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THE GROUP GENERATED BY AUTOMORPHISMS BELONGING TO GALOIS POINTS OF THE QUARTIC SURFACE

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ABSTRACT. We consider the group G generated by automorphisms belonging to Galois points of S_8, which is the quartic surface with the maximal number of Galois points. We obtain several exact sequences of groups, from which we see that the order of G is $2^5 3^2$. Moreover, we show that S_8 has a structure of C_4-fiber space, where C_4 is the quartic curve with the maximal number of Galois points.

1. INTRODUCTION

Let k be an algebraically closed field of characteristic zero. We fix it as the ground field of our discussion. Let V be a smooth curve or surface of degree d in the projective plane $\mathbb{P}^2$ or in the projective three space $\mathbb{P}^3$ respectively. Let $K = k(V)$ be the rational function field of V. For a point $P \in V$, let $\pi_P : V \cdots \to W$ be a projection of V from P to a line or hyperplane W. The rational map π_P induces the extension of fields $K/k(W)$. The structure of this extension does not depend on the choice of W, but on P, so that we write K_P instead of $k(W)$. We have been studying the structure of this extension using geometrical methods (cf. [4], [5], [10]). The point P is called a Galois point if the extension is Galois. The number of Galois points is finitely many if $d \geq 4$ (cf. [4], [10]). Hence we denote it by $\delta(V)$. An automorphism σ of V is called the one belonging to Galois point P if σ is the automorphism induced by an element of $\mathrm{Gal}(K/K_P)$. It is not only an automorphism of V over W but also a projective transformation of V (cf. [10]).

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Let $(X:Y:Z)$ [resp. $(X:Y:Z:W)$] be homogeneous coordinates on $\mathbb{P}^2$ [resp. $\mathbb{P}^3$]. Let C_4 [resp. S_8] be the curve [resp. surface] given by the equation

$$YZ^3+X^4+Y^4=0 \text{ [resp. } F(X,Y,Z,W)=XY^3+ZW^3+X^4+Z^4=0].$$

These varieties have the following special properties, which characterize them (cf. [4], [10]).

Theorem 0. *Let C [resp. S] be a smooth quartic curve [resp. surface]. Then we have that $\delta(C)\le 4$ [resp. $\delta(S)\le 8$]. Moreover $\delta(C)=4$ [resp. $\delta(S)=8$] if and only if C [resp. S] is projectively equivalent to C_4 [resp. S_8].*

Therefore C_4 and S_8 have the maximal number of automorphisms belonging to Galois points. It seems interesting to study the structure of the group generated by these automorphisms. The purpose of this article is to study the group and the structure of S_8. Especially we will obtain a new example for a maximal finite groups of symplectic automorphisms of $K3$ surfaces (cf. [6]).

We use the following notation:

- ζ : a primitive sixth root of unity
- $\langle\cdots\rangle$: the group generated by the elements of the set $\{\cdots\}$
- E : the elliptic curve with an automorphism of order three
- $\mathrm{Aut}(V)$: the automorphism group of V
- $\mathcal{L}(S_8)$: the set of automorphisms of S_8 induced by projective transformations
- Let A_i be a square matrix of size two $(i=1,2)$ and M be of size four such that

$$M=\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

Then we denote M by $A_1\oplus A_2$. Moreover, we denote $A_2\oplus A_1$ by $M^\star$, i.e.,

$$M^\star=\begin{pmatrix} A_2 & 0 \\ 0 & A_1 \end{pmatrix}.$$

2. Statement of results

Let $G(V)$ denote the group generated by the automorphisms belonging to the Galois points on $V = C_4$ or S_8. Since $G(V)$ has an injective representation in $PGL(n,k)$ ($n = 3$ or 4), *we use the same notation of an element of $G(V)$ as the projective transformation induced by it.*

2.1. The case of C_4

From [9, Proposition 5 and Lemma 11], we see easily that the coordinates of four Galois points of C_4 are $P_1 = (0:0:1)$, $P_2 = (0:\zeta:1)$, $P_3 = (0:\zeta^3:1)$ and $P_4 = (0:\zeta^5:1)$. We have the following assertion.

Lemma 1. *If σ_i ($\neq$ id) is an automorphism belonging to the Galois point P_i ($i = 1,\dots,4$), then σ_i (or $\sigma_i{}^2$) has the following representation:*

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2\zeta-1}{3} & \frac{-\zeta-1}{3} \\ 0 & \frac{4\zeta-2}{3} & \frac{\zeta+1}{3} \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2\zeta-1}{3} & \frac{-\zeta+2}{3} \\ 0 & \frac{-2\zeta+4}{3} & \frac{\zeta+1}{3} \end{pmatrix}, \qquad \sigma_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2\zeta-1}{3} & \frac{2\zeta-1}{3} \\ 0 & \frac{-2\zeta-2}{3} & \frac{\zeta+1}{3} \end{pmatrix}.$$

We put

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \rho = \begin{pmatrix} \sqrt{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Of course $\tau = \rho^2$ (in $PGL(3,k)$). Let $G(C) = G(C_4) = \langle \sigma_1, \dots, \sigma_4 \rangle$ and let l be the line $X = 0$. Then each element of $G(C)$ induces an automorphism on l, hence we put $G(l) = \{\sigma|_l \mid \sigma \in G(C)\}$.

Theorem 1. *The group $G(l)$ is isomorphic to the alternating group on four letters and there exist exact sequences of groups*

$$1 \longrightarrow \langle \tau \rangle \longrightarrow G(C) \xrightarrow{r_1} G(l) \longrightarrow 1$$

and

$$1 \longrightarrow \langle \rho \rangle \longrightarrow \mathrm{Aut}(C_4) \xrightarrow{r_2} G(l) \longrightarrow 1,$$

where the map r_i is defined as $r_i(\sigma) = \sigma|_l$ ($i = 1,2$).

2.2. THE CASE OF S_8

From [10, Proposition 2.4 and Theorem 3], we see easily that the coordinates of eight Galois points are $P_1 = (0:0:0:1)$, $P_2 = (0:0:\zeta:1)$, $P_3 = (0:0:\zeta^3:1)$, $P_4 = (0:0:\zeta^5:1)$, $P_5 = (0:1:0:0)$, $P_6 = (\zeta:1:0:0)$, $P_7 = (\zeta^3:1:0:0)$ and $P_8 = (\zeta^5:1:0:0)$.

We have the following assertion.

Lemma 2. *If $\widetilde{\sigma_i}$ ($\neq$ id) is an automorphism belonging to the Galois point P_i ($i = 1, \ldots, 8$), then $\widetilde{\sigma_i}$ (or $\widetilde{\sigma_i}^2$) has the following representation:*

$$\widetilde{\sigma_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \zeta^2 \end{pmatrix}, \quad \widetilde{\sigma_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2\zeta-1}{3} & \frac{-\zeta-1}{3} \\ 0 & 0 & \frac{4\zeta-2}{3} & \frac{\zeta+1}{3} \end{pmatrix},$$

$$\widetilde{\sigma_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2\zeta-1}{3} & \frac{-\zeta+2}{3} \\ 0 & 0 & \frac{-2\zeta+4}{3} & \frac{\zeta+1}{3} \end{pmatrix}, \quad \widetilde{\sigma_4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2\zeta-1}{3} & \frac{2\zeta-1}{3} \\ 0 & 0 & \frac{-2\zeta-2}{3} & \frac{\zeta+1}{3} \end{pmatrix},$$

and

$\widetilde{\sigma_{j+4}} = (\widetilde{\sigma_j})^\star$ $(j = 1, \ldots, 4)$.

We put $\widetilde{\tau} = I \oplus (-I)$, where I is the unit matrix of size two and put $(\widetilde{\tau})^\diamond = (\widetilde{\tau}, \widetilde{\tau})$ in $PGL(4,k) \times PGL(4,k)$.

Let $G(S) = G(S_8) = \langle \widetilde{\sigma_1}, \ldots, \widetilde{\sigma_8} \rangle$ and let l_1 [resp. l_2] be the line given by the equations $X = Y = 0$ [resp. $Z = W = 0$]. Then each element of $G(S)$ induces an automorphism on l_i ($i = 1, 2$). Hence we put $G(l_i) = \{\sigma|_{l_i} \mid \sigma \in G(S)\}$. Moreover we put $\widetilde{G_1} = \langle \widetilde{\sigma_1}, \ldots, \widetilde{\sigma_4} \rangle$ and $\widetilde{G_2} = \langle \widetilde{\sigma_5}, \ldots, \widetilde{\sigma_8} \rangle$. It is clear that $G(C) \cong \widetilde{G_1} \cong \widetilde{G_2}$. Our main results are stated as follows.

Theorem 2. *There exist exact sequences of groups*

$$1 \longrightarrow \langle \widetilde{\tau} \rangle \longrightarrow G(S) \xrightarrow{s_1} G(l_1) \times G(l_2) \longrightarrow 1$$

and

$$1 \longrightarrow \langle (\widetilde{\tau})^{\circ} \rangle \longrightarrow \widetilde{G_1} \times \widetilde{G_2} \xrightarrow{s_2} G(S) \longrightarrow 1,$$

where $s_1(\sigma) = (\sigma|_{l_1}, \sigma|_{l_2})$ *and* $s_2((\alpha_1, \alpha_2)) = \alpha_1 \cdot \alpha_2$. *Especially the order of* $G(S)$ *is* $2^5 3^2$.

We put

$$\Xi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } \Upsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}}\zeta \\ 0 & 0 & \frac{2}{\sqrt{3}}\zeta^2 & -\frac{1}{\sqrt{3}} \end{pmatrix}.$$

It is clear that Ξ and Υ are elements of $\mathcal{L}(S_8)$ and $\Xi^2 = \Upsilon^2 = \mathrm{id}$.

Theorem 3. *The order of* $\mathcal{L}(S_8)$ *is* $2^7 3^2$, *hence it is a solvable group. Moreover, there exists an exact sequence of groups*

$$1 \longrightarrow G(S) \longrightarrow \mathcal{L}(S_8) \longrightarrow H \longrightarrow 1,$$

where H *is the group generated by the cosets* $\Xi G(S)$ *and* $\Upsilon G(S)$, *that is,* H *is isomorphic to the Klein's four group.*

Here we remark that there exists automorphisms of S_8 not belonging to $\mathcal{L}(S_8)$. Indeed the following fact holds true.

Remark 3. Since S_8 is a singular $K3$ surface, the order of $\mathrm{Aut}(S_8)$ is infinite (cf. [8]).

2.3. GEOMETRY OF S_8

First we note that $C_4/\langle \sigma_i \rangle \cong \mathbb{P}^1$, $(i = 1, \dots, 4)$ and $C_4/\langle \tau \rangle \cong E$. There exists a close relation between C_4 and S_8 as we see below. The surface S_8 has the structure of E-fiber space (cf.[10]), in addition to this structure it has also a structure of C_4-fiber space. Let $H_{a,b}$ be the hyperplane given by the equation $aX + bY = 0$, which contains the line l_1. Then the set $\{S_8 \cap H_{a,b}\}$ forms a linear system Λ on S_8. The base points of Λ are just $\{P_1, \dots, P_4\}$. Let f be the rational map associated to Λ. Then, by blowing up these points, we obtain the surface $\widetilde{S_8}$ and the morphism $\widetilde{f} : \widetilde{S_8} \longrightarrow \mathbb{P}^1$.

Theorem 4. *The fibration $\widetilde{f} : \widetilde{S}_8 \longrightarrow \mathbb{P}^1$ has the following properties*:

(1) *There exist four singular fibers, each of which consists of four smooth rational curves meeting at one point with distinct tangents.*

(2) *Except the singular fibers, each fiber is isomorphic to C_4 .*

Moreover S_8 has the following structure. The automorphism $\widetilde{\tau}$ has eight fixed points that are just Galois points. Blowing up these points, we obtain the surface $\widehat{S}_8$ and the automorphism $\widehat{\tau}$ induced by $\widetilde{\tau}$. The surface $T = \widehat{S}_8/\widehat{\tau}$ is a Kummer surface $Km(E \times E)$ (cf. [2]). Clearly T has an elliptic fibration $\bar{f} : T \longrightarrow \mathbb{P}^1$ with four singular fibers, which are of type I_0^* in the sense of Kodaira's classification table in [3]. Except the singular fibers, each fiber is isomorphic to E.

Finally we mention one more special property of S_8.

Remark 4. It is known that there exist at most 64 lines on smooth quartic surfaces (see, [7]). Moreover, an example of quartic surface with 64 lines is given in [1, p. 33], which coincides with our S_8.

3. Proofs

A generator of $\mathrm{Gal}(K/K_{P_1})$ is easily found, which coincides with σ_1 in Lemma 1 or $\widetilde{\sigma_1}$ in Lemma 2, corresponding to the case of the curve or the surface. However, it is little difficult to find generators for the other Galois points, so that we use the following lemma. The proof of it may be clear from the definition.

Lemma 5. *A projective transformation M belongs to some Galois point P_i if and only if M satisfies the following three conditions*:

(1) $M(P_i) = P_i$.

(2) $M(l) = l$, *for each line l passing through P_i.*

(3) $M \in \mathcal{L}(V)$, *where $V = C_4$ or S_8.*

First, we prove Theorem 1. Since $\sigma \in G(C)$ maps a Galois point to a Galois one, it induces a permutation of the four points. Hence we get the injective representation $\phi : G(l) \hookrightarrow \mathfrak{S}_4$, where $\mathfrak{S}_4$ is the symmetric group on four letters . Indeed we have that $\phi(r_1(\sigma_1)) = (243)$ and $\phi(r_1(\sigma_3)) = (142)$. Since we have

$$\sigma_1^{-1}\sigma_3\sigma_1 = \sigma_1^2\sigma_3\sigma_1 = \sigma_4 \quad \text{and} \quad \sigma_3^{-1}\sigma_1\sigma_3 = \sigma_3^2\sigma_1\sigma_3 = \sigma_2,$$

the group $G(C)$ is generated by σ_1 and σ_3.

Moreover, we have the following relations:

$(\sigma_1)^3 = (\sigma_3)^3 = \mathrm{id}$ and
$(\sigma_1\sigma_3)^3 = (\sigma_1\sigma_3^2)^2 = (\sigma_1^2\sigma_3)^2 = (\sigma_1^2\sigma_3^2)^3 = (\sigma_3\sigma_1)^3 = (\sigma_3\sigma_1^2)^2 = (\sigma_3^2\sigma_1)^2 = (\sigma_3^2\sigma_1^2)^3 = \tau$.

We notice that τ is commutable with each element of $G(C)$, therefore we have that

$$G(C)/\langle\tau\rangle = \{\mathrm{id},\ \sigma_1,\ \sigma_3,\ \sigma_1^2,\ \sigma_3^2,\ \sigma_1\sigma_3,\ \sigma_1\sigma_3^2,\ \sigma_1^2\sigma_3,\ \sigma_1^2\sigma_3^2,\ \sigma_3\sigma_1,\ \sigma_3\sigma_1^2,\ \sigma_3^2\sigma_1,\ \sigma_3^2\sigma_1^2,\ \sigma_1\sigma_3\sigma_1,\ \sigma_3\sigma_1\sigma_3\}.$$

Computing the products of matrices, we obtain that

$$\sigma_3\sigma_1^2 = \tau\sigma_1\sigma_3^2,\ \sigma_3^2\sigma_1 = \tau\sigma_1^2\sigma_3,\ \sigma_1\sigma_3\sigma_1 = \sigma_3\sigma_1\sigma_3.$$

Therefore, we conclude that

$$G(C)/\langle\tau\rangle = \{\mathrm{id},\ \sigma_1,\ \sigma_3,\ \sigma_1^2,\ \sigma_3^2,\ \sigma_1\sigma_3,\ \sigma_1\sigma_3^2,\ \sigma_1^2\sigma_3,\ \sigma_1^2\sigma_3^2,\ \sigma_3\sigma_1,\ \sigma_3^2\sigma_1^2,\ \sigma_1\sigma_3\sigma_1\}.$$

Thus, we obtain the first exact sequence. Since each automorphism σ of the curve C_4 is the restriction of some projective transformation, σ maps a Galois point to some Galois one. If σ is in the kernel of the restriction map r_2, then it fixes each Galois point. Since $\sigma(C_4) = C_4$, σ has the representation as ρ^i $(i = 0, \ldots, 3)$. Thus, we complete the proof of Theorem 1.

Before proceeding with the proof of Theorem 2, we prove Lemma 2. To find a generator of $\mathrm{Gal}(K/K_{P_i})$ $(i = 2, 3, 4)$, we observe the following projective transformation:

$$T_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}}\zeta^{2i-3} \\ 0 & 0 & \frac{2}{\sqrt{3}}\zeta^{4i-6} & -\frac{1}{\sqrt{3}} \end{pmatrix},\ (i = 2, 3, 4).$$

The transformation T_i has the following properties: $T_i^{-1} = T_i$, $T_i(P_1) = P_i$ and $T_i \in \mathcal{L}(S_8)$. Hence we have that $\widetilde{\sigma_i} = T_i^{-1}\widetilde{\sigma_1}T_i$. By the similar way we obtain $\widetilde{\sigma_i}$ $(i = 5, 6, 7, 8)$.

Now we prove Theorem 2. Clearly s_1 is surjective, so it is sufficient to prove that $\ker s_1 = \langle\widetilde{\tau}\rangle$. If $s_1(\sigma) = \mathrm{id}$, then $\sigma|_{l_1}$ and $\sigma|_{l_2}$ are identities. Then, by Theorem 1, we conclude that $\sigma \in \langle\widetilde{\tau}\rangle$. By definition s_2 is surjective, so we prove that $\ker s_2 = \langle(\widetilde{\tau})^{\circ}\rangle$. Since each element of $\widetilde{G_1}$ and $\widetilde{G_2}$ is commutative in $G(C)$, any element of $G(C)$ can be expressed as a product $\alpha_1 \cdot \alpha_2$, where $\alpha_i \in \widetilde{G_i}$ $(i = 1, 2)$. By the same reasoning above we conclude that $(\alpha_1, \alpha_2) \in \langle(\widetilde{\tau})^{\circ}\rangle$.

Next we prove Theorem 3. First, we prove the former part. Let L_1, L_2 and L be the sets defined by

$$\left\{ \alpha \begin{pmatrix} 1 & \beta\gamma \\ 2\beta^2 & \gamma \end{pmatrix} \;\middle|\; \alpha^4 = 1/9,\ \beta^3 = -1 \text{ and } \gamma^3 = -1 \right\},$$

$$\left\{ \alpha' \begin{pmatrix} 1 & 0 \\ 0 & \beta' \end{pmatrix} \;\middle|\; \alpha'^4 = 1 \text{ and } \beta'^3 = 1 \right\} \text{ and}$$

$$\{\, A_1 \oplus A_2,\ \Xi(A_1 \oplus A_2) \mid A_1, A_2 \in L_1 \cup L_2 \,\}, \text{ respectively.}$$

Then we see $L \subset \mathcal{L}(S_8)$ and $\#L = 2^7 3^2$. Especially, we have $\#\mathcal{L}(S_8) \geq 2^7 3^2$. On the other hand, we can prove $\#\mathcal{L}(S_8) \leq 2^7 3^2$ as follows. If $\sigma \in \mathcal{L}(S_8)$, then, for a Galois point P_i, we have that $\sigma(P_i) = P_j$ for some j. Hence σ has one of the following properties:

(a) $\sigma(l_1) = l_1$ and $\sigma(l_2) = l_2$.

(b) $\sigma(l_1) = l_2$ and $\sigma(l_2) = l_1$.

Now let L_a [resp. L_b] denote the subset of $\mathcal{L}(S_8)$ consisting of elements with the property (a) [resp. (b)]. Let H_1 [resp. H_2] be the hyperplane given by the equation $Y = 0$ [resp. $W = 0$]. Then, noting that $D_i := S_8 \cap H_i$ $(i = 1, 2)$ is isomorphic to the curve C_4, we infer that if $\sigma \in L_a$, then $\sigma|_{l_i} \in G(l_i)$ $(i = 1, 2)$, since $\sigma|_{D_i} \in \mathrm{Aut}(D_i)$ and from Theorem 1. Thus, noting $\Xi L_b = L_a$, we can define the homomorphism $r : \mathcal{L}(S_8) \to \mathbb{Z}/2\mathbb{Z} \times G(l_1) \times G(l_2)$ as follows:

$$r(\sigma) = \begin{cases} (0 + 2\mathbb{Z},\ \sigma|_{l_1},\ \sigma|_{l_2}), & \text{when } \sigma \in L_a \\ (1 + 2\mathbb{Z},\ (\Xi\sigma)|_{l_1},\ (\Xi\sigma)|_{l_2}), & \text{when } \sigma \in L_b \end{cases}$$

By Theorem 1 we have the following exact sequence:

$$1 \longrightarrow \langle \rho' \rangle \longrightarrow \mathcal{L}(S_8) \xrightarrow{r} \mathbb{Z}/2\mathbb{Z} \times G(l_1) \times G(l_2),$$

where $\rho' = (\sqrt{-1}I) \oplus I$ (I is the unit matrix of size two). Therefore noting that $G(l_i)$ $(i = 1, 2)$ is isomorphic to the alternating group on four letters, we conclude $\#\mathcal{L}(S_8) \leq 2^7 3^2$, and we obtain the former assertion. The proof of the latter one is done as follows. First we recall that Ξ and Υ have order two, and note that $\Xi G(S) \Xi = G(S)$ and $\Upsilon G(S) \Upsilon = G(S)$. Looking at the components of matrices, we see that $\Xi \notin G(S)$. We now prove that $\Upsilon \notin \langle \widetilde{\sigma_1}, \cdots, \widetilde{\sigma_8}, \Xi \rangle$. Suppose the contrary. Then we have a relation that $\lambda \Upsilon = \prod \beta_i$ in $GL(4, k)$, where β_i is $\widetilde{\sigma_1}, \cdots, \widetilde{\sigma_8}$ or Ξ and $\lambda \in k \setminus 0$. Comparing the (i, j) component of both sides, where $1 \leq i, j \leq 2$, we infer that $\lambda = \pm 1$, since λI (I is the unit matrix of size two) is expressed as the product of the following matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & \zeta^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{2\zeta-1}{3} & \frac{-\zeta+2}{3} \\ \frac{-2\zeta+4}{3} & \frac{\zeta+1}{3} \end{pmatrix}.$$

Then, taking the determinant of both sides of the relation $\lambda \Upsilon = \prod \beta_i$, we get the equality $-1 = (\zeta^2)^m$ for some integer m. This is a contradiction, hence we get the exact sequence.

The proof of Theorem 4 is clear. We mention the proof of Remark 4. Choosing three lines from $S_8 \cap \{Z = 0\}$ and forming a divisor D, for example let D be given by the equations $Z = 0$ and $X^3 + Y^3 = 0$. There are four possibilities of the choices. Let g be the morphism $S \longrightarrow \mathbb{P}^1$ associated to the complete linear system $|D|$. The singular fibers of g are D and the curves given by the equations

$$\begin{cases} X & = 0 \\ Z^3 + W^3 & = 0 \end{cases}$$

and

$$\begin{cases} X - \lambda Z & = 0 \\ \lambda Y^3 + W^3 & = 0 \quad \text{where } \lambda^4 + 1 = 0. \end{cases}$$

Similarly, we consider the morphism defined by the other choice of the four lines $S_8 \cap \{Z=0\}$ and observe the singular fibers. Counting the number of the components of singular fibers, we can find 64 lines on S_8. Since the maximum number of lines lying on a quartic surface is 64 (cf. [1] or [7]), the proof of the remark is complete.

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