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On Ricci curvature of CR-submanifolds with rank one totally real distribution

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Abstract

In a recent paper, Bang-yen Chen obtained sharp inequalities between the maximum Ricci curvature and the squared mean curvature for arbitrary submanifolds in real space forms and totally real submanifolds in complex space forms ([6, 7]). In this paper we give sharp inequalities between the maximum Ricci curvature and the squared mean curvature for arbitrary submanifolds in complex space form. Moreover we investigate CR-submanifolds in complex space forms and in the nearly Kaehler six-sphere which realize the equality case of the inequalities mentioned above.

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1 Introduction

Let M^n be an n-dimensional submanifold of an m-dimensional manifold $\tilde{M}^m$. Denote by h the second fundamental form of M^n in $\tilde{M}^m$. Then the mean curvature vector $\vec{H}$ of the immersion is given by $\vec{H} = \frac{1}{n}\,\mathrm{trace}\,h$. A submanifold is said to be minimal if its mean curvature vector vanishes identically. Denote by D the linear connection induced on the normal bundle $T^{\perp}M^n$ of M^n in $\tilde{M}^m$, by R and $\tilde{R}$ the Riemann curvature tensors of M and of $\tilde{M}^m$ respectively, and by R^D the curvature tensor of the normal connection D. Then the equation of Gauss and Ricci are given respectively by

$$R(X,Y)Z \;=\; \left\langle A_{h(Y,Z)}X, W\right\rangle - \left\langle A_{h(X,Z)}Y, W\right\rangle + \tilde{R}(X,Y)Z \tag{1.1}$$

$$R^D(X,Y;\xi,\eta) = \tilde{R}(X,Y;\xi,\eta) + \langle [A_\xi, A_\eta](X), Y\rangle \tag{1.2}$$

for vectors X, Y, Z, W tangent to M and ξ, η normal to M, where A is the shape operator. For the second fundamental form h, we define the covariant derivative $\bar{\nabla}h$ of h with respect to the connection on $TM \oplus T^{\perp}M$ by

$$(\bar{\nabla}_X h)(Y,Z) = D_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{1.3}$$

The equation of Codazzi is given by

$$(\tilde{R}(X,Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z). \tag{1.4}$$

The Ricci tensor S and the scalar curvature τ at a point $p \in M^n$ are given respectively by $S(X,Y) = \sum_{i=1}^n \langle R(e_i,X)Y, e_i\rangle$ and $\tau = \sum_{i=1}^n S(e_i,e_i)$, where $\{e_1,\ldots,e_n\}$ is an orthonormal basis of the tangent space T_pM^n.

Let $\overline{Ric}$ denote the maximum Ricci curvature function on M^n defined by

$$\overline{Ric}(p) = \max\{S(X,X) | X \in T_p^1 M^n\}, \quad p \in M^n, \tag{1.5}$$

where $T_p^1 M^n$ is the unit tangent vector space of M^n at p.

When M is a submanifold of an almost Hermitian manifold $\tilde{M}$, a subspace V of T_pM is called *totally real* if JV is contained in the normal space $T_p^{\perp}M$ of M at p. The submanifold M is called *totally real* if each tangent space of M is totally real; and M is called a CR-submanifold if there exists a differential holomorphic distribution $\mathcal{H}$ on M such that the orthogonal complement $\mathcal{H}^{\perp}$ of $\mathcal{H}$ in TM is a totally real distribution ([2]). A CR-submanifold is called *proper* if it is neither totally real (i.e., $\mathcal{H}^{\perp}=TM$) nor holomorphic (i.e., $\mathcal{H}=TM$).

Let M be a $(2n+1)$-dimensional CR-submanifold with $\dim \mathcal{H}^{\perp} = 1$ and we put $\mathcal{H}^{\perp} = \mathrm{Span}\{e_{2n+1}\}$. We denote the tangential component of JX by PX. Then $(P, e_{2n+1}, \omega^1, g)$ defines an almost contact metric structure on (M, g), where $\omega^1(X) := g(e_{2n+1}, X)$ and g is an induced metric ([16]). M is said to be *normal* if the tensor field S_M defined by

$$S_M(X,Y) = [PX, PY] + P^2[X,Y] - P[X,PY] - P[PX,Y] + 2d\omega_1(X,Y)e_{2n+1} \tag{1.6}$$

vanishes ([1]).

For the maximum Ricci curvature and the squared mean curvature H^2 for n-dimensional submanifolds in m-dimensional complex space forms $\tilde{M}^m(4c)$ of constant holomorphic sectional curvature, we have the following:

$$\overline{Ric} \leq (n+2)c + \frac{n^2}{4}H^2 \qquad for \quad c \geq 0, \tag{1.7}$$

$$\overline{Ric} \leq (n-1)c + \frac{n^2}{4}H^2 \qquad for \quad c \leq 0. \tag{1.8}$$

In case $c < 0$ and $dimM = 3$, the inequality is known as Chen's basic inequality (cf. [8]). In [8] Chen has completely classified 3-dimensional proper CR-submanifold which satisfy the equality case of (1.8).

In this article, we study proper CR-submanifolds with $\dim \mathcal{H}^{\perp} = 1$ of complex space forms satifying the equality case of the inequalities (1.7) or (1.8). In particular, in case $c < 0$, we are able to establish the explicit representation of such submanifolds which are normal in an anti-de Sitter space time via Hopf's fibration, and in case $c \geq 0$, classify 3-dimensional normal CR-submanifolds satisfying the equality case of (1.7). The inequality (1.8) also holds for arbitrary submanifolds in real space forms $R^m(c)$ of constant sectional curvature c, too ([6]). In the last section, we investigate 3-dimensional CR-submanifolds in the nearly Kaehler 6-sphere which realize the equality case of the inequality.

2 Main Results

Theorem 1 *Let M be a 3-dimensional CR-submanifold with* $\dim \mathcal{H}^{\perp} = 1$ *in* $\tilde{M}^m(4c)$, $c \in \{0, 1\}$ *satisfying the equality case of* (1.7). *Then M is normal if and only if it is one of the following.*

(1) *M is an open portion of a product submanifold* $\mathbf{C} \times \mathbf{R}$ *in* $\mathbf{C}^{m-1} \times \mathbf{C}$,

(2) *M is an open portion of a geodesic sphere of radius* $\frac{\pi}{4}$ *in* $\mathbf{C}P^2(4)$.

Consider the complex number $(m+1)$-space $\mathbf{C}_1^{m+1}$ endowed with the pseudo-Euclidean metric g_0 given by $g_0 = -dz_0 d\bar{z}_0 + \sum_{j=1}^m dz_j d\bar{z}_j$, where $\bar{z}_k$ denotes the complex conjugate of z_k. On $\mathbf{C}_1^{m+1}$ we define $(z,w) = -z_0\bar{w}_0 + \sum_{k=1}^{m} z_k\bar{w}_k$. Put $H_1^{2m+1}(-r) = \{z = (z_0.z_1.\dots.z_m) \in \mathbf{C}_1^{m+1} : (z,z) = -r^2\}$, It is known that $H_1^{2m+1}(-1)$ together with the induced metric g is a pseudo-Riemannian manifold of constant sectional curvature -1, which is known as an anti-de Sitter space time.

We put $H_1^1 = \{\lambda \in \mathbf{C} : \lambda\bar{\lambda} = 1\}$. The quotient space $H_1^{2m+1}(-1)/_\sim$, under the identification induced from the action, is the complex hyperbolic space $\mathbf{C}H^m(-4)$ with constant holomorphic sectional curvature -4. The almost complex structure J on $\mathbf{C}H^m(-4)$ is induced from the canonical almost complex structure J on $\mathbf{C}_1^{m+1}$, the multiplication by i, via the totally geodesic fibration: $\pi : H_1^{2m+1}(-1) \to \mathbf{C}H^m(-4)$.

We obtain the following general property.

Theorem 2 *Let $x : M \to \mathbf{C}H^m(-4)$ be a $(2n+1)$-dimensional CR-submanifold with $\dim\mathcal{H}^\perp=1$. If M satisfies the equality case of* (1.8), *then $\vec{H}$ is parallel i.e., $D\vec{H}=0$.*

A submanifold is said to be *linearly full* in $\mathbf{C}H^m(-4)$ if it does not lie in any totally geodesic complex submanifold of $\mathbf{C}H^m(-4)$.

Theorem 3 *Let U be a domain of $\mathbf{R}^{2n}(n > 1)$. Define $z : \mathbf{R}^2 \times U \to \mathbf{C}_1^{m+1}$ by*

$$z(s,t,x_1,x_2,\dots,y_1,y_2) = (g(x_1,\dots,y_2)e^{is}, \sqrt{\frac{1}{2n-2}}e^{it}), \tag{2.1}$$

where $|g|^2 = -\frac{2n-1}{2n-2}$ and $g(x_1,\dots,y_2)e^{is}$ is a CR-submanifold of $\mathbf{C}_1^m$ such that the unit totally real vector field is $\sqrt{\frac{2n-2}{2n-1}}\frac{\partial}{\partial s}$. Then $(z,z) = -1$ and the image $z(\mathbf{R}^2 \times U)$ is invariant under the group H_1^1. Moreover the quotient space $z(\mathbf{R}^2 \times U)/_\sim$ is a $(2n+1)$-dimensional CR-submanifold with $\dim\mathcal{H}^\perp = 1$ which satisfies the equality case of (1.8) *under the condition that the shape operator A_η with respect to the unit vector field $\eta \in \mathcal{H}^\perp$ has constant principal curvatures.*

Conversely, in case $n > 1$ and $m > n+1$, up to rigid motions of $\mathbf{C}H^m(-4)$, every linearly full $(2n+1)$-dimensional CR-submanifold with $\dim\mathcal{H}^\perp = 1$ which satisfies the equality case of (1.8) *under the condition that the shape operator A_η with respect to the unit vector field $\eta \in \mathcal{H}^\perp$ has constant principal curvatures is obtained in such way.*

3 The proof of Theorem 1

For arbitrary n-dimensional submanifolds M^n in complex space forms $\tilde{M}^m(4c)$, we have the following.

Proposition 4 *If M^n is an n-dimensional submanifold of complex space forms $\tilde{M}^m(4c)$, then the maximum Ricci curvature $\overline{Ric}$ of M^n satifies the following inequalities:*

$$\overline{Ric} \le (n+2)c + \frac{n^2}{4}H^2 \qquad for \quad c \ge 0, \tag{3.1}$$

$$\overline{Ric} \le (n-1)c + \frac{n^2}{4}H^2 \qquad for \quad c \le 0. \tag{3.2}$$

The equality case of (3.1) *holds at a point $p \in M$ if and only if there exists an orthonormal basis $e_1,\dots,e_{2m}$ at p such that $e_1,\dots,e_n$ are tangent to M and*

$$(a)\overline{Ric} = S(e_n,e_n), \quad c\sum_{i=1}^{n-1}\langle Je_i, e_n\rangle^2 = c, \tag{3.3}$$

$$(b) h^s_{in} = 0, \quad \sum_{i}^{n-1} h^s_{ii} = h^s_{nn} := \mu_s, \tag{3.4}$$
$$where \quad 1 \leq i \leq n-1 \quad and \quad n+1 \leq s \leq 2m.$$

The equality case of (3.2) holds at a point $p \in M$ if and only if there exists an orthonormal basis $e_1, \ldots, e_{2m}$ at p such that $e_1, \ldots, e_n$ are tangent to M and

$$(a) \overline{Ric} = S(e_n, e_n), \quad c \sum_{i=1}^{n-1} \langle Je_i, e_n \rangle^2 = 0, \tag{3.5}$$

$$(b) h^s_{in} = 0, \quad \sum_{i}^{n-1} h^s_{ii} = h^s_{nn}, \tag{3.6}$$
$$where \quad 1 \leq i \leq n-1 \quad and \quad n+1 \leq s \leq 2m.$$

Proof: Put $\delta = \tau - n(n-1)c - \frac{n^2}{2} H^2 - 3c\|P\|^2$, where $\|P\|^2 := \sum_{i,j=1}^{n} \langle e_i, Je_j \rangle^2$. Then from the Gauss equation(1.1), we have $n^2 H^2 = 2(\delta + \|h\|^2)$, where $\|h\|^2$ is the squared norm of the second fundamental form. In a similar way to the proof of theorem 1 in [7], we have $S(e_n, e_n) \leq (n-1)c + \frac{n^2}{4} H^2 + 3c \sum_{i=1}^{n-1} \langle Je_i, e_n \rangle^2$. ∎

First we recall the following result on CR-submanifolds from [4].

Lemma 5 *Let M be a CR-submanifold of a Kaehler manifold $\tilde{M}$. Denote by $T^\perp M = J\mathcal{H}^\perp \oplus \nu$ the orthogonal decomposition of the normal bundle, where ν is a complex subbundle of $T^\perp M$. We have*

$$\langle \nabla_U Z, X \rangle = \langle J(A_{JZ} U), X \rangle, \tag{3.7}$$
$$A_{J\xi} X = -A_\xi JX, \tag{3.8}$$

for vector fields Z in $\mathcal{H}^\perp$, ξ in ν, U in TM and vector field X in the holomorphic distribution $\mathcal{H}$.

Proof of Theorem 1

Case 1: $c = 0$. In this case we consider two cases for a unit vector field $\eta \in \mathcal{H}^\perp$ to be either $\eta \in L$ or $\eta \notin L$, where L is the orthogonal complement of $\{e_3\}$ in T_pM and e_3 satisfies $\overline{Ric} = S(e_3, e_3)$.

First, we consider the case where $\eta \notin L$. If we choose e_4 in such way that $J\eta = e_4$, then we obtain that $A_{Je_3} e_3 = \mu_4 e_3$ and $\eta = e_3$ is a parallel vector field in the same way as lemma 8 in [8]. In general, for a $(2n+1)$-dimensional CR-submanifold of $\tilde{M}^m(4c)$ which satisfies the condition that $A_{Je_{2n+1}} e_{2n+1} = \mu_{2n+2} e_{2n+1}$ and e_{2n+1} is parallel, we have the following relation ([14]).

$$-2c\langle PX, Y \rangle + 2\langle A_{2n+2} P A_{2n+2} X, Y \rangle = (X\mu_{2n+2}) \langle e_{2n+1}, Y \rangle \tag{3.9}$$
$$-(Y\mu_{2n+2}) \langle e_{2n+1}, X \rangle + \mu_{2n+2} \langle P A_{2n+2} X, Y \rangle - \mu_{2n+2} \langle P A_{2n+2} Y, X \rangle,$$

where $A_{2n+2} := A_{e_{2n+2}} = A_{Je_{2n+1}}$.

We may assume that $\{e_1, e_2, e_3\}$ diagonalize the shape operator A_{Je_3} such that $Je_1 = e_2$, $A_{Je_3} e_1 = \alpha e_1$ and $A_{Je_3} e_2 = \beta e_2$. From (3.9) and proposition 4, we have $2\alpha\beta = \mu_4(\alpha + \beta)$ and $\alpha + \beta = \mu_4$. This implies that $\alpha = \beta = 0$. Then by applying (3.7), we have

$$< \nabla_{e_1} e_2 - \nabla_{e_2} e_1, e_3 > = < \nabla_{e_1} e_3, e_2 > - < \nabla_{e_2} e_3, e_1 >$$
$$= < J(A_{e_3} e_1), e_2 > - < J(A_{e_3} e_2), e_1 > = 0. \tag{3.10}$$

Therefore $\mathcal{H}$=Span$\{e_1, e_2\}$ is integrable. Hence M is a CR-product(cf. [2]) by proposition 4 and theorem 9.3 in [4]. Since the integral curve of e_3 is an open portion of real line $\mathbf{R}$ and M is normal, M is an open portion of $\mathbf{C} \times \mathbf{R}$ in $\mathbf{C}^m$.

Next, we consider the case where $\eta \in L$. We may assume that $\eta = e_1$ and $J\eta = e_4$. It follows from (3.8) and proposition 4 that $A_\xi = 0$ for $\xi \in \nu$.

It is known that M is normal if and only if $PA_{Je_1} = A_{Je_1}P$ ([1]). From this fact and proposition 4, we have $A_{Je_1}e_1 = 0$, $A_{Je_1}e_2 = \mu_4 e_2$, and $A_{Je_1}e_3 = \mu_4 e_3$. Thus we find

$$\begin{aligned}(\bar{\nabla}_{e_2}h)(e_3, e_1) &= - < \nabla_{e_2}e_1, e_3 > \mu_4 Je_1,\\ (\bar{\nabla}_{e_3}h)(e_2, e_1) &= - < \nabla_{e_3}e_1, e_2 > \mu_4 Je_1.\end{aligned} \tag{3.11}$$

The equation of Codazzi and (3.11) implies that $< \nabla_{e_2}e_3 - \nabla_{e_3}e_2, e_1 > \mu_4 = 0$. If we put $W = \{p \in M : \mu_4(p) \neq 0\}$, the above relation yields $< \nabla_{e_2}e_3 - \nabla_{e_3}e_2, e_1 >= 0$ on W, which implies that $\mathcal{H} = \mathrm{Span}\{e_2, e_3\}$ is integrable on W. Hence W is an open portion of a CR-product $\mathbf{C} \times \mathbf{R}$ and $\mu_4 = 0$. It is a contradiction. Consequently we conclude that W is empty and M is an open portion of a totally geodesic submanifold $\mathbf{C} \times \mathbf{R}$.

Case 2: $c = 1$. In this case, a unit vector field $\eta \in \mathcal{H}^\perp$ lies in L. Similarly to the proof in case 1, we have $A_\xi = 0$ for $\xi \in \nu$. Hence, using

$$-A_{Je_3}X + D_X(Je_3) = \tilde{\nabla}_X(Je_3) = J(\nabla_X e_3) + Jh(X, e_3), \tag{3.12}$$

$D_X(Je_3) = 0$ for any $X \in TM$.

Therefore, M is contained in a totally geodesic $\tilde{M}^2(4)$. Since M is normal, we have $A_{Je_1}e_1 = 0$, $A_{Je_1}e_2 = \mu_4 e_2$, and $A_{Je_1}e_3 = \mu_4 e_3$. This implies that M is a Hopf hypersurface. By virtue of theorem 8 in [5], M is an open portion of a geodesic sphere of radius $\frac{\pi}{4}$ of $\tilde{M}^2(4)$. This completes the proof of theorem 1.

In the same way as in the proof in case 1, by using (3.9), we obtain the following result.

Proposition 6 *Let M be a $(2n+1)$-dimensional CR-submanifold with $\dim \mathcal{H}^\perp = 1$ in $\mathbf{C}^m$ satisfying the equality case of* (3.1). *Then $\overline{Ric} = S(e_{2n+1}, e_{2n+1})$ for $e_{2n+1} \in \mathcal{H}^\perp$ if and only if M is an open portion of a product submanifold $N^{2n} \times \mathbf{R}$ in $\mathbf{C}^{m-1} \times \mathbf{C}$, where N^{2n} is a Kaehler submanifold in $\mathbf{C}^{m-1}$.*

4 The proof of Theorem 2

In the same way as [8, 14], we have the following result using (3.8).

Lemma 7 *Let $x : M \to \mathbf{C}H^m(-4)$ be a $(2n+1)$-dimensional CR-submanifold with $\dim\mathcal{H}^\perp$=1 . If M satisfies the equality case of* (3.2), *then the mean curvature vector $\vec{H}$ lies in $J\mathcal{H}^\perp$.*

Proof of Theorem 2

Let $\{e_1, \ldots, e_{2m}\}$ be an orthonormal frame field on M mentioned in proposition 4 such that e_{2n+2} is parallel to the mean curvature vector field and $\{e_1, \ldots, e_{2n+1}\}$ diagonalize the shape operator A_{2n+2} with respect to e_{2n+2} and moreover $e_{2l} = Je_{2l-1} (l = 1, \ldots, n)$. Under the hypothesis, we have $\vec{H} \in J\mathcal{H}^\perp$ from Lemma 7. Without loss of generality we may assume that $Je_{2n+1} = e_{2n+2}$. Then, in the same way as the proof of lemma 5.4 in [14] we obtain that Je_{2n+1} is a parallel normal vector field i.e., $D(Je_{2n+1}) = 0$. By choosing $Y = e_{2n+1}$ in (3.9), we get

$$X\mu_{2n+1} = \omega^1(X)e_{2n+1}\mu_{2n+2}. \tag{4.1}$$

Now, by differentiating (4.1) and using $(\nabla_Y \omega^1)(X) = \langle PA_{2n+2}Y, X\rangle$, we obtain

$$\begin{aligned}&Y(e_{2n+1}\mu_{2n+2})\omega^1(X) - X(e_{2n+1}\mu_{2n+2})\omega^1(Y)\\ &+e_{2n+1}\mu_{2n+2}\langle(PA_{2n+2} + A_{2n+2}P)Y, X\rangle = 0.\end{aligned} \tag{4.2}$$

By choosing $Y = e_{2n+1}$ in (4.2), we have $X(e_{2n+1}\mu_{2n+2}) = e_{2n+1}(e_{2n+1}\mu_{2n+2})\omega^1(X)$. Combining this and (4.2) yield

$$(e_{2n+1}\mu_{2n+2})\langle(PA_{2n+2} + A_{2n+2}P)Y, X\rangle = 0. \tag{4.3}$$

By choosing $X = \sum_{l=1}^n Je_{2l-1}$ and $Y = \sum_{l=1}^n e_{2l-1}$ in (4.3), we have $e_{2n+1}\mu_{2n+2}\mathrm{trace}\,(A_1^{2n+1}) = 0$. If M is nonminimal, we have $e_{2n+1}\mu_{2n+2} = 0$, since $\mathrm{trace}\,(A_1^{2n+1}) \neq 0$. Therefore this implies that μ_{2n+2} is constant. Hence we obtain $D\vec{H} = 0$. ∎

5 The proof of Theorem 3

Let $\{e_1, \ldots, e_{2n+1}\}$ be an orthonormal basis mentioned in the proof of theorem 2. From now on we shall assume that all principal curvatures of A_{2n+2} are constant. Then we have the following lemmas.

Lemma 8 *Let $\{e_1, \ldots, e_{2n}\}$ be an orthonormal frame field of $\mathcal{H}$ with $A_{2n+2}e_i = \lambda_i e_i$. Then we have for any $i \in \{1, \ldots, 2n\}$,*

$$\sum_{j=1,\ \lambda_j\neq\lambda_i}^{2n} \left(\frac{-1+\lambda_i\lambda_j}{\lambda_i - \lambda_j}(1 + 2\langle Pe_i, e_j\rangle)^2 + \frac{1}{\lambda_i - \lambda_j}\sum_{2n+3}^{m} \left(h_{ii}^r h_{jj}^r - (h_{ij}^r)^2\right)\right) = 0. \tag{5.1}$$

where $h_{ij}^r = \langle A_r e_i, e_j\rangle$.

Proof: The proof is in the same way as the proof of lemma 2 in [3]. ∎

Lemma 9 *A_{2n+2} has at most three distinct principal curvatures.*

Proof: The proof is separated into two cases.

Case 1: $\mu_{2n+2}^2 = 4$. We denote by $\sigma(\mathcal{H})$ the spectrum of $A_{2n+2}|\mathcal{H}$, and for $\lambda \in \sigma(\mathcal{H})$ by T_λ the subbundle of $\mathcal{H}$ formed by the eigenspace corresponding to the eigenvalue λ. From (3.9) we obtain for $\lambda \in \sigma(\mathcal{H})$, $X \in T_\lambda$,

$$(2\lambda - \mu_{2n+2})A_{2n+2}PX = (-2 + \lambda\mu_{2n+2})PX. \tag{5.2}$$

Assume that there exists $\lambda \in \sigma(\mathcal{H})$ with $\lambda \neq \frac{\alpha}{2}$. We obtain from (5.2) that $A_{2n+2}PX = \frac{\alpha}{2}PX$ for $X \in T_\lambda$. Hence $\frac{\alpha}{2}$ is an eigenvalue. We denote by E_j the eigenvectors corresponding to $\lambda_j \neq \frac{\alpha}{2}$.

By the way, we have $\tilde{R}(X, Y; Je_{2n+1}, \xi) = R^D(X, Y; Je_{2n+1}, \xi) = 0$ for any $\xi \in \nu$ by virtue of $D(Je_{2n+2}) = 0$. Hence, the equation of Ricci yields

$$[A_{2n+2}, A_\xi] = 0. \tag{5.3}$$

Relation (3.8) and (5.3) imply that $\langle A_r E_j, E_j\rangle \langle A_r X, X\rangle - \langle A_r E_j, X\rangle^2 = 0$ for eigenvector $X \in T_{\frac{\alpha}{2}}$. Hence we have

$$\sum_{j=1,\ \lambda_j\neq\frac{\alpha}{2}}^{2n} \frac{-1+\frac{\alpha}{2}\lambda_j}{\frac{\alpha}{2} - \lambda_j}(1 + 2\langle PX, E_j\rangle^2) = -\frac{\alpha}{2}\sum_{j=1,\ \lambda_j\neq\frac{\alpha}{2}}^{2n} (1 + 2\langle PX, E_j\rangle^2) \neq 0, \tag{5.4}$$

which contradicts (5.1). Therefore we obtain that $\sigma(\mathcal{H}) = \{\frac{\alpha}{2}\}$.

Case 2: $\mu_{2n+2}^2 \neq 4$.

Assume that $\#\sigma(\mathcal{H}) \geq 2$. Then we have the following orthogonal decomposition:

$$\mathcal{H} = T_{\alpha_1} \oplus JT_{\alpha_1} \oplus \cdots T_{\alpha_s} \oplus JT_{\alpha_s} \oplus T_\lambda \oplus T_{\mu_{2n+2}-\lambda}, \tag{5.5}$$

where JT_{α_i} is the eigenspace corresponding to $\frac{-2+\alpha_i\mu_{2n+2}}{2\alpha_i-\mu_{2n+2}}$, and $\lambda = \frac{\mu_{n+2}+\sqrt{\mu_{n+2}^2-4}}{2}$, moreover T_λ and $T_{\mu_{2n+2}-\lambda}$ are J-invariant, and $\lambda \neq \alpha_j$ from (5.2). We may assume that we can choose the eigenvalue $\beta \in \sigma(\mathcal{H})$ with $\beta > 0$ and that there are no further eigenvalues between β and $\frac{1}{\beta}$. Hence, for all eigenvalues $\gamma \in \sigma(\mathcal{H})$, we have

$$\frac{-1+\beta\gamma}{\beta-\gamma} \leq 0. \tag{5.6}$$

On the other hand by virtue of (3.8) and (5.3), we get

$$\sum_{j=1,\ \lambda_j\neq\alpha_l}^{2n} \sum_{r=2n+3}^{m} \frac{1}{\alpha_l-\lambda_j}\left(\langle A_rX,X\rangle\langle A_re_j,e_j\rangle - \langle A_re_j,X\rangle^2\right) = 0 \tag{5.7}$$

for each eigenvector X corresponding to α_l ($l = 1,\ldots,s$), and moreover, for each eigenvector Y corresponding to λ

$$\begin{aligned}&\sum_{j=1,\ \lambda_j\neq\lambda}^{2n} \sum_{r=2n+3}^{m} \frac{1}{\lambda-\lambda_j}\left(\langle A_rY,Y\rangle\langle A_re_j,e_j\rangle - \langle A_rY,e_j\rangle^2\right)\\ &= \sum_{r=2n+3}^{m}\left(\frac{1}{2\lambda-\mu_{2n+2}}\langle A_rY,Y\rangle \sum_{j=1,\ \lambda_j\neq\lambda}^{t}\left\langle A_r\tilde{E}_j,\tilde{E}_j\right\rangle\right) = 0,\end{aligned} \tag{5.8}$$

where $\tilde{E}_j$ are eigenvectors corresponding to $\mu_{2n+2}-\lambda$ and $t = \dim T_{\mu_{2n+2}-\lambda}$. Similarly, for each eigenvector Z corresponding to $\mu_{2n+2}-\lambda$

$$\begin{aligned}&\sum_{j=1,\ \lambda_j\neq\mu_{2n+2}-\lambda}^{2n} \sum_{r=2n+3}^{m} \frac{1}{\mu_{2n+2}-\lambda-\lambda_j}\left(\langle A_rZ,Z\rangle\langle A_re_j,e_j\rangle - \langle A_rZ,e_j\rangle^2\right)\\ &= \sum_{r=2n+3}^{m}\left(\frac{1}{\mu_{2n+2}-2\lambda}\langle A_rZ,Z\rangle \sum_{j=1,\ \lambda_j\neq\mu_{2n+2}-\lambda}^{s}\langle A_r\bar{E}_j,\bar{E}_j\rangle\right) = 0,\end{aligned} \tag{5.9}$$

where $\bar{E}_j$ are eigenvectors corresponding to λ and $s = \dim T_\lambda$. We obtain from (5.1), (5.6), (5.7), (5.8) and (5.9) that $-1+\beta\gamma = 0$. Therefore $\#\sigma(\mathcal{H}) = 2$. ∎

Lemma 10 *If $m > n+1$ and M is linearly full, then with respect to some suitable orthonormal frame field $\{e_1,\ldots,e_{2m}\}$, the second fundamental form of M in $\mathbf{C}H^m(-4)$ satisfies*

$$h(e_{2r-1},e_{2r-1}) = \sqrt{\frac{1}{2n-1}}Je_{2n+1} + \phi_r\xi_r, \tag{5.10}$$

$$h(e_{2r},e_{2r}) = \sqrt{\frac{1}{2n-1}}Je_{2n+1} - \phi_r\xi_r, \tag{5.11}$$

$$h(e_{2r-1},e_{2r}) = \phi_rJ\xi_r, \quad h(e_{2n+1},e_{2n+1}) = \frac{2n}{\sqrt{2n-1}}Je_{2n+1} \tag{5.12}$$

$$h(f,e_{2n+1}) = 0 \quad , \tag{5.13}$$

where $r = 1,\ldots,n$, ϕ_r are functions, $\xi_r \in \nu$ and $f \in L := \mathrm{Span}\{e_1,\ldots,e_{2n}\}$.

Proof: Suppose that $\mathcal{H} = T_\lambda \oplus T_{\mu_{2n+2}-\lambda}$. Let l and m $(l > m)$ be the dimension of T_λ and $T_{\mu_{2n+2}-\lambda}$, respectively. Then we get $(l-m)\sqrt{\mu_{2n+2}^2 - 4} = (2-l-m)\mu_{2n+2}$. But it does not hold, since l, $m > 2$.

Suppose that $\mathcal{H} = T_{\alpha_1} \oplus JT_{\alpha_1}$, where $\alpha_1 \neq \mu_{2n+2}$, λ, $\mu_{2n+2} - \lambda$. Then by using (3.8) and (5.3), we obtain that M is contained in a totally geodesic complex hyperebolic space $\mathbf{C}H^{n+1}(-4)$, since Je_{2n+1} is parallel. This is a contradiction.

Therefore, A_{2n+2} has exactly two distinct eigenvalues. We denote the eigenvector corresponding to the second eigenvalue $\alpha \neq \mu_{2n+2}$ by X. It follows from (5.2) that PX is also an eigenvector corresponding to the eigenvalue $\beta = \frac{-2+\alpha\mu_{2n+2}}{2\alpha-\mu_{2n+2}}$. Since A_{2n+2} has exactly two distinct eigenvalues, we have $\beta = \mu_{2n+2}$ or $\beta = \alpha$.

We divide the proof into two cases.

First, let us suppose that A_{2n+2} has two distinct eigenvalues μ_{2n+2} and $\frac{-2+\mu_{2n+2}^2}{\mu_{2n+2}}$ i.e. $\mu_{2n+2} = \beta$. Then, using (3.8) and (5.3), we obtain that M is contained in a totally geodesic complex hyperebolic space $\mathbf{C}H^{n+1}(-4)$. This is a contaradiction.

Next, we consider the case where A_{2n+2} has two distinct eigenvalues μ_{2n+2} and $\alpha = \beta$. Then from (5.5) we have $\mathcal{H} = T_\lambda$ or $T_{\mu_{2n+2}-\lambda}$.

Consequentry, from proposition 3, replace e_{2n+1} by $-e_{2n+1}$ if necessary, we obtain that $\alpha = \frac{1}{\sqrt{2n-1}}$ and $\mu_{2n+2} = \frac{2n}{\sqrt{2n-1}}$. ∎

Let $\hat{M} = \pi^{-1}(M)$ denote the inverse image of M via the Hopf fibration $\pi : H_1^{2m+1} \to \mathbf{C}H^m(-4)$. Then $\hat{M}$ is a principal circle bundle over M with time-like totally geodesic fibers. Let $z : \hat{M} \to H_1^{2m+1}(-1) \subset \mathbf{C}_1^{m+1}$ denote the immersion of $\hat{M}$ in $\mathbf{C}_1^{m+1}$. Let $\tilde{\nabla}$ and $\hat{\nabla}$ denote the metric connections of $\mathbf{C}_1^{m+1}$ and $\hat{M}$, respectively. We denote by X^* the horizontal lift of a tangent vector X of $\mathbf{C}H^m(-4)$. Then we have (cf. [9])

$$\tilde{\nabla}_{X^*}Y^* = (\nabla_X Y)^* + (h(X,Y))^* + \langle JX, Y\rangle V + \langle X, Y\rangle z, \tag{5.14}$$
$$\tilde{\nabla}_{X^*}V = \tilde{\nabla}_V X^* = (JX)^*, \tag{5.15}$$
$$\tilde{\nabla}_V V = -z, \tag{5.16}$$

for vector fields X, Y tangent to M, where z is the position vector of $\hat{M}$ in $\mathbf{C}_1^{2m+1}$ and $V = iz \in T_z H_1^{2m+1}(-1)$.

Let $E_1, \ldots, E_{2n+1}, \xi_r^*$ be the horizontal lifts of $e_1, \ldots, e_{2n+1}, \xi_r$, respectively and let $E_{2n+2} = iz$, and let $\{\omega_i^j\}$ be connection forms of $\hat{M}$. Then, from lemma 10, (5.14), (5.15) and (5.16), we obtain

$$\tilde{\nabla}_{E_{2r-1}}E_{2r-1} = \sum_{j=1}^{2n} \omega_{2r-1}^j(E_{2r-1})E_j + \alpha i E_{2n+1} + \phi_r \xi_r^* - iE_{2n+2}, \tag{5.17}$$

$$\tilde{\nabla}_{E_{2r-1}}E_{2r} = \sum_{j=1}^{2n} \omega_{2r}^j(E_{2r-1})E_j - \alpha E_{2n+1} + i\phi_r \xi_r^* + E_{2n+2}, \tag{5.18}$$

$$\tilde{\nabla}_{E_{2r}}E_{2r-1} = \sum_{j=1}^{2n} \omega_{2r-1}^j(E_{2r})E_j + \alpha E_{2n+1} + i\phi_r \xi_r^* - E_{2n+2}, \tag{5.19}$$

$$\tilde{\nabla}_{E_{2r}}E_{2r} = \sum_{j=1}^{2n} \omega_{2r}^j(E_{2r})E_j + i\alpha E_{2n+1} - \phi_r \xi_r^* - iE_{2n+2}, \tag{5.20}$$

$$\tilde{\nabla}_{E_{2r-1}}E_{2n+1} = \alpha E_{2r}, \tag{5.21}$$
$$\tilde{\nabla}_{E_{2r}}E_{2n+1} = -\alpha E_{2r-1}, \tag{5.22}$$
$$\tilde{\nabla}_{E_{2n+1}}E_{2n+1} = 2n\alpha i E_{2n+1} - iE_{2n+2}, \tag{5.23}$$

$$\tilde{\nabla}_{E_{2r-1}} E_{2n+2} = \tilde{\nabla}_{E_{2n+2}} E_{2r-1} = E_{2r}, \tag{5.24}$$
$$\tilde{\nabla}_{E_{2r}} E_{2n+2} = \tilde{\nabla}_{E_{2n+2}} E_{2r} = -E_{2r-1}, \tag{5.25}$$
$$\tilde{\nabla}_{E_{2n+1}} E_{2n+2} = \tilde{\nabla}_{E_{2n+2}} E_{2n+1} = iE_{2n+1}, \tag{5.26}$$
$$\tilde{\nabla}_{E_{2n+2}} E_{2n+2} = iE_{2n+2}, \tag{5.27}$$

where $r = 1, \ldots, n$, $\alpha = \sqrt{\frac{1}{2n-1}}$ and

By using the above equations, we obtain the following lemma.

Lemma 11 *$\hat{M}$ is a Riemannian product $\hat{M}_1 \times \hat{M}_2$, where M_1, M_2 are integral submanifolds of $D_1 := \mathrm{Span}\{E_1, \ldots, E_{2n}, \alpha E_{2n+1} - E_{2n+2}\}$ and $D_2 := \mathrm{Span}\{E_{2n+1} - \alpha E_{2n+2}\}$, respectively.*

Proof: For $X', Y' \in D_1$, we have

$$\hat{\nabla}_{X'}(E_{2n+1} - \alpha E_{2n+2}) = 0, \quad \hat{\nabla}_{E_{2n+1}-\alpha E_{2n+2}}(E_{2n+1} - \alpha E_{2n+2}) = 0,$$
$$\hat{\nabla}_{X'}Y' \in D_1, \quad \hat{\nabla}_{E_{2n+1}-\alpha E_{2n+2}} X' \in D_1.$$

Hence, D_1 and D_2 are totally geodesic in $\hat{M}$ and parallel. ∎

Moreover we obtain from (5.21)-(5.27) that

$$\tilde{\nabla}_{E_{2r-1}}(E_{2n+1} - \alpha E_{2n+2}) = \tilde{\nabla}_{E_{2r}}(E_{2n+1} - \alpha E_{2n+2}) = 0,$$
$$\tilde{\nabla}_{\alpha E_{2n+1}-E_{2n+2}}(E_{2n+1} - \alpha E_{2n+2}) = (2n\alpha^2 - \alpha^2 - 1)iE_{2n+1} = 0.$$

Hence, $Z := E_{2n+1} - \alpha E_{2n+2}$ is a constant vector in $\mathbf{C}_1^{m+1}$ along each integral manifold $\hat{M}_1$ of D_1.

From lemma 11, there exist coordinates $\{s, t, x_1, y_1, \ldots, x_n, y_n\}$ such that $\frac{\partial}{\partial s}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial y_n}$ are tangent to integral manifolds $\hat{M}_1$ of D_1, $\frac{\partial}{\partial s} = \alpha E_{2n+1} - E_{2n+2}$ and $\frac{\partial}{\partial t} = E_{2n+1} - \alpha E_{2n+2}$. Without loss of generality, we may assume that $\hat{M}_1$ is defined by $t = 0$. We put $Z_0 := Z|_{t=0}$.

Then we may assume $Z_0 = (0, \ldots, 0, \sqrt{1-\alpha^2})$, up to rigid motions. Since (z, Z_0) is constant along $\hat{M}_1$, we can write

$$z(s, 0, x_1, y_1, \ldots, x_n, y_n) = (\Psi_1, \ldots, \Psi_m, c), \tag{5.28}$$

where c is a constant determined by the initial conditions and $\Psi_1, \ldots, \Psi_m$ are functions.

Since $z_s + (1-\alpha^2)iz = \alpha E_{2n+1} - E_{2n+2} + (1-\alpha^2)E_{2n+2} = \alpha(E_{2n+1} - \alpha E_{2n+2}) = \alpha Z$, we have

$$\frac{\partial \Psi_j}{\partial s} + (1-\alpha^2)i\Psi_j = 0, \quad c(1-\alpha^2)i = \alpha\sqrt{1-\alpha^2}, \quad (1-\alpha^2)iz_2 = \alpha\frac{\partial z_2}{\partial t}, \tag{5.29}$$

where z_2 is a position vector of $\hat{M}_2$ in $\mathbf{C}_1^{m+1}$. Thus we have

$$z = (g(x_1, \ldots, y_n)e^{-(1-\alpha^2)is}, \frac{\alpha\sqrt{1-\alpha^2}}{1-\alpha^2}e^{\frac{1-\alpha^2}{\alpha}it}) \tag{5.30}$$

Since $(z, z) = -1$, we have

$$-|g|^2 + \frac{\alpha^2}{1-\alpha^2} = -1. \tag{5.31}$$

We put $\tilde{E}_{2n+1} = \frac{1}{\sqrt{1-\alpha^2}}(\alpha E_{2n+1} - E_{2n+2})$ and $\tilde{E}_{2n+2} = \frac{1}{\sqrt{1-\alpha^2}}(E_{2n+1} - \alpha E_{2n+2})$. It follows from (5.17)-(5.27) that $\hat{M}_1$ is a CR-submanifold of $\mathbf{C}_1^m$ such that the unit totally real vector field is $\frac{1}{\sqrt{1-\alpha^2}}\frac{\partial}{\partial s}$.

Conversely, we consider the immersion mentioned in Theorem 3. We put $\tilde{E}_{2n+2} = (0, \sqrt{2n-2}\frac{\partial}{\partial t})$, $\tilde{E}_{2n+1} = (-\sqrt{\frac{2n-2}{2n-1}}\frac{\partial}{\partial s}, 0)$, $E_{2n+1} = -\sqrt{\frac{1}{2n-2}}\tilde{E}_{2n+1} + \sqrt{\frac{2n-1}{2n-2}}\tilde{E}_{2n+2}$ and $E_{2n+2} = -\sqrt{\frac{2n-1}{2n-2}}\tilde{E}_{2n+1} + \sqrt{\frac{1}{2n-2}}\tilde{E}_{2n+2}$. Then by straight-forward computaions we can see that $\{E_1, \dots, E_{2n}, E_{2n+1}, E_{2n+2}\}$ is an orthonormal basis of $z(\mathbf{R}^2 \times U)$ and the second fundamental form of $z(\mathbf{R}^2 \times U)$ in $\mathbf{C}_1^{m+1}$ satisfies

$$\tilde{h}(E_{2r-1}, E_{2r-1}) = \sqrt{\frac{1}{2n-1}}iE_{2n+1} - iE_{2n+2} + \phi_r\tilde{\xi}_r, \tag{5.32}$$

$$\tilde{h}(E_{2r-1}, E_{2r-1}) = \sqrt{\frac{1}{2n-1}}iE_{2n+1} - iE_{2n+2} - \phi_r\tilde{\xi}_r, \tag{5.33}$$

$$\tilde{h}(E_{2r-1}, E_{2r-1}) = i\phi_r\tilde{\xi}_r, \quad \tilde{h}(X, E_{2n+1}) = 0, \tag{5.34}$$

$$\tilde{h}(E_{2n+1}, E_{2n+1}) = \frac{2n}{\sqrt{2n-1}}iE_{2n+1} - iE_{2n+2}, \tag{5.35}$$

$X \in \mathrm{Span}\{E_1, \dots, E_{2n}\}$, ϕ_r are functions and $\tilde{\xi}_r$ are unit normal vector fields perpendicular to iE_{2n+1}, iE_{2n+2}.

Since iz is always tangent to $z(\mathbf{R}^2 \times U)$, the image is invariant under the action of H_1^1. Hence, $z(\mathbf{R}^2 \times U)$ is projectable via π. The image $\pi(z(\mathbf{R}^2 \times U))$ is a (2n+1)-dimensional proper CR-submanifold of $\mathbf{C}H^m(-4)$ whose holomorphic ditribution $\mathcal{H}$ is spanned by $e_1 = \pi_*(E_1), \dots, e_n = \pi_*(E_{2n})$ and $\mathcal{H}^\perp$ is spanned by $e_{2n+1} = \pi_*(E_{2n+1})$. From (5.32)-(5.35), we obtain that $e_1, \dots, e_n, e_{2n+1}$ and $\xi_r = \pi_*(\xi_r)$ satisfy (5.10)-(5.13). This completes the proof of theorem 2.

In the rest of this section we shall determine normal CR-submanifolds in a complex hyperbolic space satisfying the equality case of (3.2).

Corollary 12 *In case* $n > 1$ *and* $m > n+1$, *every linearly full* $(2n+1)$*-dimensional normal CR-submanifold with* $\dim \mathcal{H}^\perp = 1$ *in* $\mathbf{C}H^m(-4)$ *satisfying the equality case of* (3.2) *is obtained in the same way as in theorem* 3.

Proof: By using (3.9) and relation $PA_{2n+2} = AP_{2n+2}$, we obtain that the shape operator A_{2n+2} has at most three distinct constant eigenvalues μ_{2n+2}, $\frac{\mu_{n+2}+\sqrt{\mu_{n+2}^2-4}}{2}$ and $\frac{\mu_{n+2}-\sqrt{\mu_{n+2}^2-4}}{2}$. The assertion follows immediatly from theorem 3. ∎

6 CR-submanifolds in the nearly Kaehler six-sphere

It is well known that the unit six-sphere $S^6(1)$ has a nearly Kaehler structure J in the sense that $(\tilde{\nabla}_X J)(X) = 0$, for any vector field X tangent to $S^6(1)$, where $\tilde{\nabla}$ denote the Levi-Civita connection related to the standard metric on $S^6(1)$ ([10]). For the maximum Ricci curvature $\overline{Ric}$ of a 3-dimensional submanifold in $S^6(1)$, we have

$$\overline{Ric} \le 2 + \frac{9}{4}H^2. \tag{6.1}$$

F. Dillen and L. Vrancken have completely classified totally real submanifolds in the nearly Kaehler six-sphere satisfying the equality case of (6.1) ([11]). An n-dimensional Riemannian manifold is called *quasi-Einstein* if Ricci tensor has an eigenvalue of multiplicity at least $n-1$. R. Deszcz, F. Dillen, L. Verstraelen and L. Vrancken proved that 3-dimensional totally real submanifolds in $S^6(1)$ satisfying the equality case of (6.1) are quasi-Einstein ([12]). For proper CR-submanifolds, we obtained the following.

Theorem 13 *Let M^3 be a 3-dimensional proper CR-submanifold in $S^6(1)$. If M^3 satisfies the equality case of* (6.1), *then M^3 is minimal quasi-Einstein.*

Proof: By virtue of main theorem in [14], $\overline{Ric} \neq S(\eta, \eta)$ for a unit vector field $\eta \in \mathcal{H}^\perp$. Let $\{e_1, e_2, e_3\}$ be an orthonormal frame field on M^3 such that $\overline{Ric} = S(e_3, e_3)$. We may assume that $\eta = e_2$. Since $\langle A_\xi JX, X\rangle = -\langle A_\xi X, X\rangle$ for any vector field $X \in \mathcal{H}^\perp$ and $\xi \in \nu$, we obtain that the second fundamental form satisfies

$$h(e_1, e_1) = aJe_2, \quad h(e_2, e_2) = bJe_2, \quad h(e_3, e_3) = (a + b)Je_2, \tag{6.2}$$
$$h(e_1, e_2) = cJe_2 + d\xi, \quad h(e_1, e_3) = h(e_2, e_3) = 0, \tag{6.3}$$

where a, b, c and d are functions and $\xi \in \nu$. From $(\bar{\nabla}_{e_2} h)(e_1, e_3) = (\bar{\nabla}_{e_1} h)(e_2, e_3)$, we get

$$\langle \nabla_{e_2} e_3, e_2\rangle d = \langle \nabla_{e_1} e_3, e_1\rangle d, \tag{6.4}$$
$$\langle \nabla_{e_1} e_2 - \nabla_{e_2} e_1, e_3\rangle h(e_3, e_3) - \langle \nabla_{e_2} e_3, e_1\rangle h(e_1, e_1) + \langle \nabla_{e_1} e_3, e_2\rangle h(e_2, e_2) = 0. \tag{6.5}$$

By using $(\tilde{\nabla}_X J)(Y) = -(\tilde{\nabla}_Y J)(X)$, we have the following:

$$-A_{Je_2} e_1 + D_{e_1} Je_2 = \tilde{\nabla}_{e_1}(Je_2) = -\nabla_{e_2} e_3 + J(\nabla_{e_2} e_1 + \nabla_{e_1} e_2 + 2h(e_1, e_2)), \tag{6.6}$$
$$-A_{Je_2} e_3 + D_{e_3} Je_2 = \tilde{\nabla}_{e_3}(Je_2) = \nabla_{e_2} e_1 + h(e_1, e_2) + J(\nabla_{e_2} e_3 + \nabla_{e_3} e_2), \tag{6.7}$$
$$J(\nabla_{e_2} e_2) + Jh(e_2, e_2) = \tilde{\nabla}_{e_2}(Je_2) = -A_{Je_2} e_2 + D_{e_2} Je_2. \tag{6.8}$$

It follows from (6.6), (6.7) and (6.8) that an orthonormal basis $\{e_1, e_2, e_3\}$ satisfies

$$\langle \nabla_{e_2} e_1, e_3\rangle = 0, \quad \langle \nabla_{e_1} e_2, e_1\rangle = 0, \quad \langle \nabla_{e_1} e_2, e_3\rangle = -a, \quad \langle \nabla_{e_2} e_3, e_2\rangle = -c,$$
$$\langle \nabla_{e_3} e_2, e_3\rangle = 0, \quad \langle \nabla_{e_3} e_2, e_1\rangle = -a - b, \quad D_{e_3} Je_2 = 0. \tag{6.9}$$

From $(\bar{\nabla}_{e_1} h)(e_3, e_3) = (\bar{\nabla}_{e_3} h)(e_1, e_3)$, we obtain

$$(a + b)D_{e_1} Je_2 = 0, \quad e_1(a + b)Je_2 = -\langle \nabla_{e_3} e_1, e_3\rangle (a + b)Je_3 - \langle \nabla_{e_3} e_3, e_1\rangle h(e_1, e_1). \tag{6.10}$$

We put $M_0 := \{p \in M^3 | (a + b)(p) \neq 0\}$. Then $D_{e_1} Je_2 = 0$ on M_0, which implies that $h(e_1, e_2) = d\xi = 0$ by (6.6). If $d = 0$, (6.7) yields $DJe_2 = 0$. Since $h(X, Y) \in \mathrm{Span}\{Je_2\}$ for any tangent vector X, Y, we obtain that M_0 is contained in a totally geodesic $S^4(1)$. Hence $TS^4(1)|_{M_0}$ is spanned by $\{e_1, e_2, e_3, Je_2\}$. A result of Gray in [13] shows that this is impossible. Therefore, $a + b = 0$ on M^3. Moreover by using $(\bar{\nabla}_{e_3} h)(e_1, e_1) = (\bar{\nabla}_{e_1} h)(e_3, e_1)$, we have $ac = 0$. It follows from the equation of Gauss that $c = 0$, $a^2 = d^2 = 1$ and $S(X, Y) = 2\langle X, e_3\rangle \langle Y, e_3\rangle$ for any tangent vector X, Y. This proves the required result. ∎

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