\\&\qquad+\{2dh(Y)-\beta g(AY,\phi U)\}\{2dh(X)-\beta g(AX,\phi U)\}\big]AX\equiv 0\\&\pmod{\xi,U,\phi U}.\end{aligned}$$

Let us write the above equation in such a way that

$$\lambda(X,Y)\phi AX+\mu(X)AY+\nu(X,Y)AX\equiv 0 \pmod{\xi,U,\phi U},$$

where $\lambda(X,Y)$,(resp. $\mu(X)$ and $\nu(X,Y)$) denotes the corresponding coefficients of ϕAX (resp. AY and AX). Moreover, by virtue of our assumption we know that these coefficients can not be simultaneously vanishing. So naturally let us consider the following equation for any $X, Y \in T'$

$$\phi AX \equiv \kappa(X,Y)AY + \rho(X,Y)AX \pmod{\xi, U, \phi U},$$
$$\phi AY \equiv \kappa(Y,X)AX + \rho(Y,X)AY \pmod{\xi, U, \phi U}$$

where $\kappa(X,Y) = -\frac{\mu(X)}{\lambda(X,Y)}$ and $\rho(X,Y) = -\frac{\nu(X,Y)}{\lambda(X,Y)}$. The second equation mentioned above is just obtained by interchanging vector fields X and Y in T' in the first equation. Now applying ϕ to the first equation, then it follows that

$$AX \equiv -\kappa(X,Y)\phi AY - \rho(X,Y)\phi AX \pmod{\xi, U, \phi U}.$$

From this, substituting the first and the second equation into the right side, and from our assumption we know that the coefficients of two vectors AX and AY can not be vanishing simutaneously. So we are able to assert the following

$$AX \equiv a(X,Y)AY \pmod{\xi, U, \phi U} \tag{4.6}$$

for any vector fields X and Y in T', where $a(X,Y)$ is a smooth function depends on X and Y. Let $X_1, ..., X_m$ be an orthonormal basis of $T'(x)$ at any point x in M_0. Then we see by (4.6)

$$A(X_i - a_i X_1) \equiv 0 \pmod{\xi, U, \phi U}, \qquad i \geq 2.$$

Since $\{X_i - d_i X_1; i \geq 2\}$ is a basis of an $(n-1)$-dimensional subspace in $T_0(x)$ with respect to X_1, we get $AX \equiv 0 \pmod{\xi, U, \phi U}$ for any vector X in this subspace. Accordingly, again by (4.6) we obtain (4.3). It completes the proof. □

Now we want to show that the distribution T' is A-invariant and ϕ-invariant, which is crucial and important to prove that our hypersurface is a pseudo-Einstein ruled one. Namely, we have the following

Lemma 4.2. *Under the conditions* (I) *and* (II) *we have*

$$\begin{cases} A\xi = \alpha\xi + \beta U, \\ AU = \beta\xi + \gamma U + \delta\phi U, \\ A\phi U = \delta U - \gamma\phi U \end{cases} \tag{4.7}$$

on the open subset M_0. Namely, the subbundle T' is A-invariant and ϕ- invariant on M_0.

Proof. On the non-empty open set M_0, we may suppose that we put

$$AU = \beta\xi + \gamma U + \delta\phi U + \epsilon V,$$

where $\xi, U, \phi U$ and V are orthonormal vector fields. So V is contained in T'. Then we have

$$g(AV, U) = g(AU, V) = \epsilon,$$
$$g(AV, \phi U) = g(V, A\phi U) = -g(V, \phi AU) = 0$$

from which together with Lemma 4.1 it follows that

$$AV = \epsilon U, \quad A\phi V = -\epsilon\phi U. \tag{4.8}$$

On the other hand, in the covariant derivative of the Ricci tensor S, we replace Y and Z with V and ϕV. By the assumption (I) we have

$$g((\nabla_X S)V, \phi V) = 0 \tag{4.9}$$

for any vector field X in T_0. Thus we have

$$\begin{aligned} dh(X)g(AV, \phi V) + hg((\nabla_X A)V, \phi V) - g((\nabla_X A)V, A\phi V) \\ - g((\nabla_X A)\phi V, AV) = 0 \end{aligned} \tag{4.10}$$

for any vector field X in T_0.

On the other hand, combining Lemma 4.1 with (4.4), we have

$$hg((\nabla_X A)Y, \phi Z) = 0, \tag{4.11}$$

for any vector fields $X \in T_0$, Y and Z in T'. Accordingly, the covariant derivative of the Ricci tensor S implies that

$$g((\nabla_X A)Y, AZ) + g((\nabla_X A)Z, AY) = 0$$

for any vector fields X, Y and Z in T'. This shows that the first term is skew-symmetric with respect to Y and Z. Furthermore, let us take the skew-symmetric part of the above equation with respect to X and Y in T' , we have

$$g((\nabla_X A)Z, AY) = g((\nabla_Y A)Z, AX)$$

by the Codazzi equation (2.3). Hence it turns out to be symmetric with respect to Y and Z. So we have

$$g((\nabla_X A)Y, AZ) = 0$$

for any vector fields X, Y and Z in T', from which together with (4.8) it follows that

$$g((\nabla_X A)Y, AV) = \epsilon g((\nabla_X A)Y, U) = 0 \tag{4.12}$$

for any vector fields X and Y in T'. Thus the assumption (I) we see that

$$
\begin{aligned}
g((\nabla_U S)V, \phi V) =& dh(U)g(AV, \phi V) + hg((\nabla_U A)V, \phi V) \\
& - g((\nabla_U A)V, A\phi V) - g((\nabla_U A)AV, \phi V) \\
=& hg((\nabla_U A)V, \phi V) \\
& + \epsilon\{g((\nabla_U A)V, \phi U) - g((\nabla_U A)U, \phi V)\} \\
=& 0.
\end{aligned} \tag{4.13}
$$

From this together with (4.2) and (4.8) it follows that for $U \in T_0$ and $V \in T'$

$$hg((\nabla_U A)V, \phi V) - \beta\epsilon^2 = 0.$$

Then by the equation of Codazzi (2.3) and the second formula in (4.12), we get

$$\epsilon = 0$$

on M_0, where we have used that $g((\nabla_X A)Y, Z)$ is symmetric with respect to any vector fields X, Y and Z in T_0. Thus it completes the proof. □

Lemma 4.3. *Under the conditions* (I) *and* (II), *we have* $AY = 0$ *for any vector field* Y *in* T' *on* M_0.

Proof. By Lemma 4.2 the subbundle T' is A-invariant on M_0. Furthermore Lemma 4.1 implies that we have $AY \equiv 0$ (mod $\xi, U, \phi U$) for any vector field Y in T'. By the above facts it turns out to be $AY = 0$ for any vector field Y in T'. □

Under these preparations of Lemmas 4.1, 4.2 and 4.3 we are in a position to prove our main theorems in the introduction.

Proof of Theorem 1. Suppose that the interior of $M - M_0$ is not empty. On the interior the function β vanishes identically and therefore ξ is prinicipal. Thus we have

$$(A\phi + \phi A)\xi = 0.$$

For any principal vector X in T_0 with principal curvature λ, the condition (II) is reduced to $A\phi X = -\lambda\phi X + \theta(X)\xi$. From $A\xi = \alpha\xi$ the linear product of $A\phi X$ and ξ gives us to $\theta(X) = 0$. This means that

$$A\phi + \phi A = 0 \tag{4.14}$$

on the interior of $M - M_0$. It is seen in Ki and Suh [7] that (4.14) holds on M, then we have $c = 0$. Since this property is local, we have $c = 0$ on the interior of $M - M_0$, a contradiction. Thus the interior of $M - M_0$ must be empty and hence the open set M_0 is dense. Accordingly, under the condition of Theorem 1 we see that $AX = 0$ for any X in T' on M_0. By the continuity of principal curvatures we see that the shape operator also satisfies such kind of properties on the whole M.

Since the distribution T_0 is integrable on M by the definition, the integral manifold of T_0 can be regarded as the submanifold of codimension 2 in $M_n(c)$ whose normal vectors are ξ and C. By the definition of the second fundamental form, we see

$$g(\bar{\nabla}_X Y, C) = -g(\bar{\nabla}_X C, Y) = g(A_C X, Y) = g(AX, Y),$$
$$g(\bar{\nabla}_X Y, \xi) = -g(\bar{\nabla}_X \xi, Y) = g(A_\xi X, Y)$$

for any vector fields X and Y in T_0, where $\bar{\nabla}$ denotes the Riemannian connection of $M_n(c)$ and A_ξ or A_C denotes the shape operator of $M(t)$ in $M_n(c)$ in the direction of the normal ξ or C respectively. Namely, it is seen that these shape operators satisfy

$$\begin{aligned}\bar{\nabla}_X Y &= \nabla_X Y + g(AX, Y)C \\ &= \nabla^t_X Y + g(A_\xi X, Y)\xi + g(A_C X, Y)C\end{aligned}$$

where ∇^t denotes the Riemannian connection of the integral submanifold of T_0. Thus we see

$$A_C X = AX + g(A_C X - AX, \xi)\xi = AX - \beta g(X, U)\xi, \quad X \in T_0,$$
$$A_\xi X = -\phi AX, \quad X \in T_0,$$

on M_0, because we have

$$g(\bar{\nabla}_X Y, \xi) = -g(\bar{\nabla}_X \xi, Y) = -g(\phi AX, Y) \ X, Y \in T_0,$$

by (2.1). Since T_0 is ϕ-invariant and therefore it is also J-invariant, its integral manifold is a complex hypersurface. Since the open subset M_0 is dense in M, by means of the continuity of principal curvatures, we have

$$\text{(4.15)} \qquad \begin{aligned}&AU = \beta\xi + \gamma U + \delta\phi U, \ A\phi U = \delta U - \gamma\phi U, \\ &AX = 0, \quad X \in T'\end{aligned}$$

on M and therefore it is seen that another shape operator A_ξ of the integral submanifold $M(t)$ of T_0 satisfies

$$\text{(4.16)} \qquad A_\xi X = \begin{cases} \delta U - \gamma\phi U, & X = U \\ -\gamma U - \delta\phi U, & X = \phi U \\ 0, & X \in T' \end{cases}$$

on $M(t)$ and it is also seen that another shape operator A_C of the integral submanifold of T_0 satisfies

$$(4.17) \qquad A_C X = \begin{cases} \gamma U + \delta\phi U, & X = U \\ \delta U - \gamma\phi U, & X = \phi U \\ 0, & X \in T'. \end{cases}$$

By combining (4.16) with (4.17) and by the direct calculation, it is trivial that we have

$$(A_\xi^2 + A_C^2)X = 2(\gamma^2 + \delta^2)X, \quad X = U \text{ and } \phi U.$$

In the case where X are in T', we see

$$(A_\xi^2 + A_C^2)X = 0.$$

This shows that an integral submanifold is pseudo-Einstein. Thus M is a pseudo-Einstein ruled real hypersurface.

Conversely, it is trivial by the fundamental properties discussed in section 3 that a pseudo-Einstein ruled real hypersurface M of $M_n(c)$ satisfies the integrability condition (II) and the Ricci condition (I). So it completes the proof. □

Proof of Theorem 2. Now we are going to prove Theorem 2. Let us denote by M' a subset of M_0 consisting of points x in M_0 at which $(\gamma^2 + \delta^2)(x) \neq 0$.

Lemma 4.4. *Under the conditions* (I) *and* (II) *if the mean curvature is non-vanishing, then the subset M' is empty, i.e., the smooth functions $\gamma = g(AU, U)$ and $\delta = g(AU, \phi U)$ vanish identically on M_0.*

Proof. Suppose that M' is not empty. From Lemma 4.3 it follows that $AY = 0$ for any vector field Y in T'. Consequently, we see

$$g(\nabla_X U, Y) = 0, \quad X, Y \in T'$$

on M'.

In fact, differentiating $AY = 0$ with respect to $X \in T'$ covariantly, we get

$$(\nabla_X A)Y + A\nabla_X Y = 0,$$

which yields that

$$g((\nabla_X A)Y, Z) + g(\nabla_X Y, AZ) = 0$$

for any vector fields X and Y in T' and Z in T_0.

For any $X, Y \in T'$ Lemma 4.3 gives $AX = AY = 0$. Moreover, by the condition (I) and the equation of Codazzi, we have for any $X, Y \in T'$ and $Z \in T_0$ in (2.6)

$$0 = hg((\nabla_Z A)X, Y) = hg((\nabla_X A)Y, Z).$$

From this it follows that

$$(\nabla_X A)Y = 0 \ (\mathrm{mod}\ \xi)$$

for any X, Y in T' when the mean curvature is non-vanishing. So the first term in above equation vanishes identically, we have $g(\nabla_X Y, AZ) = 0$ for any vector fields X and Y in T' and Z in T_0. By (4.7), according as $Z = U$ or ϕU, we have

$$\gamma g(\nabla_X Y, U) + \delta g(\nabla_X Y, \phi U) = 0$$

or

$$\delta g(\nabla_X Y, U) - \gamma g(\nabla_X Y, \phi U) = 0$$

for any vector fields X and Y in T' on M'. Since the determinant of the coefficients in the above system of linear equations is given by $\gamma^2 + \delta^2 \neq 0$, we have

$$g(\nabla_X Y, U) = 0, \quad g(\nabla_X Y, \phi U) = 0$$

for any vector fields X and Y in T' on M'. Thus it means that the vector field $\nabla_X U$ is expressed as the linear combination of ξ, U and ϕU on M'.

On the other hand, the equation of Codazzi (2.3) gives us that

$$(\nabla_X A)\xi - (\nabla_\xi A)X = -\frac{c}{4}\phi X$$

for any X in T_0. Then by the direct calculation of the left side of the above relation, we have

$$\begin{aligned} &d\alpha(X)\xi + d\beta(X)U + \frac{1}{4}c\phi X + \alpha\phi AX + \beta\nabla_X U \\ &- A\phi AX - \nabla_\xi(AX) + A\nabla_\xi X = 0 \end{aligned}$$

for any vector field X in T_0. Accordingly, we have

$$d\alpha(X)\xi + d\beta(X)U + \frac{1}{4}c\phi X + \beta\nabla_X U + A\nabla_\xi X = 0 \tag{4.18}$$

for any vector field X in T'. By Lemma 4.2 and the property that $AX = 0$, $X \in T'$ the last term in (4.18) is expressed as the linear combination of ξ, U and ϕU, because $g(A\nabla_\xi X, Y) = 0$ for any vector field $Y \in T'$. This shows that $c\phi X = 0$, a contradiction. It means that the subset M' becomes empty and the smooth functions γ and δ vanish identically on the open subset M_0. It completes the proof. □

Now, from Lemmas 4.2 and 4.4, we get

$$\begin{cases} A\xi = \alpha\xi + \beta U, \\ AU = \beta\xi, \\ A\phi U = 0. \end{cases} \tag{4.19}$$

On the other hand, Theorem 1 implies that M is a pseudo-Einstein ruled real hypersurface of $M_n(c)$, $c\neq 0$. Furthermore, its integral submanifold is a pseudo-Einstein codimension 2 submanifold in $M_n(c)$ with orthonormals C and ξ. Moreover, from (4.19) together with (4.16) and (4.17) we know $A_\xi = 0$ and $A_C = 0$. So it becomes totally geodesic ruled ones.

Conversely, it is evident that a ruled real hypersurface M in $M_n(c)$ in the sense of Kimura [9] and in Ahn, Lee and the second author [1] satisfies the Ricci condition (I) and the condition (II). From this we complete the proof of Theorem 2. □

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