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An extended class of nonlinear groups and its applications to the generalized Kortweg-deVries equations

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ABSTRACT. The initial value problem for the generalized Kortweg-deVries equation

$$u_t + (f(u))_x + u_{xxx} = 0, \qquad t,\, x \in \mathbb{R}$$

is treated in terms of a recent theory of nonlinear operator semigroups associated with semilinear evolution equations in Banach spaces. Two operators A and B are introduced to represent the linear and nonlinear differential operators in the equation and convert the initial-value problem to a semilinear evolution problem

$$\text{(SP)} \qquad u'(t) = (A+B)\,u(t), \quad t > 0; \qquad u(0) = v$$

in the Sobolev space $H^2(\mathbb{R})$. Five energy functionals are then employed to restrict stability properties of $A + B$ as well as the growth of mild solutions to (SP). The solution operators to (SP) are obtained by applying a generation theorem for groups of locally Lipschitzian operators. Here the main point of our argument is to make a precise investigation of the resolvents of $A + B$ and construct a group of locally Lipschitzian operators $G(t)$ on $H^2(\mathbb{R})$ which provides mild solutions to the problem. Also, regularized equations of the form

$$u_t + (f(u))_x + u_{xxx} - \mu u_{txx} = 0, \qquad t,\, x \in \mathbb{R},$$

μ being a positive parameter, are studied by means of the same approach and the convergence of the associated groups $G_\mu(t)$ to the group $G(t)$ is discussed.

1 Introduction

This paper is concerned with the initial value problem for the generalized Kortweg-deVries equations

$$\begin{aligned} &u_t + (f(u))_x + u_{xxx} = 0, && t,\, x \in \mathbb{R}; \\ &u(0,x) = v(x), && x \in \mathbb{R}, \end{aligned}$$

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where f is a nonlinear function of class $\mathscr{C}^3(\mathbb{R})$ which satisfies the normalization conditions $f(0) = 0$ and the growth condition

$$\overline{\lim_{|u|\to\infty}} f'(u) / |u|^p < \infty \quad \text{for some} \quad p \in [0,4),$$

and v is a given initial function in the Sobolev space $H^2(\mathbb{R})$.

In the case of $f(u) = u + u^2/2$, the above equation is known as the Kortweg-deVries equation (usually, abbreviated to K-dV equation), which is understood to be a general model for the unidirectional propagation of long waves of small amplitude. In fact, u represents the height of a wave at position x and time t with respect to the standard level. The K-dV equation is also formulated to describe nonlinear phenomena such as magnetohydrodynamical waves and interaction of solitons.

In this paper, we convert the initial-value problem for the generalized K-dV equation to a semilinear evolution problem of the form

$$\text{(SP)} \qquad u'(t) = (A+B)\,u(t), \qquad t \in \mathbb{R}; \qquad u(0) = v,$$

and treat the initial value problem in an operator theoretic fashion. Here A represents the third-order differential operator $-\partial_x^3$ and B stands for the nonlinear first-order differential operator $-\partial_x \circ f$. We then apply a recent theory for evolution equations goverend by nonlinear quasidissipative operators developped in [7], [14], [17], [18] to this semilinear operator $A+B$ and construct a group $G = \{G(t); t \in R\}$ of nonlinear operators on $H^2(\mathbb{R})$ which provides mild solutions to (SP) in the sense that

$$G(t)v = U(t)v + \int_0^t U(t-s)BG(s)v\,ds$$

for $t \in \mathbb{R}$ and $v \in H^2(\mathbb{R})$, where U is the unitary group generated in $L^2(\mathbb{R})$ by A. One of the main features of our argument is that we make use of five energy functionals φ_k, $k = 0,1,2,3,4$ and investigate the growth of the mild solutions and their qualitative properties. More precisely, we show that the group G enjoys exponential type growth conditions with respect to the functionals φ_k, and that the regularity of the mild solutions $u(t) \equiv G(t)v$ is obtained by means of φ_k. In order to apply a recent theory for groups of locally Lipschitzian operators, we necessitate investigating the ranges of $I - \lambda(A+B)$, $|\lambda| < \lambda_0$ and their resolvents $(I - \lambda(A+B))^{-1}$ in the Sobolev space $H^3(\mathbb{R})$ of higher order through a fixed point argument. The aimed group $G(t)$ is constructed through the exponential formula

$$G(t)v = H^2\text{-}\lim_{\lambda\to\pm 0}(I - \lambda(A+B))^{-[t/\lambda]}v, \quad \pm t \geq 0\,,\ v \in H^2(\mathbb{R}).$$

We next consider the initial value problem for the pseudoparabolic regularization of the generalized Kortweg-deVries equations

$$\begin{aligned} &u_t + (f(u))_x + u_{xxx} - \mu u_{txx} = 0, \qquad t, x \in \mathbb{R} \\ &u_0(0,x) = v(x)\,, \qquad x \in \mathbb{R}, \end{aligned}$$

where μ is a positive parameter. Because of the regularizing effect of the term $-\mu u_{txx}$, a nonlinear group of Fréchet differentiable operators which provides mild solutions to the regularized problem is constructed on $H^1(\mathbb{R})$. For this problem it is assumed that $\mu > 0$, f is a nonlinear function of class $\mathscr{C}^1(\mathbb{R})$ satisfying $f(0) = 0$ and v is an initial function given in $H^1(\mathbb{R})$. We also discuss the convergence of the groups G_μ to G as $\mu \to 0$.

2 A generation theorem for nonlinear groups

It is widely recognized that the theory of semilinear evolution equations plays an important role in the systematic studies of many important problems arising from various fields. In this section, we follow the lines of [7], and make an attempt to outline a generation theory for nonlinear groups of locally Lipschitzian operators associated with a general class of semilinear evolution problems. For the related Hille-Yosida type theorems in the semilinear case, we refer to [14], [17] and [18]. See also [12], [13], [21] for the generation theory for nonlinear evolution operators under more general assumptions.

Let $(X, |\cdot|)$ be a real Banach space, D a subset of X and $\varphi : X \to [0, \infty]$ a l.s.c. functional such that $D \subset D(\varphi) = \{v \in X; \varphi(v) < \infty\}$. We denote by X^* the dual of X, and given $v \in X$ and $v^* \in X^*$, the value of v^* at v is written as $\langle v, v^* \rangle$. We also denote by $D_\alpha = \{v \in D; \varphi(v) \le \alpha\}$ a level set of D with respect to φ. The duality mapping of X is the function $F : X \to 2^{X^*}$ defined by

$$Fv = \left\{v^* \in X^*; \langle v, v^* \rangle = |v|^2 = |v^*|^2\right\}.$$

This is well-defined by the Hahn-Banach theorem.

We then define the lower and upper inner products $\langle \cdot, \cdot \rangle_i$ and $\langle \cdot, \cdot \rangle_s$ on $X \times X$ by

$$\langle w, v \rangle_i = \inf\left\{\langle w, v^* \rangle, v^* \in Fv\right\},$$

and

$$\langle w, v \rangle_s = \sup\left\{\langle w, v^* \rangle, v^* \in Fv\right\}, \qquad \text{respectively.}$$

A nonlinear operator $B : D \subset X \to X$ is said to be locally quasidissipative (respectively, strongly locally quasidissipative) on $D(B)$ with respect to φ if for each $\alpha \ge 0$ there exists $\omega_\alpha \in \mathbb{R}$ such that

$$\langle Bv - Bw, v - w \rangle_i \le \omega_\alpha |v - w|^2 \quad \text{for } v, w \in D_\alpha,$$

$$\text{(respectively,} \qquad \langle Bv - Bw, v - w \rangle_s \le \omega_\alpha |v - w|^2 \quad \text{for } v, w \in D_\alpha).$$

For basic properties of the duality mapping and those of quasidissipative operators, see [6] and [20].

By a locally Lipschitzian group on D with respect to φ, we mean a one-parameter family $\mathcal{G} = \{G(t); t \in \mathbb{R}\}$ of (possibly nonlinear) operators from D into itself such that the following three conditions hold:

(G1) For $v \in D$ and $s, t \in \mathbb{R}$, $\quad G(t)\,G(s)\,v = G(t+s)\,v \quad$ and $\quad G(0)\,v = v$.

(G2) For $v \in D$, $G(\cdot)\,v \in \mathscr{C}(\mathbb{R}; X)$.

(G3) For each $\alpha > 0$ and each $\tau > 0$ there exists $\omega = \omega(\alpha, \tau) \in \mathbb{R}$ such that

$$|G(t)v - G(t)w| \le e^{\omega|t|}|v - w|$$

for v, $w \in D_\alpha = \{v \in D; \varphi(v) \le \alpha\}$ and $t \in [0, \tau]$.

We now consider the semilinear evolution problem

$$\text{(SP)} \qquad u'(t) = (A + B)u(t), \qquad t \in \mathbb{R}; \qquad u(0) = v \in D,$$

where we impose the following hypotheses on the operators A, B and the class of initial data D:

(A) $A : D(A) \subset X \to X$ generates a (C_0)-group $U = \{U(t); t \in \mathbb{R}\}$ such that $|U(t)v| \le e^{\omega t}|v|$ for $v \in X$, $t \in \mathbb{R}$ and some $\omega \in \mathbb{R}$.

(B) The level set D_α is closed for each $\alpha \ge 0$ and $B : D \subset X \to X$ is continuous on each D_α.

Since the semilinear problem (SP) does not necessarily admit strong solutions, the variation of constants formula is employed to introduce solutions in a generalized sense. We say that a function $u(\cdot) \in \mathscr{C}(\mathbb{R}); X)$ is a mild solution to (SP) if $u(t) \in D$ for $t \ge 0$, $Bu(\cdot) \in \mathscr{C}(\mathbb{R}); X)$ and $u(\cdot)$ satisfies the integral equation

$$u(t) = U(t)v + \int_0^t U(t - s)Bu(s)\,ds$$

for all $t \in \mathbb{R}$. We also say that a group $\mathcal{G}$ is associated with (SP), if it provides mild solutions to (SP) in the sense that for each $v \in D$ the function $u(\cdot) \equiv G(\cdot)v$ is a mild solution to (SP).

Under the above hypotheses one can obtain a semilinear Hille-Yosida theorem for locally Lipschitzian groups associated with (SP) as follows.

Theorem 2.1. *Let $a, b \ge 0$, A a linear operator in X satisfing condition* (A), *and let B be a nonlinear operator on D which satisfies condition* (B) *with respect to an l.s.c. functional φ on X with $D \subset D(\varphi)$. Then the following statements are equivalent:*

(I) *There is a group $\mathcal{G} = \{G(t); t \in \mathbb{R}\}$ of locally Lipschitzian operators on D satisfying the conditions below:*

(I.1) $G(t)v = U(t)v + \int_0^t U(t - s)BG(s)v\,ds$ *for $t \in \mathbb{R}$ and $v \in D$.*

(I.2) *For $\alpha > 0$ and $\tau > 0$ there is $\omega = \omega(\alpha, \tau) \in \mathbb{R}$ such that*

$$|G(t)v - G(t)w| \le e^{\omega(\alpha,\tau)|t|}|v - w|$$

for v, $w \in D_\alpha$.

(I.3) $\varphi(G(t)v) \le e^{a|t|}(\varphi(v) + b|t|)$ *for $t \in \mathbb{R}$ and $v \in D$.*

(II) *The subtangential condition and semilinear stability condition are satisfied in the following sense:*

(II.1) *For* $v \in D$ *and* $\varepsilon > 0$ *there exist* $(h_1, v_{h_1}) \in (0, \varepsilon] \times D$ *and* $(h_2, v_{h_2}) \in [-\varepsilon, 0) \times D$ *such that*

$$(1/h_i)\,|U(h_i)\,v + h_i Bv - v_{h_i}| \leq \varepsilon, \quad \varphi(v_{h_i}) \leq e^{a|h_i|}\left(\varphi(v) + (b+\varepsilon)\,h_i\right), \quad i = 1, 2.$$

(II.2) *For each* $\alpha > 0$ *there is* $\omega_\alpha \in \mathbb{R}$ *such that*

$$\varliminf_{h \to 0}\,(1/|h|)\left[|U(h)(v-w) + h(Bv - Bw)| - |v - w|\right] \leq \omega_\alpha\,|v - w|$$

for $v,\ w \in D_\alpha$.

If in particular D *and* φ *are both convex, then the above statements are equivalent to:*

(III) *The following denseness, quasidissipativity and range conditions are satisfied:*

(III.1) $D(A) \cap D$ *is dense in* D.

(III.2) *For* $\alpha > 0$ *there is* $\omega_\alpha \in \mathbb{R}$ *such that*

$$\langle (A+B)v - (A+B)w, v - w\rangle_i \leq \omega_\alpha\,|v - w|^2,$$
$$\langle (A+B)v - (A+B)w, v - w\rangle_s \geq -\omega_\alpha\,|v - w|^2.$$

(III.3) *To* $\alpha > 0$ *and* $\varepsilon > 0$ *there corresponds* $\lambda_0 = \lambda_0(\alpha) > 0$ *and for* $v \in D_\alpha$ *and* $\lambda \in \mathbb{R}$ *with* $|\lambda| < \lambda_0(\alpha)$ *there exist* $v_\lambda \in D(A) \cap D$ *and* $z_\lambda \in X$ *such that* $|z_\lambda| < \varepsilon$,

$$v_\lambda - \lambda(A+B)\,v_\lambda = v + \lambda z_\lambda \quad \text{and} \quad \varphi(v_\lambda) \leq (1 - |\lambda|\,a)^{-1}\left(\varphi(v) + (b+\varepsilon)\,|\lambda|\right).$$

It should be noted here that the implication from (III) to (I) does not require the convexity of D and φ. Also, if X is a Hilbert space and $(Av, v) = 0$ for each $v \in D(A)$ as in the case of K-dV equation, only the inequality $|(Bv - Bw, v - w)| \leq \omega_0(\alpha)\,|v - w|^2$ should be checked to verify (III.2). Moreover, if B is a locally Lipschitzian operator, then the denseness condition (III.1) is not necessary for the derivation of (I) from (III).

3 Semilinear evolution problems for the generalized Kortweg-deVries equations

In this section we construct a nonlinear group which provides mild solutions to the initial value problem for the generalized K-dV equation

$$u_t + (f(u))_x + u_{xxx} = 0, \qquad t, x \in \mathbb{R}, \tag{3.1}$$
$$u(0, x) = v(x), \qquad x \in \mathbb{R}. \tag{3.2}$$

Here $\mathbb{R} = (-\infty, \infty)$, f in (3.1) is a nonlinear function of class $\mathscr{C}^3(\mathbb{R})$ satisfying $f(0) = 0$ and v in (3.2) is an initial function given in $H^2(\mathbb{R})$. We also assume that f satisfies the growth condition

$$\varlimsup_{|u| \to \infty} f'(u)\,/\,|u|^p < \infty \tag{3.3}$$

for some real number $p \in [0, 4)$, where f' denotes the derivative of f.

For studies in K-dV equations and their generalizations through compactness methods, we refer the reader to, for instance, Kametaka [11], Tsutsumi and Mukasa [24], Bona and Smith [5]. See also [8] for a discussion on K-dV equation through the related methods. In this paper we apply the generation theorem stated in the previous section and establish an existence and uniqueness theorem for a nonlinear group of locally Lipschitzian operators on $H^2(\mathbb{R})$ which provides mild solutions to the initial-value problem (3.1)-(3.2).

In what follows, H^k stands for the Sobolev space $H^k(\mathbb{R})$ for each nonnegative integer k. The inner product and norm of H^k are expressed by $(\cdot,\cdot)_k$ and $|\cdot|_k$, respectively.

In particular, H^0 denotes the ordinary Lebesgue space $L^2 = L^2(\mathbb{R})$ with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. By $\mathscr{C}(\mathbb{R}, H^k)$ is meant the space of H^k–valued continuous functions on $\mathbb{R}$. For each positive integer m we write $\mathscr{C}^m(\mathbb{R}; H^k)$ for the space of H^k–valued functions which are m times continuously differentiable on $\mathbb{R}$. Also $\mathscr{C}_c^\infty(\mathbb{R})$ represents the space of infinitely many times differentiable functions with compact supports in $\mathbb{R}$. In what follows, we write $w_n \to w$ in H^k if a sequence $\{w_n\}$ in H^k converges strongly in H^k to w. Likewise, we wite $w_n \rightharpoonup w$ in H^k converges weakly in H^k to w.

Let ∇ be the differential operator d/dx from H^1 into L^2. It is obvious that

$$(\nabla v, w) = -(v, \nabla w) \text{ and } (\nabla v, v) = 0 \text{ for } v,\ w \in H^1. \tag{3.4}$$

The following inequality is well-known, see [22].

Lemma 3.1. *Let $2 \le q \le \infty$. For $v \in H^1$, the inequality*

$$|v|_{L^q} \le 2^r |\nabla v|^r |v|^{1-r} \tag{3.5}$$

is valid, where $r = (q-2)/2q$ and $|\cdot|_{L^q}$ denotes the norm of the Lebesgue space $L^q(\mathbb{R})$.

Since we aim to reformulate equation (3.1) as an abstract semilinear evolution equation in L^2 of the form

$$(d/dt)\, u(t) = (A + B)\, u(t) \qquad t \in \mathbb{R}, \tag{3.6}$$

we first introduce a densely defined and closed linear operator from H^3 into L^2 by

$$Av = -\nabla^3 v \qquad \text{for } v \in H^3. \tag{3.7}$$

It is seen that A is the infinitesimal generator of a group $\mathscr{U} = \{U(t)\,; t \ge 0\}$ of linear isometries on L^2. More precisely, each of $U(t)$ maps H^k into itself and satisfies the identities

$$|U(t)\, v|_k = |v|_k \qquad \text{for } v \in H^k \text{ and } k \ge 0. \tag{3.8}$$

Secondly, we define a nonlinear operator B from H^1 into L^2 by

$$Bv = -\nabla f(v) = -f'(v)\, \nabla v \qquad \text{for } v \in H^1. \tag{3.9}$$

The idea which motivates this approach is that B, as a lower order differential operator, may be regarded as a continuous perturbation of A via restrictions to appropriate subsets of H^k. The same viewpoint is also appropriate for showing quasidissipativity, as seen in the following result.

Proposition 3.1. *The following assertions hold:*

(i) *Let $v \in H^1$ and let $\{v_n\}_{n\geq 1}$ be a sequence in H^1 such that $\sup_{n\geq 1} |v_n|_1 < \infty$. If $v_n \rightharpoonup v$ in L^2, then $Bv_n \rightharpoonup Bv$ in L^2.*

(ii) *Let $v \in H^2$ and let $\{v_n\}_{n\geq 1}$ be a sequence in H^2 such that $\sup_{n\geq 1} |v_n|_2 < \infty$. If $v_n \to v$ in L^2, then $Bv_n \to Bv$ in L^2.*

(iii) *For each $\alpha \geq 0$, there is a number $\omega_0(\alpha) \geq 0$ such that*

$$|(Bv - Bw, v - w)| \leq \omega_0(\alpha)\,|v - w|^2 \tag{3.10}$$

for v, $w \in H^2$ with $|v|_2 \leq \alpha$ and $|w|_2 \leq \alpha$.

(iv) *For each $\alpha \geq 0$, there is a number $\omega_1(\alpha) \geq 0$ such that*

$$|(Bv - Bw, v - w)|_1 \leq \omega_1(\alpha)\,|v - w|_1^2 \tag{3.11}$$

for v, $w \in H^3$ with $|v|_3 \leq \alpha$ and $|w|_3 \leq \alpha$.

Proof. First, for each $u \in H^1$, we have

$$u^2(x) = \int_{-\infty}^{x} u(s)\,u'(s)\,ds - \int_{x}^{\infty} u(s)\,u'(s)\,ds \quad \text{for a.e. } x \in \mathbb{R},$$

and so

$$\left|u^2(x)\right| \leq \int_{-\infty}^{\infty} |u(s)|\,|u'(s)|\,ds \leq |u|\,|u'| \text{ for a.e. } x \in \mathbb{R},$$

which implies $|u|_{L^\infty} \leq \left(1/\sqrt{2}\right)|u|_1$. In what follows, we use the inequality

$$|u|_{L^\infty} \leq |u|_1 \text{ for each } u \in H^1 \tag{3.12}$$

for the sake of simplicity.

(i): Let $v \in H^1$ and $\{v_n\}_{n\geq 1}$ a sequence in H^1 such that $\sup_{n\geq 1} |v_n|_1 < \infty$ and $v_n \rightharpoonup v$ in L^2. Denoting $\sup_{n\geq 1} |v_n|_1$ by M_1, a simple computation implies

$$|Bv_n|^2 = \int_{\mathbb{R}} |f'(v_n)\nabla v_n|^2\,dx \leq \left(\sup_{|s|\leq M_1} |f'(s)|\right)^2 \int_{\mathbb{R}} |\nabla v_n|^2\,dx,$$

from which we infer that $\sup_{n\geq 1} |Bv_n| < \infty$ and $v_n \to v$ in the Fréchet space $L^2_{\text{loc}}(\mathbb{R})$.

Put $K_1 = \sup\{|f'(s)| : |s| \leq M_1\}$ and let $\varphi \in L^2$. Since $\mathscr{C}_c^\infty(\mathbb{R})$ is dense in L^2, one may construct a sequence $\{\varphi_m\}_{m\geq 1}$ such that $\varphi_m \in \mathscr{C}_c^\infty(\mathbb{R})$ and $\varphi_m \to \varphi$ in L^2 as $m \to \infty$. In view of this, we see from (3.4) that

$$\begin{aligned}\langle Bv_n - Bv, \varphi_m\rangle &= \langle f(v_n) - f(v), \nabla\varphi_m\rangle \\ &= \int_{C_m} (f(v_n) - f(v))\,\nabla\varphi_m dx\end{aligned}$$

$$\leq |f(v_n) - f(v)|_{L^2(C_m)} |\nabla \varphi_m|_{L^2(C_m)}$$
$$\leq K_1 |v - v_n|_{L^2(C_m)} |\nabla \varphi_m|_{L^2(C_m)},$$

where m is an arbitrary nonnegative integer, C_m denotes the (compact) support of φ_m for each fixed $m \in \mathbb{N}$. Thus it follows that for each $m \in \mathbb{N}$

$$\langle Bv_n - Bv, \varphi_m \rangle \to \infty \text{ as } n \to \infty. \tag{3.13}$$

Since

$$\langle Bv_n - Bv, \varphi \rangle = \langle Bv_n - Bv, \varphi - \varphi_m \rangle + \langle Bv_n - Bv, \varphi_m \rangle$$
$$\leq \left(\sup_{n\geq 1} |Bv_n| + |Bv| \right) |\varphi - \varphi_m| + \langle Bv_n - Bv, \varphi_m \rangle,$$

the desired result now follows from (3.13).

(ii): Let $v \in H^2$ and $\{v_n\}_{n\geq 1}$ a sequence in H^2 such that $M_2 \equiv \sup_{n\geq 1} |v_n|_2 < \infty$ and $v_n \to v$ in L^2 as $n \to \infty$. Put $K_2 = \max\{\sup_{|\xi|\leq M_2} |f'(\xi)|, \sup_{|\xi|\leq M_2} |f''(\xi)|\}$. Since $|f(v_n) - f(v)| \leq K_2 |v_n - v|$ for $n \geq 1$, we infer that $f(v_n) \to f(v)$ in L^2. In view of the estimate

$$|Bv_n - Bv|^2 \leq |f(v) - f(v_n)| \left|\nabla^2 f(v_n) - \nabla^2 f(v)\right|, \tag{3.14}$$

it is sufficient to show that $\sup_{n\geq 1} |\nabla^2 f(v_n) - \nabla^2 f(v)| < \infty$. To this end, we first observe that

$$\left|\nabla^2 f(v_n)\right| = \left|f''(v_n)(\nabla v_n)^2 + f'(v_n)\nabla^2 v_n\right|$$
$$\leq \left|f''(v_n)(\nabla v_n)^2\right| + \left|f'(v_n)\nabla^2 v_n\right|$$
$$\leq \sup_{|\xi|\leq M_2} |f''(\xi)| |\nabla v_n|_{L^4}^2 + \sup_{|\xi|\leq M_2} |f'(\xi)| \left|\nabla^2 v_n\right|.$$

Since

$$\left|\nabla^2 f(v_n)\right| \leq K_2 \left(\left|\nabla^2 v_n\right|^{1/2} |\nabla v_n|^{3/2} + \left|\nabla^2 v_n\right| \right),$$

by Lemma 3.1 and since $\{v_n\}_{n\geq 1}$ is bounded in H^2, it follows that

$$\sup_{n\geq 1} \left|\nabla^2 f(v_n) - \nabla^2 f(v)\right| < \infty. \tag{3.15}$$

Combining (3.14) and (3.15), we obtain the desired result.

(iii): Let $\alpha \geq 0$ and define $\omega_0(\alpha)$ by

$$\omega_0(\alpha) = (1/2) \sup \left\{ |f''(v)\nabla v|_{L^\infty(\mathbf{R})} ; v \in H^2, |v|_2 \leq \alpha \right\}. \tag{3.16}$$

Suppose that $v, w \in H^2$, $|v|_2 \leq \alpha$, $|w|_2 \leq \alpha$, and $\theta \in [0,1]$. Let $z_\theta(\cdot) = \theta v(\cdot) + (1-\theta) w(\cdot)$. Then by (3.4) and integration by parts we have

$$(Bv - Bw, v - w) = (f(v) - f(w), \nabla(v - w))$$

$$= \left(\int_0^1 d/d\theta \left(f\left(z_\theta \right) \right) d\theta, \nabla \left(v - w \right) \right)$$
$$= \left(\int_0^1 f'\left(z_\theta \right) d\theta, \nabla \left(1/2 \left(v - w \right)^2 \right) \right)$$
$$= (-1/2) \left(\int_0^1 f''\left(z_\theta \right) \left(\nabla z_\theta \right) d\theta, \left(v - w \right)^2 \right).$$

From this relation we deduce that $|(Bv - Bw, v - w)| \leq \omega_0(\alpha) |v - w|^2$. This proves the third assertion.
(iv): Let $\alpha \geq 0$ and define $\omega_1(\alpha)$ by

$$\begin{aligned} \text{(3.17)} \quad \omega_1(\alpha) = \max \{ & \omega_0(\alpha), (\sup |f'''(v)|_{L^\infty}) (\sup |\nabla v|_{L^\infty})^2 \\ & + (\sup |f''(v)|_{L^\infty}) (\sup |\nabla^2 v|_{L^\infty}) + (3/2) (\sup |f''(v)|_{L^\infty}) (\sup |\nabla v|_{L^\infty}) \}, \end{aligned}$$

where the suprema are taken over the set $\{v \in H^3, |v|_3 \leq \alpha\}$. As the first step, we observe that

$$\begin{aligned} & (\nabla (Bv - Bw), \nabla (v - w)) = (f'(v) \nabla v - f'(w) \nabla w, \nabla^2 (v - w)) \\ & = \left(\int_0^1 (d/d\theta) (f'(z_\theta) \nabla z_\theta) d\theta, \nabla^2 (v - w) \right) \\ & = \left(\int_0^1 f''(z_\theta) (v - w) \nabla z_\theta d\theta, \nabla^2 (v - w) \right) + \left(\int_0^1 f'(z_\theta) (\nabla v - \nabla w) d\theta, \nabla^2 (v - w) \right), \end{aligned}$$

where $z_\theta = \theta v + (1 - \theta) w$. We denote the first and second terms on the right-hand side of the above inequality by J_1 and J_2, respectively. In view of (3.4), J_1 and J_2 are estimated as

$$\begin{aligned} |J_1| & = \left| \left(\left(\int_0^1 f''(z_\theta) \nabla z_\theta d\theta \right) (v - w), \nabla^2 (v - w) \right) \right| \\ & = \left| \left(\nabla \left(\left(\int_0^1 f''(z_\theta) \nabla z_\theta d\theta \right) (v - w) \right), \nabla (v - w) \right) \right| \\ & \leq \left| \left(\left(\int_0^1 f'''(z_\theta) (\nabla z_\theta)^2 d\theta \right) (v - w), \nabla (v - w) \right) \right| \\ & \quad + \left| \left(\left(\int_0^1 f''(z_\theta) \nabla^2 z_\theta d\theta \right) (v - w), \nabla (v - w) \right) \right| \\ & \quad + \left| \left(\left(\int_0^1 f''(z_\theta) \nabla z_\theta d\theta \right) \nabla (v - w), \nabla (v - w) \right) \right| \\ & \leq [(\sup |f'''(v)|_{L^\infty}) (\sup |\nabla v|_{L^\infty})^2 + (\sup |f''(v)|_{L^\infty}) (\sup |\nabla^2 v|_{L^\infty})] \\ & \quad \times |v - w| |\nabla (v - w)| + (\sup |f''(v)|_{L^\infty}) (\sup |\nabla v|_{L^\infty}) |\nabla (v - w)|^2 \end{aligned}$$

and

$$|J_2| = \left| \left(\left(\int_0^1 f'(z_\theta) d\theta \right) \nabla (v - w), \nabla^2 (v - w) \right) \right|$$

$$= \left| \left(\int_0^1 f'(z_\theta)\, d\theta, (1/2) \nabla \left((\nabla (v - w))^2 \right) \right) \right|$$
$$= (1/2) \left| \left(\int_0^1 f''(z_\theta) \nabla z_\theta d\theta, (\nabla (v - w))^2 \right) \right|$$
$$\leq (1/2) \left(\sup |f''(v)|_{L^\infty} \right) \left(\sup |\nabla v|_{L^\infty} \right) |\nabla (v - w)|^2 .$$

Since $|\nabla (v - w)|^2$ and $|w - v| \, |\nabla (v - w)|$ are bounded by $|v - w|_1^2$, we obtain the desired result. □

From the above proposition, it is seen that the nonlinear differential operator B is continuous and quasidissipative on bounded sets $\{v \in H^2; |v|_2 \leq \alpha\}$, $\alpha \geq 0$, in H^2. In view of this fact, we can employ the notion of mild solution as defined in Section 2.

Let $v \in H^2$ and $u(\cdot) \in C(\mathbb{R}; H^2)$. As easily seen, $u(\cdot)$ is a mild solution of (3.6) with $u(0) = v$ if and only if it satisfies the integral equations

$$(3.18) \qquad (u(t), w) = (v, w) + \int_0^t \left\{ \left(\nabla^2 u(s), \nabla w \right) - (\nabla f(u(s)), w) \right\} ds$$

for $t \in \mathbb{R}$ and $w \in H^1$. By Theorem 2.1 and Proposition 3.1, we obtain the following result which guarantees the uniqueness of mild solutions.

Proposition 3.2. *Let $u(\cdot), \hat{u}(\cdot)$ be mild solutions of* (3.6) *with initial data $u(0) = v$ and $\hat{u}(0) = \hat{v}$, respectively. Then for each $\tau > 0$ we have*

$$(3.19) \qquad |u(t) - \hat{u}(t)| \leq e^{\omega_0(\alpha)|t|} |v - \hat{v}| \qquad \textit{for } t \in [-\tau, \tau].$$

where α is chosen such that $|u(t)|_2 \leq \alpha$ and $|\hat{u}(t)|_2 \leq \alpha$ for each $t \in [-\tau, \tau]$ and $\omega_0(\alpha)$ is the constant given for α by Lemma 3.1.

Next, in order to construct a nonlinear group of locally Lipschitzian operators on H^2 which provide mild solutions to (3.6), we employ five energy functionals that are l.s.c. functionals $\varphi_k : H^k \to \mathbb{R}$, $k = 0, \cdots, 4$. It should be mentioned that these functionals are also applied to discuss regularity properties of mild solutions. We define

$$(3.20) \qquad \begin{aligned} \varphi_0(v) &= |v|, & v &\in L^2; \\ \varphi_1(v) &= (1/2) |\nabla v|^2 - \int_{-\infty}^{\infty} \int_0^{v(x)} f(\xi)\, d\xi dx, & v &\in H^1; \\ \varphi_2(v) &= (1/2) \left| \nabla^2 v \right|^2 + (5/6) \left(f(v), \nabla^2 v \right), & v &\in H^2; \\ \varphi_3(v) &= \left| \nabla^3 v + \nabla f(v) \right|, & v &\in H^3; \\ \varphi_4(v) &= \left| \nabla^3 f(v) + \nabla f(v) \right|_1, & v &\in H^4. \end{aligned}$$

Proposition 3.1 asserts the continuity and quasidissipativity of B on bounded sets of H^2 with respect to the standard Sobolev norm. We then show that the boundedness with respect to φ_0, $\{\varphi_0, \varphi_1\}$, $\{\varphi_0, \varphi_1, \varphi_2\}$ are equivalent to the boundedness with respect to the Sobolev norms $|\cdot|_0$, $|\cdot|_1$ and $|\cdot|_2$, respectively. However, the functionals φ_k appear to be more intimately related to the physical structure of the model since, as will be seen in Theorem 4.2, φ_1 and φ_2 are invariants for the generalized Kortweg-deVries equation (3.1). On the other hand, it is also important to observe that φ_3 and φ_4 are rather abstract, although these are useful for the regularity argument. In fact we obtain the following:

Lemma 3.2. (i) *For α_0, $\alpha_1 \geq 0$, there is $\beta_1 = \beta_1(\alpha_0, \alpha_1) \geq 0$ such that $v \in H^1$, $\varphi_0(v) \leq \alpha_0$ and $\varphi_1(v) \leq \alpha_1$ imply $|\nabla v| \leq \beta_1$.*

(ii) *For α_0, α_1, $\alpha_2 \geq 0$ there is $\beta_2 = \beta_2(\alpha_0, \alpha_1, \alpha_2) \geq 0$ such that if $v \in H^2$, $\varphi_0(v) \leq \alpha_0$, $\varphi_1(v) \leq \alpha_1$ and $\varphi_2(v) \leq \alpha_2$, then $|\nabla^2 v| \leq \beta_2$.*

Proof. (i): In view of the growth condition (3.3), one finds constants $\overline{C}_1$ and $\overline{C}_2 \in \mathbb{R}$ such that $f'(s) \leq \overline{C}_1 + \overline{C}_2 |s|^p$ for $s \in \mathbb{R}$. Using $f(0) = 0$ and integrating both sides, we have $\int_0^s f(\xi)\, d\xi \leq C_1 |s|^2 + C_2 |s|^{p+2}$ for $s \in \mathbb{R}$ and some constants C_1 and C_2. The application of Lemma 3.1 implies the estimate

$$\begin{aligned} \int_{-\infty}^{\infty} \int_0^{v(x)} f(\xi)\, d\xi dx &\leq C_1 |v|^2 + C_2 |v|_{L^{p+2}}^{p+2} \\ &\leq C_1 |v|^2 + C_2 2^{p/2} |\nabla v|^{p/2} |v|^{(p+4)/2} \end{aligned} \tag{3.21}$$

for each $v \in H^1$. Invoking Young's inequality, one obtains

$$C_2 2^{p/2} |\nabla v|^{p/2} |v|^{(p+4)/2} \leq (1/4) |\nabla v|^2 + (1/4)(4-p) \left(C_2 (4p)^{p/4} |v|^{(p+4)/2} \right)^{4/(4-p)}.$$

Hence we get the estimate

$$|\nabla v|^2 \leq 4 (\alpha_1 + C_1 \alpha_0)^2 + (4-p) \left(C_2 (4p)^{p/4} \alpha_0^{(p+4)/2} \right)^{4/(4-p)}, \tag{3.22}$$

from which the desired result follows.
(ii): From (3.4) and (3.20) it is seen that

$$|\nabla^2 v|^2 = 2\varphi_2(v) - (5/3) \left(f'(v), (\nabla v)^2 \right),$$

and therefore

$$|\nabla^2 v|^2 \leq 2\alpha_2 + (5/3) C_3 \beta_1^2 \tag{3.23}$$

Here, $C_3 \equiv C_3(f, \alpha_0, \beta_1)$ denotes a positive constant which depends only on f, α_0 and β_1. □

In view of Theorem 2.1, it is necessary to show that the range condition (III.3) is verified with respect to $\varphi = \varphi_2$. Therefore, to proceed further, we need the following technical lemma, which gives an estimate for intermediate terms arising from the computation of the values of φ_2.

Lemma 3.3. *For each $\alpha_0 \geq 0$ and $\alpha_1 \geq 0$ there exist $a \equiv a(\alpha_0, \alpha_1) \geq 0$ and $b \equiv b(\alpha_0, \alpha_1) \geq 0$ such that*

$$(3.24)\qquad \left|(1/6)\left(f'''(v)(\nabla v)^3, \nabla^2 v\right) + (5/6)\left(f'(v)\nabla^2 v, f'(w)\nabla w\right)\right| \leq a\varphi_2(v) + b$$

for v, $w \in H^2$ satisfying $\varphi_0(v) \leq \alpha_0$, $\varphi_1(v) \leq \alpha_1$, $\varphi_0(w) \leq \alpha_0$ and $\varphi_1(w) \leq \alpha_1$.

Proof. Let α_0, α_1 be positive numbers and let v, $w \in H^2$ satisfy $\varphi_0(v) \leq \alpha_0$, $\varphi_0(w) \leq \alpha_0$, $\varphi_1(v) \leq \alpha_1$ and $\varphi_1(w) \leq \alpha_1$. First, inequality (3.12) now implies that there is $\gamma_1 = \gamma_1(\alpha_0, \alpha_1)$ so that $|v|_{L^\infty} \leq \gamma_1$, $|w|_{L^\infty} \leq \gamma_1$. Put $C_4 \equiv \sup_{|\xi| \leq \gamma_1} |f'''(\xi)|$. We have

$$\begin{aligned}
&\left|(1/6)\left(f'''(v)(\nabla v)^3, \nabla^2 v\right) + (5/6)\left(f'(v)\nabla^2 v, f'(w)\nabla w\right)\right| \\
&\quad \leq (1/6)\left|f'''(v)(\nabla v)^3\right|\left|\nabla^2 v\right| + (5/6)\left|f'(v)\nabla^2 v\right|\left|f'(w)\nabla w\right|
\end{aligned}$$

We denote by J_1 and J_2, respectively, the first and second terms on the right hand side of the above inequality. Then, by Lemma 3.2, there is $\beta_1 \equiv \beta_1(\alpha_0, \alpha_1) \geq 0$ such that

$$\begin{aligned}
J_1 = (1/6)\left|f'''(v)(\nabla v)^3\right|\left|\nabla^2 v\right| &\leq (1/6)\, C_3 \left|\nabla v\right|_{L^6}^3 \left|\nabla^2 v\right| \\
&\leq (1/3)\, C_4 \left|\nabla^2 v\right|^2 \left|\nabla v\right|^2 \\
&\leq (1/3)\, C_4 \beta_1^2 \left|\nabla^2 v\right|^2
\end{aligned}$$

and

$$\begin{aligned}
J_2 = (5/6)\left|f'(v)\nabla^2 v\right|\left|f'(w)\nabla w\right| &\leq (5/6)\, C_5^2 \left|\nabla^2 v\right|\left|\nabla w\right| \\
&\leq (5/12)\, C_5^2 \beta_1 \left(\left|\nabla^2 v\right|^2 + 1\right),
\end{aligned}$$

where $C_5 \equiv C_5(f, \alpha_0, \alpha_1) = \sup_{|\xi| \leq \gamma_1} |f'(\xi)|$. Using (3.23), one obtains

$$\begin{aligned}
&\left|(1/6)\left(f'''(v)(\nabla v)^3, \nabla^2 v\right) + (5/6)\left(f'(v)\nabla^2 v, f'(w)\nabla w\right)\right| \\
&\quad \leq (2/3)\left(\mathscr{C}\beta_1^2 + (5/4)\overline{\mathscr{C}}^2\beta_1\right)\left(\varphi_2(v) + (5/6)\mathscr{C}\beta_1^2\right) + (5/12)\overline{\mathscr{C}}^2\beta_1 \\
&\quad = a(\alpha_0, \alpha_1)\,\varphi_2(v) + b(\alpha_0, \alpha_1).
\end{aligned}$$

□

4 Resolvents of the semilinear operator $A + B$

In this section we employ the five functionals φ_0, φ_1, φ_2, φ_3 and φ_4 defined by (3.20). The next result shows that a generalized form of the range condition is fulfilled for the semilinear operator $A + B$.

Theorem 4.1. *Let $v \in H^3$, $\varepsilon > 0$ and suppose that $\alpha_0, \alpha_1, \alpha_2 > 0$ are chosen so that $\varphi_0(v) + \varepsilon < \alpha_0$, $\varphi_1(v) + \varepsilon < \alpha_1$, and $e^{2a}(|\varphi_2(v)| + (b + \varepsilon)) < \alpha_2$, where $a = a(\alpha_0, \alpha_1)$ and $b = b(\alpha_0, \alpha_1)$ are numbers determined for α_0 and α_1 in Lemma 3.3. Then there* *is a number* $\lambda_0 = \lambda_0(\alpha_1, \alpha_2, \alpha_3, \varepsilon)$ *such that* $0 < \lambda_0 < \min\{1, 1/2a, 1/\omega_0\}$ *and for each* $\lambda \in (-\lambda_0, \lambda_0)$ *there is a unique element* $v_\lambda \in H^3$ *satisfying*

$$v_\lambda - \lambda(A+B)v_\lambda = v, \tag{4.1}$$

and

$$\begin{aligned} \varphi_0(v_\lambda) &\le \varphi_0(v) + |\lambda|\varepsilon, \\ \varphi_1(v_\lambda) &\le \varphi_1(v) + |\lambda|\varepsilon, \\ \varphi_2(v_\lambda) &\le (1-|\lambda|a)^{-1}(\varphi_2(v) + |\lambda|(b+\varepsilon)), \\ \varphi_3(v_\lambda) &\le (1-|\lambda|\omega_0)^{-1}\varphi_3(v). \end{aligned} \tag{4.2}$$

Furthermore, if $v \in H^4$ *then* $v_\lambda \in H^4$ *and satisfies the growth condition*

$$\varphi_4(v_\lambda) \le (1-|\lambda|\omega_1)^{-1}\varphi_4(v).$$

Proof. Let $v \in H^3$, $\varepsilon > 0$ and suppose that α_0, α_1, α_2 are numbers as indicated above. By virtue of Lemma 3.2 we may choose β_0, β_1, $\beta_2 > 0$ so that

$$\{w \in H^2, \varphi_k(w) \le \alpha_k, k=0,1,2\} \subset \{w \in H^2, |\nabla^k w| \le \beta_k, k = 0,1,2\}. \tag{4.3}$$

We then write

$$N_0 = \sup\{|Bw|; w \in H^1, |w| \le \beta_0, |\nabla w| \le \beta_1\}; \tag{4.4}$$

$$N_1 = \sup\{|\nabla Bw|; w \in H^2, |\nabla^k w| \le \beta_k, k=0,1,2\}. \tag{4.5}$$

Further, we fix any $\beta_3 > 0$ satisfying $|\nabla^3 v| + 2N_0 \le \beta_3$ and set

$$N_2 = \sup\{|\nabla^2 Bw|; w \in H^3, |\nabla^k w| \le \beta_k, k=0,1,2,3\}. \tag{4.6}$$

By (4.3) and Proposition 3.1, there is a number $\delta = \delta(|v|_3, \varepsilon) > 0$ such that $w \in H^3$, $|w-v| < \delta$ and $|\nabla^k w| \le \max\{\beta_k, |\nabla^k v| + N_k\}$, $k=0,1,2$, together imply

$$\begin{aligned} &|Bw - Bv| \le \varepsilon/2, \\ &|f(w) - f(v)|\beta_3 \le \varepsilon/2, \\ &|\nabla Bw - \nabla Bv|\beta_3 \le \varepsilon/5, \text{ and} \\ &|f'(w) - f'(v)|(|\nabla^2 v| + N_2)\beta_3 \le \varepsilon/5. \end{aligned} \tag{4.7}$$

We now demonstrate through a fixed point argument that (4.1) and (4.2) are obtained for $|\lambda|$ sufficient small. Set

$$\lambda_0 = \min\{1,\ \delta/\beta_3,\ \varepsilon/(2\beta_3),\ 1/(2a),\ 1/\omega_0\} \tag{4.8}$$

and let $\lambda \in (-\lambda_0, \lambda_0)\backslash\{0\}$. Let K be a subset of H^3 defined by

$$K = \{w \in H^3;\ |w-v| \le |\lambda|\beta_3,\ |\nabla^k w| \le \beta_k,\ k=0,1,2,3\}. \tag{4.9}$$

First, it is easily seen that K is convex, bounded and closed in L^2. Since L^2 is reflexive, K is weakly compact in L^2. We then define a mapping $\Gamma : K \to H^3$ by

$$\Gamma w = (I - \lambda A)^{-1} (v + \lambda B w) \qquad \text{for } w \in K. \tag{4.10}$$

Since the resolvent $(I - \lambda A)^{-1}$ is bounded and linear in L^2, it is weakly continuous. Hence Proposition 3.1 implies that Γ is weakly continuous on K. We next show that Γ maps K into itself. To this end, let $w \in K$ and put $z = \Gamma w$. It is seen that

$$\begin{aligned} |z - v|^2 &= (z - v, \lambda A z) + (z - v, \lambda B w) \\ &= (-v, \lambda A z) + (z - v, \lambda B w) \\ &= \lambda (Av, z) + \lambda (z - v, Bw) \\ &\le |\lambda| \, |z - v| \left(|\nabla^3 v| + |Bw| \right), \end{aligned}$$

and so

$$|z - v| \le |\lambda| \left(|\nabla^3 v| + N_0 \right) \le |\lambda| \, \beta_3. \tag{4.11}$$

Also, since

$$\lambda A z = z - v - \lambda B w, \tag{4.12}$$

by (4.10), it follows from (4.4) that

$$\left|\nabla^3 z\right| \le (1/|\lambda|) \, |z - v| + |Bw| \le \left|\nabla^3 v\right| + 2N_0 \le \beta_3. \tag{4.13}$$

We next prove that $\left|\nabla^k z\right| \le \left|\nabla^k v\right| + N_k$ for $k = 0, 1, 2$, which enable us to apply the estimates in (4.7). Using (3.4) again, we obtain

$$|z|^2 = (z, v + \lambda A z + \lambda B w) = (z, v) + \lambda (z, Bw) \le |z| \left(|v| + |\lambda| \, |Bw| \right).$$

This estimate, together with (4.4) and (4.8), implies $|z| \le |v| + |\lambda| \, N_0$. Repeating the same argument as above, it is seen that

$$\begin{aligned} |\nabla z|^2 &= \left(-\nabla^2 z, v + \lambda A z + \lambda B w \right) \\ &= -\left(\nabla^2 z, v + \lambda B w \right) = (\nabla z, \nabla v + \lambda \nabla B w) \\ &\le |\nabla z| \left(|\nabla v| + |\lambda| \, |\nabla B w| \right), \end{aligned}$$

and Hence that $|\nabla z| \le |\nabla v| + |\lambda| \, N_1$ by (4.5) and (4.8). Let $z_n \in \mathscr{C}_c^\infty(\mathbb{R})$ and $z_n \to z$ in H^2. Choose $z_n \in \mathscr{C}_c^\infty(\mathbb{R})$ so that $z_n \to z$ in H^2. By (3.4) and (4.12), we have

$$\begin{aligned} \left(\nabla^2 z, \nabla^2 z \right) &= \left(\nabla^2 z, \nabla^2 z - \nabla^2 z_n \right) + \left(\nabla^2 z, \nabla^2 z_n \right) \\ &\le \left|\nabla^2 z\right| \left|\nabla^2 z - \nabla^2 z_n\right| + \left(v + \lambda A z + \lambda B w, \nabla^4 z_n \right). \end{aligned} \tag{4.14}$$

Since $Az \in H^2$by (4.12), we infet from (3.4) that

$$\left(v + \lambda A z + \lambda B w, \nabla^4 z_n \right) \to \left(\nabla^2 v, \nabla^2 z \right) + \lambda \left(\nabla^2 A z, \nabla^2 z \right) + \lambda \left(\nabla^2 B w, \nabla^2 z \right)$$

as $n \to \infty$. Combining this with (4.14), one obtains $|\nabla^2 z| \le |\nabla^2 v| + N_2$. We next prove that $\varphi_k(z) \le \alpha_k$ for $k = 0,1,2$, and so that (4.3) would imply $|\nabla^k z| \le \beta_k$ for $k = 0,1,2$. In view of (3.4), (4.12) and the fact that $(Bz, z) = 0$ for $z \in H^1$ we obtain

$$\begin{aligned} |z|^2 &= (z, v + \lambda Az + \lambda Bw) \\ &= (z, v) + \lambda(z, Bw - Bz) \\ &\le |z| \left(|v| + |\lambda| \, |Bw - Bz|\right), \end{aligned}$$

and so

$$|z| \le |v| + |\lambda| \left(|Bw - Bv| + |Bz - Bv|\right).$$

Therefore, by (4.7), z satisfies the inequality

$$|z| \le |v| + |\lambda| \, \varepsilon < \alpha_0. \tag{4.15}$$

We next give the estimate for $\varphi_1(z)$. By (3.20) we have

$$\begin{aligned} \varphi_1(z) - \varphi_1(v) &= (1/2) \left|\nabla^2 z\right|^2 - (1/2) \left|\nabla^2 v\right|^2 - \int_{-\infty}^{\infty} \int_{v(x)}^{z(x)} f(\xi) \, d\xi dx \\ &= (1/2) \left|\nabla^2 z\right|^2 - (1/2) \left|\nabla^2 v\right|^2 \\ &\quad - \int_{-\infty}^{\infty} \int_0^1 f(\theta z(x) + (1-\theta) v(x)) \, d\theta \, (z(x) - v(x)) \, dx. \end{aligned} \tag{4.16}$$

Put $w_\theta(\cdot) = \theta z(\cdot) + (1-\theta) v(\cdot)$. From (4.8) we infer that

$$|w_\theta - v| = |\theta(z - v)| \le |\lambda| \, \beta_3 < \delta \quad \text{for} \theta \in [0,1]. \tag{4.17}$$

Since $|w - v| < \delta$, relation (4.7) leads us to the estimate

$$|(f(w_\theta) - f(w), z - v)| \le \varepsilon / \beta_3 \, |z - v| \le |\lambda| \, \varepsilon. \tag{4.18}$$

Therefore, by (4.16) and (4.18), it follows that

$$\varphi_1(z) - \varphi_1(v) \le (1/2) |\nabla z|^2 - (1/2) |\nabla v|^2 - (f(w), z - v) + |\lambda| \, \varepsilon.$$

Since $(f(w), z - w) = \lambda (f(w), Az + Bw)$ and $(f(w), Bw) = 0$, we have

$$\varphi_1(z) - \varphi_1(v) \le (1/2) |\nabla z|^2 - (1/2) |\nabla v|^2 - \lambda (\nabla Bw, \nabla z) + |\lambda| \, \varepsilon \tag{4.19}$$

which implies that $\varphi_1(z) \le \varphi_1(v) + |\lambda| \, \varepsilon < \alpha_1$.

To estimate $\varphi_2(z)$, we first observe by virtue of (3.4) that,

(4.20)

$$\begin{aligned} \left(f(z), \nabla^2 z\right) - \left(f(v), \nabla^2 v\right) &= \left(f_1(z) - f(v), \nabla^2 z\right) + \left(f(v), \nabla^2 z - \nabla^2 v\right) \\ &= \left(\int_0^1 f'(w_\theta) \, d\theta \, (z - v), \nabla^2 z\right) - (\nabla Bv, z - v) \end{aligned}$$

$$
\begin{aligned}
&= \left(\int_0^1 (f'(w_\theta) - f'(z))\, d\theta\, (z-v), \nabla^2 z\right) + \left(f'(z)(z-v), \nabla^2 z\right) \\
&\quad - (\nabla Bv - \nabla Bw, z-v) - (\nabla Bw, z-v) \\
&= \int_0^1 \left((f'(w_\theta) - f'(z)), \nabla^2 z\,(z-v)\right) d\theta + \left(f'(z)\nabla^2 z, z-v\right) \\
&\quad - (\nabla Bv - \nabla Bw, z-v) - (\nabla Bw, z-v),
\end{aligned}
$$

where w_θ denotes the convex combination $\theta z + (1-\theta)v$. Since

$$
\left|\left(f'(w_\theta) - f'(z), \left(\nabla^2 z\right)(z-v)\right)\right| \le \left(|f'(w_\theta) - f'(v)| + |f'(v) - f'(z)|\right)\left|\nabla^2 z\right| |z-v|,
$$

the estimates in (4.17) imply

$$
|f'(w_\theta) - f'(v)| \le \varepsilon / \left(5\beta_3\left(\left|\nabla^2 v\right| + N_2\right)\right),
$$

$$
|f'(z) - f'(v)| \le \varepsilon / \left(5\beta_3\left(\left|\nabla^2 v\right| + N_2\right)\right).
$$

Hence $|((f'(w_\theta) - f'(z)), (\nabla^2 z)(z-v))| \le 2\varepsilon |\lambda| |\nabla^2 z| / ((|\nabla^2 v| + L_2))$, for $\theta \in [0,1]$. Since $|\nabla^2 z| \le |\nabla^2 v| + N_2$ we have

$$
\left|\left((f'(w_\theta) - f'(z)), \left(\nabla^2 z\right)(z-v)\right)\right| \le 2\varepsilon |\lambda| / 5. \tag{4.21}
$$

From (4.7) and (4.11), we obtain

$$
|(\nabla Bv - \nabla Bw, z-v)| \le |\nabla Bv - \nabla Bw| |z-v| \le \varepsilon |\lambda| / 5. \tag{4.22}
$$

We now estimate the term $(f'(z)\nabla^2 z, z-v) - (\nabla Bw, z-v)$ on the right-hand side of (4.20). Applying (3.4) and (4.12), we have

$$
\begin{aligned}
\left(f'(z)\nabla^2 z, z-v\right) - (\nabla Bw, z-v) &= \left(f'(z)\nabla^2 z, \lambda(Az + Bw)\right) - \lambda(\nabla Bw, Az + Bw) \\
&= \lambda\left(f'(z)\nabla^2 z, \lambda(Az + Bw)\right) - \lambda(\nabla Bw, Az) \\
&= -\lambda\left(f'(z)\nabla^2 z, \nabla^3 z + f'(w)\nabla w\right) - \lambda(\nabla Bw, Az).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\left(\nabla^2 z, \nabla^2 z\right) &= \left(\nabla^2 z, \nabla^2 v + \lambda\nabla^2 Az + \lambda\nabla^2 Bw\right) \\
&= \left(\nabla^2 z, \nabla^2 v\right) + \lambda\left(\nabla^2 z, \nabla^2 Bw\right) \\
&= \left(\nabla^2 z, \nabla^2 v\right) + \lambda(Az, \nabla Bw),
\end{aligned}
$$

from which we infer that

$$
\begin{aligned}
\left(f'(z)\nabla^2 z, z-v\right) + (\nabla Bw, z-v) = &- \lambda\left(f'(z)\nabla^2 z, \nabla^3 z + f'(w)\nabla w\right) \\
&+ \left(\nabla^2 z, \nabla^2 v\right) - \left(\nabla^2 z, \nabla^2 z\right).
\end{aligned}
$$

In order to estimate the right-hand side, we need the following useful identity

$$
5\left(f'(z)\nabla^2 z, \nabla^3 z\right) = (\nabla Bz, Az) + \left(f'''(z)(\nabla z)^3, \nabla^2 z\right). \tag{4.23}
$$

To prove this, we first observe that

$$
(4.24) \qquad \begin{aligned}
\left(f'(z)\nabla^2 z, \nabla^3 z\right) &= -\left(\nabla\left(f'(z)\nabla^2 z\right), \nabla^2 z\right) \\
&= -\left(f''(z)\nabla z\nabla^2 z, \nabla^2 z\right) - \left(f'(z)\nabla^3 z, \nabla^2 z\right) \\
&= -\left(f''(z)\nabla z\nabla^2 z, \nabla^2 z\right) - \left(f'(z)\nabla^2 z, \nabla^3 z\right).
\end{aligned}
$$

This implies

$$
\begin{aligned}
2\left(f'(z)\nabla^2 z, \nabla^3 z\right) &= -\left(f''(z)\nabla z\nabla^2 z, \nabla^2 z\right) \\
&= -(1/2)\left(\nabla\left(f''(z)(\nabla z)^2\right) - f'''(z)(\nabla z)^3, \nabla^2 z\right) \\
&= (1/2)\left(f''(z)(\nabla z)^2, \nabla^3 z\right) + (1/2)\left(f'''(z)(\nabla z)^3, \nabla^2 z\right) \\
&= (1/2)\left(\nabla\left(f'(z)\nabla z\right) - f'(z)\nabla^2 z, \nabla^3 z\right) \\
&\quad + (1/2)\left(f'''(z)(\nabla z)^3, \nabla^2 z\right) \\
&= -(1/2)\left(\nabla Bz, \nabla^3 z\right) - (1/2)\left(f'(z)\nabla^2 z, \nabla^3 z\right) \\
&\quad + (1/2)\left(f'''(z)(\nabla z)^3, \nabla^2 z\right),
\end{aligned}
$$

from which the desired equality follows. Note that the formula (4.23) justifies the choice of the functional φ_2 in (3.20).

Thus it follows from (4.20) through (4.23) that

$$
\begin{aligned}
&\left(f(z), \nabla^2 z\right) - \left(f(v), \nabla^2 v\right) \\
&\quad \le 3|\lambda|\varepsilon/5 - \lambda\left(f'(z)\nabla^2 z, f'(w)\nabla w\right) + \left(\nabla^2 v, \nabla^2 z\right) - \left(\nabla^2 z, \nabla^2 z\right) \\
&\qquad - \lambda\left((1/5)(\nabla Bz, Az) + \left(f'''(z)(\nabla z)^3, \nabla^2 z\right)\right) \\
&\quad \le 3|\lambda|\varepsilon/5 - (\lambda/5)(\nabla Bz, Az) + \left(\nabla^2 v, \nabla^2 z\right) - \left(\nabla^2 z, \nabla^2 z\right) \\
&\qquad - (\lambda/5)\left[\left(f'(z)\nabla^2 z, f'(w)\nabla w\right) + 5\left(f'''(z)(\nabla z)^3, \nabla^2 z\right)\right].
\end{aligned}
$$

We now use the identity $\lambda(\nabla Bw, Az) = (\nabla^2 z, \nabla^2 z) - (\nabla^2 v, \nabla^2 z)$ in the above estimate, we have

(4.25)

$$
\begin{aligned}
&\left(f(z), \nabla^2 z\right) - \left(f(v), \nabla^2 v\right) \\
&\quad \le 3|\lambda|\varepsilon/5 - (\lambda/5)\left[\left(f'(z)\nabla^2 z, f'(w)\nabla w\right) + 5\left(f'''(z)(\nabla z)^3, \nabla^2 z\right)\right] \\
&\qquad - (\lambda/5)(\nabla Bz - \nabla Bw, Az) - (6/5)\left(\left(\nabla^2 z, \nabla^2 z\right) - \left(\nabla^2 v, \nabla^2 z\right)\right).
\end{aligned}
$$

Also, we may apply (4.7) to get

$$
(4.26) \qquad |(\nabla Bz - \nabla Bw, Az)| \le (|\nabla Bz - \nabla Bv| + |\nabla Bv - \nabla Bw|)\,|Az| \le 2\varepsilon/5.
$$

We therefore infer from (4.25), (4.26) and Lemma 3.3 that

$$
\begin{aligned}
\left(f(z), \nabla^2 z\right) - \left(f(v), \nabla^2 v\right) + (6/5)\left(\left(\nabla^2 z, \nabla^2 z\right) - \left(\nabla^2 v, \nabla^2 z\right)\right)& \\
\le \varepsilon|\lambda| + |\lambda|\left(a\varphi_2(z) + b\right)/5.&
\end{aligned}
$$

Since

$$\left(\nabla^2 z, \nabla^2 z\right) - \left(\nabla^2 v, \nabla^2 z\right) \geq (1/2)\left(\left(\nabla^2 z, \nabla^2 z\right) - \left(\nabla^2 v, \nabla^2 v\right)\right),$$

we have

$$\varphi_2(z) - \varphi_2(v) \leq (5/6)\,\varepsilon\,|\lambda| + |\lambda|\,(a\varphi_2(z) + b).$$

And so

$$\begin{aligned}\varphi_2(z) &\leq (1 - |\lambda|\,a)^{-1}\,(\varphi_2(v) + (5/6)\,\varepsilon\,|\lambda| + |\lambda|\,b)\\ &< (1 - |\lambda|\,a)^{-1}\,(\varphi_2(v) + |\lambda|\,(b+\varepsilon)).\end{aligned}$$

Noting that (4.8) implies $(1 - |\lambda|\,a)^{-1} \leq 1 + 2\,|\lambda|\,a < e^{2a}$, one obtains

$$\varphi_2(z) \leq e^{2a}\left(|\varphi_2(v)| + |\lambda|\,(b+\varepsilon)\right) < \alpha_2, \tag{4.27}$$

which shows that $z \in K$. Applying Tihonov's Fixed Point Theorem, we get the existence of a fixed point v_λ satisfying (4.1).

As seen from Proposition 3.1, the operator $A + B - \omega_0 I$ is dissipative on the set $\{v \in H^2; |v|_2 \leq \alpha_2\}$. This implies the uniqueness of v_λ. Since $A + B$, $-A - B$ are also quasidissipative, we see from (4.1) that

$$|\lambda|\,\varphi_3(v_\lambda) \leq (1 - |\lambda|\,\omega_0)^{-1}\,|\lambda\,(A+B)\,v| = |\lambda|\,(1 - |\lambda|\,\omega_0)^{-1}\,\varphi_3(v),$$

and so that $\varphi_3(v_\lambda) \leq (1 - |\lambda|\,\omega_0)^{-1}\,\varphi_3(v)$. It is easily seen that if $v \in H^4$ then $v_\lambda \in H^4$. Finally, by the same reasoning as above, we have $\varphi_4(v_\lambda) \leq (1 - |\lambda|\,\omega_1)^{-1}\,\varphi_4(v)$. This completes the proof. □

We can now employ the generation theorem stated in Section 2 and obtain the existence of a nonlinear group of locally Lipschitzian operators on H^2 which provides mild solutions to the initial-value problem for the generalized Kortweg-de Vries equation (3.1).

Theorem 4.2. *There exists a nonlinear group $G = \{G(t)\,; t \in \mathbb{R}\}$ on H^2 such that the following properties are satisfied:*

(i) *For each $v \in H^2$, $G(\cdot)\,v \in C(\mathbb{R}; H^2)$ and $G(\cdot)\,v$ satisfies*

$$G(t)\,v = U(t)\,v + \int_0^t U(t-s)\,BG(s)\,v ds \qquad \text{for } t \in \mathbb{R}. \tag{4.28}$$

(ii) *$\varphi_0(G(t)\,v) = \varphi_0(v)$ and $\varphi_1(G(t)\,v) = \varphi_1(v)$ for $t \in \mathbb{R}$ and $v \in H^2$.*

(iii) *For each $\alpha_0, \alpha_1 \geq 0$ there exist positive numbers $a = a(\alpha_0, \alpha_1)$ and $b = b(\alpha_0, \alpha_1)$ such that*

$$\varphi_2(G(t)\,v) \leq e^{a|t|}\,(\varphi_2(v) + b\,|t|) \tag{4.29}$$

for $t \in \mathbb{R}$ and $v \in H^2$ with $\varphi_0(v) \leq \alpha_0$ and $\varphi_1(v) \leq \alpha_1$.

(iv) *Each of $G(t)$ maps H^3 into itself and H^4 into itself.*

(v) *For any* $\alpha_k \geq 0$, $k = 0,1,2$, *and any* $\tau > 0$, *there exists a positive number* $\omega = \omega(\alpha_0, \alpha_1, \alpha_2, \tau)$ *such that*

$$\varphi_3(G(t)v) \leq e^{\omega_0|t|}\varphi_3(v) \tag{4.30}$$

for $t \in [-\tau, \tau]$ *and* $v \in H^3$ *with* $\varphi_k(v) \leq \alpha_k$, $k = 0,1,2$. *Therefore, if* $v \in H^3$, *then* $G(\cdot)v \in C(\mathbb{R}; H^3) \cap C^1(\mathbb{R}; L^2)$.

(vi) *For any* $\alpha_k \geq 0$, $k = 0,1,2,3$, *and any* $\tau > 0$, *there exists a positive number* $\overline{\omega} = \overline{\omega}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \tau)$ *such that*

$$\varphi_4(G(t)v) \leq e^{\overline{\omega}|t|}\varphi_4(v) \tag{4.31}$$

for $t \in [-\tau, \tau]$ *and* $v \in H^4$ *with* $\varphi_k(v) \leq \alpha_k$, $k = 0,1,2,3$. *Therefore, for each* $v \in H^4$, $G(\cdot)v \in \mathscr{C}(\mathbb{R}; H^4) \cap \mathscr{C}^1(\mathbb{R}; H^1)$ *and the function* $u(t,x) = [G(t)v](x)$ *satisfies equation* (3.1) *pointwise on* $\mathbb{R} \times \mathbb{R}$.

Proof. Let $v \in H^2$. Since the range condition has been verified for $v \in H^3$ in Theorem 4.1, we show that the domain $D(A+B)$ is dense in H^2. We first choose a sequence $\{v_n\}$ in H^3 such that $v_n \to v$ in H^2 as $n \to \infty$. Let $\varepsilon \in (0,1)$ and choose α_0, α_1, $\alpha_2 > 0$ so that

$$\sup_{n\geq 1} \varphi_k(v_n) + \varepsilon < \alpha_k,\ k = 0,1 \ \text{ and } \ e^{2a}\left(\sup_{n\geq 1}|\varphi_2(v_n)| + (b+\varepsilon)\right) < \alpha_2,$$

where $a = a(\alpha_0, \alpha_1)$ and $b = b(\alpha_0, \alpha_1)$ are numbers as specified in Lemma 3.3. Applying Theorem 4.1, one finds $\lambda_0 > 0$ such that to $\lambda \in (-\lambda_0, \lambda_0)$ and $n \geq 1$ there corresponds a unique element $v_{\lambda,n} \in K$ satisfying

$$v_{\lambda,n} - \lambda(A+B)v_{\lambda,n} = v_n, \tag{4.32}$$

and

$$\begin{aligned} \varphi_0(v_{\lambda,n}) &\leq \varphi_0(v_n) + |\lambda|\varepsilon, \\ \varphi_1(v_{\lambda,n}) &\leq \varphi_1(v_n) + |\lambda|\varepsilon, \\ \varphi_2(v_{\lambda,n}) &\leq (1 - |\lambda|a)^{-1}(\varphi_2(v_n) + |\lambda|(b+\varepsilon)), \\ \varphi_3(v_{\lambda,n}) &\leq (1 - |\lambda|\omega_0)^{-1}\varphi_3(v_n). \end{aligned} \tag{4.33}$$

Since, as seen from Proposition 3.1, B is quasidissipative on bounded sets of H^2, (4.32) implies $|v_{\lambda,n} - v_{\lambda,m}| \leq (1 - |\lambda|\omega_0)^{-1}|v_n - v_m|$, and so $v_{\lambda,n}$ converges in L^2 to some v_λ. Since K is weakly compact in H^3, one can extract a subsequence v_{λ,n_k} such that $v_{\lambda,n_k} \rightharpoonup v_\lambda$ in H^3. We now prove that this implies $v_{\lambda,n_k} \to v_\lambda$ in H^2. By (3.4), one obtains

$$(\nabla(v_{\lambda,n_k} - v_\lambda), \nabla(v_{\lambda,n_k} - v_\lambda)) = -\left(v_{\lambda,n_k} - v_\lambda, \nabla^2(v_{\lambda,n_k} - v_\lambda)\right),$$

which implies that $|\nabla(v_{\lambda,n_k} - v_\lambda)| \to 0$ and hence $v_{\lambda,n_k} \to v_\lambda$ in H^1. Also,

$$\left(\nabla^2(v_{\lambda,n_k} - v_\lambda), \nabla^2(v_{\lambda,n_k} - v_\lambda)\right) = -\left(\nabla(v_{\lambda,n_k} - v_\lambda), \nabla^3(v_{\lambda,n_k} - v_\lambda)\right),$$

from which we obtain the required convergence in H^2 of v_{λ,n_k} to v_λ. From Proposition 3.1 (ii), we see that $Av_{\lambda,n_k} \rightharpoonup Av_\lambda$ and $Bv_{\lambda,n_k} \to Bv_\lambda$. Passing to the limit as $k \to \infty$ in the relation $v_{\lambda,n_k} - \lambda (A+B) v_{\lambda,n_k} = v_{n_k}$, we get

$$v_\lambda - \lambda (A+B) v_\lambda = v.$$

Moreover, $\varphi_0 (v_{\lambda,n_k}) \le \varphi_0 (v_{n_k}) + |\lambda| \varepsilon$ and $v_{n_k} \to v$ in H^2. Hence $\varphi_0 (v_\lambda) \le \varphi_0 (v) + |\lambda| \varepsilon$. We also infer from (4.33) that

$$(4.34) \qquad (1/2) |\nabla v_{\lambda,n_k}|^2 - \int_{-\infty}^{\infty} \int_0^{v_{\lambda,n_k}(x)} f(\xi)\, d\xi dx \le (1/2) |\nabla v_{n_k}|^2 - \int_{-\infty}^{\infty} \int_0^{v_{n_k}(x)} f(\xi)\, d\xi dx + |\lambda| \varepsilon.$$

Now, we have

$$(4.35) \qquad \left| \int_{-\infty}^{\infty} \int_{v_{\lambda,n_k}(x)}^{v_\lambda(x)} f(\xi)\, d\xi dx \right| = \left| \int_{-\infty}^{\infty} \left(\int_0^1 f'(\theta v_\lambda(x) + (1-\theta) v_{\lambda,n_k}(x))\, d\theta \right) (v_\lambda(x) - v_{\lambda,n_k}(x))\, dx \right| \le \left| \int_0^1 f'(\theta v_\lambda(\cdot) + (1-\theta) v_{\lambda,n_k}(\cdot))\, d\theta \right| |v_\lambda - v_{\lambda,n_k}|.$$

Since the first term on the right-hand side is uniformly bounded with respect to k, we let $k \to \infty$ in (4.34) to get

$$(4.36) \qquad (1/2) |\nabla v_\lambda|^2 - \int_{-\infty}^{\infty} \int_0^{v_\lambda(x)} f(\xi)\, d\xi dx \le (1/2) |\nabla v|^2 - \int_{-\infty}^{\infty} \int_0^{v(x)} f(\xi)\, d\xi dx + |\lambda| \varepsilon.$$

Thus, it follows that $\varphi_1 (v_\lambda) \le \varphi_1 (v) + |\lambda| \varepsilon$. We then demonstrate that $\varphi_2(v_\lambda) \le (1 - |\lambda| a)^{-1}(\varphi_2(v) + |\lambda|(b+\varepsilon))$. From (3.4) we have $(5/6) (f(v_{\lambda,n_k}), \nabla^2 v_{\lambda,n_k}) = (5/6) (Bv_{\lambda,n_k}, \nabla v_{\lambda,n_k})$, and so we may apply the continuity of B on bounded sets in H^2 to assert that

$$(4.37) \qquad (5/6) \left(f(v_{\lambda,n_k}), \nabla^2 v_{\lambda,n_k}\right) \to (5/6) \left(f(v_\lambda), \nabla^2 v_\lambda\right) \qquad \text{as } k \to \infty.$$

Combining (4.37) and (4.33), we deduce that $\varphi_2 (v_\lambda) \le (1 - |\lambda| a)^{-1} (\varphi_2 (v) + |\lambda| (b+\varepsilon))$ as required. Applying now Theorem 2.1 to the operator $A+B$, we conclude that there exists a nonlinear group $G = \{G(t); t \in \mathbb{R}\}$ of locally Lipschitz operators on H^2 such that for each $v \in H^2$ the function $G(\cdot) v \in C(\mathbb{R}; L^2)$ satisfies (4.28) and the growth conditions

$$(4.38) \qquad \varphi_0 (G(t) v) \le \varphi_0 (v), \quad \varphi_1 (G(t) v) \le \varphi_1 (v)$$

for $v \in H^2$ and $t \in \mathbb{R}$,

$$(4.39) \qquad \varphi_2 (G(t) v) \le e^{a|t|} (\varphi_2 (v) + b |t|)$$

for $t \in \mathbb{R}$ and $v \in H^2$ with $\varphi_0(v) \le \alpha_0$, $\varphi_1(v) \le \alpha_1$.

We now show that $G(\cdot)v$ belongs to $\mathscr{C}(\mathbb{R}; H^2)$ for each $v \in H^2$. Let $v \in H^2(\mathbb{R})$, $t \in \mathbb{R}$ and let $\{t_n\}$ be a sequence such that $t_n \to t$ as $n \to \infty$. By Lemma 3.2 (ii), (4.38) and (4.39), there is $\beta_2 = \beta_2(\{t_n\}, t)$ such that $|\nabla^2 G(t_n)v| \le \beta_2$ for $n \ge 1$. Since

$$\begin{aligned}|\nabla G(t_n)v - \nabla G(t)v|^2 &= \left|\left(\nabla^2 G(t_n)v - \nabla^2 G(t)v, G(t_n)v - G(t)v\right)\right| \\ &\le M\,|G(t_n)v - G(t)v|,\end{aligned}$$

it follows that $G(\cdot)v \in \mathscr{C}(\mathbb{R}, H^1)$. To show that $G(\cdot)v$ belongs to $\mathscr{C}(\mathbb{R}; H^2)$, we first prove its continuity with respect to the weak topology of H^2 and then use the exponential growth condition (4.39) to prove the continuity of $|\nabla^2 G(\cdot)|$. The desired strong continuity in H^2 will then follow from a criterion for the strong convergence in uniformly convex spaces. We first note that

$$\left(\nabla^2 G(t_n)v, \psi\right) = -\left(\nabla G(t_n)v, \nabla\psi\right) \to -\left(\nabla G(t)v, \nabla\psi\right)$$

as $n \to \infty$, for each $\psi \in \mathscr{C}_c^\infty(\mathbb{R})$. Since $(\nabla G(t)v, \nabla\psi) = -(\nabla^2 G(t)v, \psi)$, one can see that $G(t_n)v \rightharpoonup G(t)v$ in H^2 as $n \to \infty$. Moreover, from (4.39) and the group property of G, we obtain

$$\varphi_2(G(t_n)v) = \varphi_2(G(t_n - t)G(t)v) \le e^{a|t_n - t|}\left(\varphi_2(G(t)v) + b\,|t_n - t|\right).$$

Therefore we assert that

$$\varlimsup_{n\to\infty} \varphi_2(G(t_n)v) \le \varphi_2(G(t)v). \tag{4.40}$$

On the other hand, $G(t_n)v \rightharpoonup G(t)v$ in H^2, $G(t_n)v \to G(t)v$ in H^1 and

$$(5/6)\left(f(G(t_n)v), \nabla^2 G(t_n)v\right) = (5/6)\left(BG(t_n)v, \nabla G(t_n)v\right).$$

Hence we have

$$\varliminf_{n\to\infty} \varphi_2(G(t_n)v) \ge \varphi_2(G(t)v). \tag{4.41}$$

Combining (4.40) and (4.41) implies

$$\lim_{n\to\infty} \varphi_2(G(t_n)v) = \varphi_2(G(t)v).$$

Since

$$\begin{aligned}\varphi_2(G(t_n)v) &= (1/2)\left|\nabla^2 G(t_n)v\right| + (5/6)\left(f\left(G(t_n)v, \nabla^2 G(t_n)v\right)\right) \\ &= (1/2)\left|\nabla^2 G(t_n)v\right| + (5/6)\left(BG(t_n)v, \nabla G(t_n)v\right),\end{aligned}$$

and $G(t_n)v \to G(t)v$ in H^1 as $n \to \infty$, it follows that

$$\left|\nabla^2 G(t_n)v\right| \to \left|\nabla^2 G(t)v\right| \qquad \text{as } n \to \infty. \tag{4.42}$$

Since $G(t_n)v \rightharpoonup G(t)v$ in H^2, (4.42) implies via a criterion for the strong convergence uniformly convex spaces that $\nabla^2 G(t_n)v \to \nabla^2 G(t)v$ in L^2, and so $G(t_n)v \to G(t)v$ in H^2 as requested. Also, the inequalities in (4.38) imply

$$\varphi_0(G(t)v) \leq \varphi_0(v) = \varphi_0(G(-t)G(t)v) \leq \varphi_0(G(t)v)$$

and

$$\varphi_1(G(t)v) \leq \varphi_1(v) = \varphi_1(G(-t)G(t)v) \leq \varphi_1(G(t)v),$$

hence $\varphi_0(G(t)v) = \varphi_0(v)$ and $\varphi_1(G(t)v) = \varphi_1(v)$ for each $v \in H^2$.

By Theorems 2.1 and 4.2, each of $G(t)$ maps H^3 into itself and (4.32) easily implies (4.36). Next, let us suppose that $v \in H^3$. It is seen that

$$e^{-\omega_0|t_n-t|}\varphi_3(G(t_n)v) \leq \varphi_3(G(t)v) \leq e^{\omega_0|t_n-t|}\varphi_3(G(t_n)v),$$

which implies in turn

$$\lim_{n\to\infty}\varphi_3(G(t_n)v) = \varphi_3(G(t)v). \tag{4.43}$$

Since

$$\left(\nabla^3 G(t_n)v + \nabla f(G(t_n)v), \psi\right) = -\left(\nabla^2 G(t_n)v + f(G(t_n)v), \nabla\psi\right)$$

for each $\psi \in C_c^\infty(\mathbb{R})$, using the H^2-continuity of $G(\cdot)v$ one obtains that

$$\nabla^3 G(t_n)v + \nabla f(G(t_n)v) \rightharpoonup \nabla^3 G(t)v + \nabla f(G(t)v) \qquad \text{as } n\to\infty$$

and this, together with (4.43), implies

$$\nabla^3 G(t_n)v + \nabla f(G(t_n)v) \to \nabla^3 G(t)v + \nabla f(G(t)v) \qquad \text{as } n\to\infty. \tag{4.44}$$

But $\nabla f(G(t_n)v) = -BG(t_n)v \to -BG(t)v = \nabla f(G(t)v)$, so

$$\nabla^3 G(t_n)v \to \nabla^3 G(t)v \qquad \text{as } n\to\infty,$$

and therefore $G \in \mathscr{C}(\mathbb{R}; H^3) \cap \mathscr{C}^1(\mathbb{R}; L^2)$. In the same way one can get that, for $v \in H^4$, $G \in \mathscr{C}(\mathbb{R}; H^4) \cap \mathscr{C}^1(\mathbb{R}; H^1)$, and in this further case $u(t,x)$ satisfies the equation (3.1) pointwise on $\mathbb{R}\times\mathbb{R}$. □

Remark 4.1. It is easily seen that differentiating a solution of (3.1) with respect to t one reduces its regularity from $\mathscr{C}(\mathbb{R}; H^3)$ to $\mathscr{C}(\mathbb{R}; L^2)$, that is, with three x-derivatives. With regard to this, it should be mentioned that if the function f in (3.1) is of class $\mathscr{C}^\infty(\mathbb{R})$ and satisfies (3.3), then $G(\cdot)v \in \bigcap_{m=0}^{[k/3]} \mathscr{C}^m\left(\mathbb{R}; H^{k-3m}\right)$ for each $v \in H^k$ and for any integer $k \geq 3$. It is also interesting to note that the regularity of the group $G = \{G(t); t\in\mathbb{R}\}$ with respect to t is established with the aid of the l.s.c. functionals φ_2, φ_3 and φ_4.

5 Regularized dispersive equations

In this section we establish the existence of nonlinear groups $G_\mu = \{G_\mu(t); t \in \mathbb{R}\}$ of Fréchet differentiable operators on H^1 which provide mild solutions to the initial-value problems for nonlinear dispersive equations of the form

$$u_t + (f(u))_x + u_{xxx} - \mu u_{txx} = 0, \qquad t, x \in \mathbb{R} \tag{5.1}$$

$$u(0,x) = v(x), \qquad x \in \mathbb{R}. \tag{5.2}$$

Here $\mu > 0$, f is merely a nonlinear function of class $\mathscr{C}^1(\mathbb{R})$ satisfying $f(0) = 0$, and v is an initial function given in H^1. The convergence of $G_\mu = \{G_\mu(t); t \in \mathbb{R}\}$ to the group $G = \{G(t); t \in \mathbb{R}\}$ obtained in Theorem 4.2 will be also discussed in the next section.

Equation (5.1) is regarded as a pseudoparabolic regularization of the generalized K-dV equation (3.1). An equation related to (5.1) is the long wave equation

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{5.3}$$

which was proposed as a substitute for K-dV equation by Benjamin, Bona and Mahony in [2]. A derivation of equation (5.3) is also described in Benjamin [3]. Since then, tremendous work has been devoted to the study of equations of type (5.3). We refer the reader to for instance Iwamiya, Oharu and Takahashi [10], Medeiros and Menzala [15], Medeiros and Miranda [16]. See also Avrin and Goldstein [1], Goldstein, Kajikiya and Oharu [9] for the discussions on equation (5.3) in several space variables, Tsutsumi and Mukasa [24] for the other types of parabolic regularizations of (3.1), Bona and Chen [4], Bona and Smith [5] and Takahashi [23] for more problems related to (5.1). Also, initial-boundary value problems for a class of equations which significantly generalize (5.3) were treated in [19] by Oharu and Takahashi using nonlinear operator theory.

In order to derive a semilinear evolution equation in H^1 which is equivalent to (5.1), we begin by defining some operators and stating their properties.

Let Δ be the one-dimensional Laplace operator defined by $\Delta v = \nabla^2 v$ for $v \in H^2$. Let $\mu > 0$. It is easy to see that $I - \mu\Delta$ has a bounded inverse $(I - \mu\Delta)^{-1}$ on L^2 which satisfies the relation

$$\left((I - \mu\Delta)^{-1} v, w\right) = \left(v, (I - \mu\Delta)^{-1} w\right), \qquad \text{for } v, w \in L^2. \tag{5.4}$$

Moreover, by (3.4),

$$\left((I - \mu\Delta)^{-1} \nabla v, v\right) = -\left((I - \mu\Delta)^{-1} v, \nabla v\right) \qquad \text{for } v \in H^1,$$

and so (5.4) implies

$$\left((I - \mu\Delta)^{-1} \nabla v, v\right) = 0 \qquad \text{for } v \in H^1. \tag{5.5}$$

We then introduce a closed linear operator A_μ densely defined in L^2 by

$$A_\mu v = (1/\mu)\left(\nabla v - (I - \mu\Delta)^{-1} \nabla v\right) \qquad \text{for } v \in H^1 \tag{5.6}$$

and a nonlinear operator B_μ from H^1 into H^2 defined by

$$B_\mu v = -(I-\mu\Delta)^{-1}\nabla f(v) \qquad \text{for } v \in H^1. \tag{5.7}$$

Moreover, we define new scalar products on H^1 and H^2, respectively, by

$$\begin{aligned} (u,v)_{1,\mu} &= (u,v) + \mu(\nabla u, \nabla v) && \text{for } u, v \in H^1, \\ (u,v)_{2,\mu} &= (u,v) + 2\mu(\nabla u, \nabla v) + \mu^2(\nabla^2 u, \nabla^2 v) && \text{for } u, v \in H^2, \end{aligned}$$

and the associated norms $|\cdot|_{1,\mu}$ on H^1 and $|\cdot|_{2,\mu}$ on H^2.

Let k be an arbitrary positive integer. It is easy to see that A_μ maps H^{k+1} into H^k and

$$\left(\nabla^k A_\mu u, \nabla^k u\right) = (1/\mu)\left(\nabla^{k+1}u, \nabla^k u\right) - (1/\mu)\left((I-\mu\Delta)^{-1}\nabla^{k+1}u, \nabla^k u\right) \tag{5.8}$$

for $u \in H^{k+1}$. Therefore it follows from (5.5) and (5.8) that $\left(\nabla^k A_\mu u, \nabla^k u\right) = 0$ for $u \in H^{k+1}$. Hence $(A_\mu u, u)_k = 0$ for each $v \in H^{k+1}$, and so A_μ is the generator of a (C_0)-group $T_\mu = \{T_\mu(t); t \in \mathbb{R}\}$ on L^2 such that each of $T_\mu(t)$ maps H^k into itself and satisfies the identity

$$|T_\mu(t)v|_k = |v|_k \qquad \text{for } t \in \mathbb{R} \text{ and } v \in H^k.$$

Also, we observe that

$$\left|(I-\mu\Delta)^{-1}w\right|_{2,\mu} = |w| \qquad \text{for } w \in H^2. \tag{5.9}$$

Now equation (5.1) is rewritten as a semilinear evolution equation in $\left(H^1, |\cdot|_\mu\right)$ of the form

$$(d/dt)\, u_\mu(t) = (A_\mu + B_\mu)\, u_\mu(t), \qquad t \in \mathbb{R}. \tag{5.10}$$

Our purpose is to construct a group of the solution operators to (5.10) on H^1 by applying using the generation theorem stated in Section 2. To this end, we need to establish further regularity properties of the operators B_μ, stated in the next proposition.

Proposition 5.1. *Let $\mu \in (0,1)$. For the nonlinear operator B_μ, the following statements hold:*

(i) *For each $\alpha \geq 0$ there is a number $\omega = \omega(\alpha,\mu) \geq 0$ such that*

$$|B_\mu v - B_\mu w|_{1,\mu} \leq \omega\, |v - w|$$

for $v, w \in H^1$ with $|v|_{1,\mu} \leq \alpha$ and $|w|_{1,\mu} \leq \alpha$.

(ii) *B_μ is continuously Fréchet differentiable on H^1 and its Fréchet derivative $B'_\mu v$ at $v \in H^1$ is given by*

$$B'_\mu(v)\,w = -\nabla\left((I-\mu\Delta)^{-1}\left(f'(v)\,w\right)\right) \qquad \textit{for } w \in H^1.$$

Proof. Let $\alpha \geq 0$ and set

$$\omega = \mu^{-1/2} \sup \left\{ |f'(v)|_{L^\infty} : v \in H^1, |v|_{1,\mu} \leq \alpha \right\}.$$

By (3.4) and the definition of B_μ, we see that

$$\begin{aligned} |B_\mu v - B_\mu w|_\mu^2 &= (B_\mu v - B_\mu w, B_\mu v - B_\mu w) - \mu (\Delta (B_\mu v - B_\mu w), B_\mu v - B_\mu w) \\ &= -(\nabla f(v) - \nabla f(w), B_\mu v - B_\mu w) \\ &= (f(v) - f(w), \nabla (B_\mu v - B_\mu w)). \end{aligned}$$

Hence $|B_\mu v - B_\mu w|_{1,\mu}^2 \leq \omega |v - w| \mu^{1/2} |\nabla (B_\mu v - B_\mu w)|$, from which the desired estimate follows. To get (ii), we first observe that

$$\begin{aligned} &\left| B_\mu (v+w) - B_\mu v + \nabla (I - \mu\Delta)^{-1} f'(v) w \right|_{1,\mu}^2 \\ &= -\left(B_\mu (v+w) - B_\mu (v) + \nabla \left((I - \mu\Delta)^{-1} \right) f'(v) w, \nabla f(v+w) - \nabla f(v) \right) \\ &\quad - \left(\nabla \left(B_\mu (v+w) - \nabla B_\mu v + \nabla (I - \mu\Delta)^{-1} f'(v) w \right), f'(v) w \right) \\ &= \left(\nabla \left(B_\mu (v+w) - B_\mu (v) - \nabla \left((I - \mu\Delta)^{-1} \right) f'(v) w \right), f(v+w) - f(v) - f'(v) w \right), \end{aligned}$$

which implies

$$\left| B_\mu (v+w) - B_\mu v - \nabla (I - \mu\Delta)^{-1} (f'(v) w) \right|_{1,\mu} \leq \mu^{-1/2} |f(v+w) - f(v) - f'(v) w|.$$

This shows that $\left| B_\mu (v+w) - B_\mu v - \nabla (I - \mu\Delta)^{-1} (f'(v) w) \right|_{1,\mu} = o\left(|w|_{1,\mu} \right)$, and so (ii) is proved.

□

We are now in a position to state the main result of this section.

Theorem 5.1. *For each $\mu > 0$ there exists a nonlinear group $G_\mu = \{G_\mu(t); t \in \mathbb{R}\}$ of locally Lipschitzian operators on H^1 which has the properties below:*

(i) *If $v \in H^1$, then $G_\mu(\cdot) v \in \mathscr{C}(\mathbb{R}; H^1) \cap \mathscr{C}^1(\mathbb{R}; L^2)$ and*

$$G_\mu(t) v = U_\mu(t) v + \int_0^t U_\mu(t-s) B_\mu G_\mu(s) v ds,$$

for $t \in \mathbb{R}$ and $v \in H^1$.

(ii) *If $v \in H^2$, then $G_\mu(\cdot) v \in C(\mathbb{R}; H^2) \cap C^1(\mathbb{R}; H^1) \cap C^2(\mathbb{R}; L^2)$ and satisfies the equation in $C(\mathbb{R}; H^1)$*

$$(d/dt) G_\mu(t) v = (A_\mu + B_\mu) G_\mu(t) v \qquad \text{for } t \in \mathbb{R}.$$

(iii) *Each of $G_\mu(t)$ is continuously Fréchet differentiable on H^1.*

(iv) *$\varphi_{0,\mu}(G_\mu(t) v) = \varphi_{0,\mu}(v)$ for $t \in \mathbb{R}$ and $v \in H^1$, where the functional $\varphi_{0,\mu}$ is defined by*

$$\varphi_{0,\mu}(v) = |v|_{1,\mu} \text{ for } v \in H^1.$$

(v) *$\varphi_1(G_\mu(t) v) = \varphi_1(v)$ for $t \in \mathbb{R}$ and $v \in H^1$, where φ_1 is the functional on H^1 defined by* (3.20).

Proof. One may show that $A_\mu + B_\mu$ satisfies the following range condition: For each $\alpha \geq 0$ there is a number $\lambda_\mu = \lambda_\mu(\alpha) > 0$ such that for $v \in H^1$ with $|v|_{1,\mu} \leq \alpha$ and $\lambda \in (-\lambda_\mu, \lambda_\mu)$ there is an element $v_\lambda \in H^1$ such that

$$\begin{aligned} &v_\lambda - \lambda (A_\mu + B_\mu) v_\lambda = v, \\ &\varphi_{0,\mu}(v_\lambda) \leq \varphi_{0,\mu}(v) + |\lambda| \varepsilon, \\ &\varphi_1(v_\lambda) \leq \varphi_1(v) + |\lambda| \varepsilon. \end{aligned}$$

Therefore the proof is obtained in a way similar to that of Theorem 6.1, with the aid of Theorem 2.1 and Proposition 5.1. □

Remark 5.1. Note that the differentiation of a solution reduces its Sobolev regularity from $\mathscr{C}(\mathbb{R}; H^1)$ to $\mathscr{C}(\mathbb{R}; L^2)$. If in particular $f \in \mathscr{C}^\infty(\mathbb{R})$, then $G_\mu(\cdot) v \in \bigcap_{m=0}^{k} \mathscr{C}^m(\mathbb{R}; H^{k-m})$ for each $v \in H^k$ and each integer $k \geq 1$.

6 A convergence theorem for nonlinear groups

In the previous sections we have obtained the existence of the nonlinear groups $G = \{G(t); t \geq 0\}$ and $G_\mu = \{G_\mu(t); t \geq 0\}$ of locally Lipschitzian operators on H^2 and H^1, respectively. The group $\{G(t)\}$ provides mild solutions to the initial value problems for the generalized K-dV equation (3.1) and thegroups $\{G_\mu(t)\}$, $\mu > 0$, provide mild solutions to its pseudoparabolic regularizations (5.1). Here we discuss the convergence of G_μ to G under the assumption that the nonlinear function f is of class $\mathscr{C}^3(\mathbb{R})$ and satisfies (3.3).

In what follows, $\varphi_{k,\mu}$, $k = 0, 1, 2, 3$ denote the functionals

$$\begin{aligned} (6.1) \qquad &\varphi_{0,\mu}(v) = |v|_{1,\mu} = \left(|v|^2 + \mu |\nabla v|^2\right)^{1/2}, && v \in H^1; \\ &\varphi_{1,\mu}(v) = \varphi_1(v) = (1/2)|\nabla v|^2 - \int_{-\infty}^{\infty} \int_0^{v(x)} f(\xi)\, d\xi dx, && v \in H^1; \\ &\varphi_{2,\mu}(v) = (1/2)\left|\nabla^2 v\right|^2 + (\mu/12)\left|\nabla^3 v\right|^2 + (5/6)\left(f(v), \nabla^2 v\right) \\ &\qquad - (5\mu/12)\left(f'(v), \left(\nabla^2 v\right)^2\right), && v \in H^3; \\ &\varphi_{3,\mu}(v) = \varphi_3(v) = \left|\nabla^3 v + \nabla f(v)\right|, && v \in H^3. \end{aligned}$$

Following the argument in Section 3, we first establish relations between $\varphi_{k,\mu}$-boundedness and norm boundedness.

Lemma 6.1. *Let $\mu \in (0,1)$. For α_0, α_1, $\alpha_2 \geq 0$, there is $\beta_2 = \beta_2(\alpha_0, \alpha_1, \alpha_2) \geq 0$, independent of μ, such that $v \in H^3$ and $\varphi_{0,\mu}(v) \leq \alpha_0$, $\varphi_{1,\mu}(v) \leq \alpha_1$ and $\varphi_{2,\mu}(v) \leq \alpha_2$ imply $|\nabla^2 v| \leq \beta_2$ and $\mu^{1/2}|\nabla^3 v| \leq \beta_2$.*

Proof. Let $\mu \in (0,1)$, $\alpha_0, \alpha_1, \alpha_2 \geq 0$, and let $v \in H^3$ be such that $\varphi_{0,\mu}(v) \leq \alpha_0$, $\varphi_{1,\mu}(v) \leq \alpha_1$, $\varphi_{2,\mu}(v) \leq \alpha_2$. Then, in a way similar to the proof of Lemma 3.2, we see that there exist numbers $\beta_0 = \beta_0(\alpha_0) \geq 0$ and $\beta_1 = \beta_1(\alpha_0, \alpha_1) \geq 0$ such that $|v| \leq \beta_0$ and $|\nabla v| \leq \beta_1$. We further note that

(6.2) $$C\left|\nabla^2 v\right|^2 \leq (1/2)\left(C^2|\nabla v|^2 + \left|\nabla^3 v\right|^2\right) \qquad \text{for each } v \in H^3 \text{ and } C \in \mathbb{R}_+.$$

Since $\varphi_{2,\mu}(v) \leq \alpha_2$, (6.1) implies the estimate

$$\begin{aligned}(1/2)\left|\nabla^2 v\right|^2 &+ (\mu/12)\left|\nabla^3 v\right|^2 \\ &\leq \alpha_2 + (5/6)\left(f'(v), (\nabla v)^2\right) + (5\mu/12)\left(f'(v), \left(\nabla^2 v\right)^2\right).\end{aligned}$$

Since $|w|_{L^\infty} \leq |w|_1$ for each $w \in H^1$, one finds a constant $\gamma_1 = \gamma_1(\alpha_0, \alpha_1) \geq 0$ such that $|v|_{L^\infty} \leq \gamma_1$. We then define

$$C = C(f', \alpha_0, \alpha_1) = \sup\{|f'(x)|\,;\,|x| \leq \gamma_1\}.$$

Therefore, combining (6.1) and (6.2) implies the following estimate:

$$\begin{aligned}(1/2)\left|\nabla^2 v\right|^2 + (\mu/12)\left|\nabla^3 v\right|^2 &\leq \alpha_2 + (5/6)\,C\,|\nabla v|^2 + (5\mu/12)\,C\left|\nabla^2 v\right|^2 \\ &\leq \alpha_2 + (5/6)\,C\,|\nabla v|^2 + (5\mu/72)\left(9C^2|\nabla v|^2 + \left|\nabla^3 v\right|^2\right).\end{aligned}$$

Thus

$$(1/2)\left|\nabla^2 v\right|^2 + (\mu/72)\left|\nabla^3 v\right|^2 \leq \alpha_2 + (5/24)\,C\beta_1^2\,(4 + 3C),$$

and the proof of Lemma 6.1 is complete. □

In order to apply Theorem 2.1, we need the quasidissipativity of the operators B_μ on level sets with respect to φ_k, $k = 0, 1, 2$.

Lemma 6.2. *Let $\mu \in (0,1)$. For $\alpha_0, \alpha_1, \alpha_2 \geq 0$ there exists a number $\omega_0 = \omega_0(\alpha_0, \alpha_1, \alpha_2)$, independent of μ, such that*

(6.3) $$\left|(B_\mu v - B_\mu w, v - w)_{1,\mu}\right| \leq \omega_0\,|v - w|^2$$

and

(6.4) $$\left|(B_\mu v - B_\mu w, v - w)_{2,\mu}\right| \leq \omega_0\,|v - w|^2$$

for $v, w \in H^3$ with $\varphi_{k,\mu}(v) \leq \alpha_k$ and $\varphi_{k,\mu}(w) \leq \alpha_k$, $k = 0, 1, 2$.

Proof. Let $\alpha_0, \alpha_1, \alpha_2 \geq 0$ and let v, $w \in H^3$ be such that $\varphi_{k,\mu}(v) \leq \alpha_k$ and $\varphi_{k,\mu}(w) \leq \alpha_k$, $k = 0, 1, 2$. It is clear that

$$\begin{aligned}(B_\mu v - B_\mu w, v - w)_{1,\mu} &= ((I - \mu\Delta)(B_\mu v - B_\mu w), v - w) \\ &= (f(v) - f(w), \nabla(v - w))\end{aligned}$$

and

$$\begin{aligned}&(B_\mu v - B_\mu w, v - w)_{2,\mu} \\ &\quad= ((I - \mu\Delta)(B_\mu v - B_\mu w), v - w) - \mu\left((I - \mu\Delta)(B_\mu v - B_\mu w), \nabla^2(v - w)\right) \\ &\quad= (f(v) - f(w), \nabla(v - w)) + \mu\left(\nabla(f(v) - f(w)), \nabla^2(v - w)\right).\end{aligned}$$

From this we obtain the required conclusion. □

We next prove the range condition for the operators $A_\mu + B_\mu$.

Theorem 6.1. *Let* $\varepsilon > 0$, $v \in H^3$, *and suppose that* $\alpha_0, \alpha_1, \alpha_2 \geq 0$ *are chosen so that* $\varphi_{0,\mu}(v) + \varepsilon < \alpha_0$, $\varphi_{1,\mu}(v) + \varepsilon < \alpha_1$ *and* $e^{2a}\{|\varphi_{2,\mu}(v)| + 1 + \varepsilon\} < \alpha_2$ *for all* $\mu \in (0,1)$. *Let* $a = a(\alpha_0, \alpha_1)$ *and* $b = b(\alpha_0, \alpha_1)$ *be positive numbers as specified in Lemma 3.3, and let* $\omega_0 = \omega_0(\alpha_0, \alpha_1, \alpha_2)$ *be a positive number as stated in Lemma 6.2. Then there are numbers* $\mu_0 = \mu_0(\alpha_0, \alpha_1, \alpha_2) > 0$ *and* $\hat{\lambda}_0 = \hat{\lambda}_0(|v|_3, \varepsilon)$, *such that* $0 < \hat{\lambda}_0 \leq \min\{1, 1/2a, 1/\omega_0\}$ *and for each* $\mu \in (0, \mu_0)$ *and each* $\lambda \in \left(-\hat{\lambda}_0, \hat{\lambda}_0\right)$ *there exists a unique element* $v_{\lambda,\mu} \in H^3$ *satisfying*

$$v_{\lambda,\mu} - \lambda(A_\mu + B_\mu)v_{\lambda,\mu} = v, \tag{6.5}$$

and

$$\begin{aligned} \varphi_{0,\mu}(v_{\lambda,\mu}) &\leq \varphi_{0,\mu}(v) + |\lambda|\varepsilon, \\ \varphi_{1,\mu}(v_{\lambda,\mu}) &\leq \varphi_{1,\mu}(v) + |\lambda|\varepsilon, \\ \varphi_{2,\mu}(v_{\lambda,\mu}) &\leq (1 - |\lambda| a)^{-1}[\varphi_{2,\mu}(v) + |\lambda|(b + 1 + \varepsilon)], \\ \varphi_{3,\mu}(v_{\lambda,\mu}) &\leq (1 - |\lambda| a)^{-1}\varphi_{3,\mu}(v). \end{aligned} \tag{6.6}$$

Proof. The proof is obtained in a way similar to that of Theorem 4.1. We first choose $\beta_k > 0$, $k = 0, 1, 2$, so that

$$\{w \in H^3; \varphi_{k,\mu}(w) \leq \alpha_k,\ k = 0,1,2\} \subset \{w \in H^3; |\nabla^k w| \leq \beta_k,\ k = 0,1,2\}$$

for $\mu \in (0,1)$, and $|\nabla^3 v| + 2N_0 \leq \beta_3$, where

$$N_0 = \sup\{|Bw|; w \in H^1, |w| \leq \beta_0, |\nabla w| \leq \beta_1\}.$$

We also employ the same bounds N_1 and N_2 as in the proof of Theorem 4.1 and put

$$M_1 = \sup\{|f'(w)|_{L^\infty}; |w| \leq \beta_0, |\nabla w| \leq \beta_1\}, \tag{6.7}$$
$$M_2 = \sup\{|f''(w)|_{L^\infty}; |w| \leq \beta_0, |\nabla w| \leq \beta_1\}. \tag{6.8}$$

It follows that there exists a positive number $\delta = \delta(|v|_3, \varepsilon)$ such that if $w \in H^3$, $|w - v| < \delta$ and $|\nabla^k w| \leq \max\{\beta_k, |\nabla^k v| + N_k\}$, $k = 0, 1, 2$, then the inequalities in (4.7) are valid. We now define

$$\hat{\lambda}_0 = \min\{1, \delta/\beta_3, \varepsilon/(2\beta_3), 1/\omega_0, 1/(2a)\}$$

and $\mu_0 = \mu_0(\alpha_0, \alpha_1, \alpha_2)$ to be a positive number such that $5\mu_0 M_1/6 \leq 1$ and

$$a\mu_0(4 + 25M_1^2\beta_2^3)/48 + 5M_2\beta_2^2\mu_0^{1/2}(\beta_2 + \mu_0^{1/2}N_0) \leq 1,$$

where ω_0 is a positive number as specified in Lemma 6.2.

Let $\lambda \in \left(-\hat{\lambda}_0, \hat{\lambda}_0\right)$, $\lambda \neq 0$, $\mu \in (0, \mu_0)$, and define a subset $K_{\lambda,\mu}$ of H^3 by

$$K_{\lambda,\mu} = \{w \in H^3; |v - w|_{2,\mu} \leq |\lambda|\beta_3, |\nabla^k w| \leq \beta_k,\ k = 0,1,2,3\}. \tag{6.9}$$

We then define an operator $\Gamma_{\lambda,\mu} : K_{\lambda,\mu} \to H^3$ by

$$\Gamma_{\lambda,\mu} w = (I - \lambda A_\mu)^{-1} (v + \lambda B_\mu w) \qquad \text{for } w \in K_{\lambda,\mu}. \tag{6.10}$$

We wish to show that $\Gamma_{\lambda,\mu}$ is a self map of $K_{\lambda,\mu}$ and has a fixed point. To this end, let $w \in K_{\lambda,\mu}$ and write $z = \Gamma_{\lambda,\mu} w$ for simplicity in notation. We have

$$z = v + \lambda A_\mu z + \lambda B_\mu w. \tag{6.11}$$

Since

$$|z - v|_{2,\mu} = |(I - \mu\Delta)(z - v)|^2, \tag{6.12}$$

it follows that

$$\begin{aligned} |z - v|_{2,\mu} &= \lambda \left((I - \mu\Delta)(z - v), Az + Bw\right) \\ &= -\lambda \left((I - \mu\Delta) v, Az\right) + \lambda \left((I - \mu\Delta)(z - v), Bw\right). \end{aligned}$$

Hence

$$\begin{aligned} |z - v|_{2,\mu} &= \lambda \left(Av, (I - \mu\Delta) z\right) + \lambda \left((I - \mu\Delta)(z - v), Bw\right) \\ &= \lambda \left(Av, (I - \mu\Delta)(z - v)\right) + \lambda \left((I - \mu\Delta)(z - v), Bw\right) \\ &= \lambda \left((I - \mu\Delta)(z - v), Av + Bw\right), \end{aligned}$$

and, in view of (6.12), we deduce the estimate

$$\left(|z - v|^2 + 2\mu |\nabla (z - v)|^2 + \mu^2 |\nabla^2 (z - v)|^2\right)^{1/2} \le |\lambda| \left(|\nabla^3 v| + N_0\right) \le |\lambda| \beta_3. \tag{6.13}$$

On the other hand, $|z - v|_{2,\mu} = |(I - \mu\Delta) A_\mu z|^2$, and so the application of Minkowski's inequality implies

$$\begin{aligned} |\nabla^3 z| &= |A_\mu z|_{2,\mu} \\ &\le |\lambda|^{-1} \left(|z - v|_{2,\mu} + |B_\mu w|_{2,\mu}\right) \\ &\le |\nabla^3 v| + 2N_0 \le \beta_3. \end{aligned} \tag{6.14}$$

Since $|B_\mu w| \le |Bw|$ for $w \in H^1$, the estimates $|\nabla^k z| \le |\nabla^k v| + \lambda N_k$, $k = 0, 1, 2$, are obtained in the same way as in the proof of Theorem 4.1. By (5.6), (5.7) and (6.11), we see that

$$(z - v, z) + \mu (\nabla (z - v), \nabla z) = \lambda (Az + Bw, z).$$

Therefore, it follows from the relations $(Az, z) = (Bz, z) = 0$, that

$$(z, z) + \mu (\nabla z, \nabla z) = (v, z) + \mu (\nabla v, \nabla z) + \lambda (Bw - Bz, z).$$

In view of (4.7), we conclude that

$$\varphi_{0,\mu}(z) \le \varphi_{0,\mu}(v) + |\lambda| \varepsilon. \tag{6.15}$$

We then demonstrate that the second inequality in (6.6) is valid. Noting that

$$\left|\int_{-\infty}^{\infty}\int_{v(x)}^{z(x)} f(\xi)\,d\xi dx - \int_{-\infty}^{\infty} f(w(x))(z(x)-v(x))\,dx\right| \le \left|\int_0^1 f(\theta v+(1-\theta)z)\,d\theta - f(w)\right| |z-v|,$$

we may apply (6.13) to obtain the inequality

$$(\nabla z,\nabla z) - \int_{-\infty}^{\infty}\int_0^{z(x)} f(\xi)\,d\xi dx \le (\nabla v,\nabla z) - \int_{-\infty}^{\infty}\int_0^{v(x)} f(\xi)\,d\xi dx + |\lambda|\,\varepsilon,$$

and hence

$$\varphi_{1,\mu}(z) \le \varphi_{1,\mu}(v) + |\lambda|\,\varepsilon. \tag{6.16}$$

We next derive the estimate

$$\varphi_{2,\mu}(z) \le (1-|\lambda|\,a)^{-1}\left\{\varphi_{2,\mu}(v) + |\lambda|\,(b+1+\varepsilon)\right\}. \tag{6.17}$$

The application of the Mean-Value Theorem implies

$$\begin{aligned}\left(f(z),\nabla^2 z\right) - \left(f(v),\nabla^2 v\right) &= \left(f(z),\nabla^2 z\right) - \left(f(v),\nabla^2 z\right) + \left(f(v),\nabla^2 z\right) - \left(f(v),\nabla^2 v\right)\\ &= \left(f'(w_\theta)\nabla^2 z, z-v\right) - (\nabla Bv, z-v)\\ &= \left((f'(w_\theta)-f'(z))\,\nabla^2 z, z-v\right) + \left(f'(z)\nabla^2 z, z-v\right)\\ &\quad + (\nabla Bw - \nabla Bv, z-v) - (\nabla Bw, z-v),\end{aligned}$$

where $w_\theta(\cdot) = \theta(\cdot)z(\cdot) + (1-\theta(\cdot))v(\cdot)$. In view of (4.23), it is easy to check that

$$\begin{aligned}\left(f'(z)\nabla^2 z, z-v\right) &- \mu\left(f'(z)\nabla^2 z, \nabla^2 z - \nabla^2 v\right)\\ &= \left(f'(z)\nabla^2 z, \lambda(Az+Bw)\right)\\ &= (-\lambda/5)\left[5\left(f'(z)\nabla^2 z, \nabla^3 z\right) - \left(f'''(z)\,(\nabla z)^3, \nabla^2 z\right)\right]\\ &\quad - (\lambda/5)\left[\left(f'''(z)\,(\nabla z)^3, \nabla^2 z\right) + 5\left(f'(z)\nabla^2 z, f'(w)\nabla w\right)\right].\end{aligned}$$

Furthermore, (3.4) and (6.11) together imply

$$(\nabla Bw, z-v) = \lambda\left(\nabla^2 B_\mu w, \nabla^2 z\right) = \left(\nabla^2 z, \nabla^2 z\right) - \left(\nabla^2 z, \nabla^2 v\right).$$

Since $(\nabla A_\mu z, Az) = 0$, this gives the identity

$$\lambda\mu\left(\nabla^3 B_\mu w, \nabla^3 z\right) = -\left[\left(\nabla^2 z, \nabla^2 z\right) - \left(\nabla^2 z, \nabla^2 v\right)\right] + \lambda\left(\nabla Bw, \nabla^3 z\right).$$

We are now ready to show the estimate (6.17). Applying (4.23) and Lemma 3.3, we obtain

$$(1-|\lambda|\,a)\varphi_{2,\mu}(z) \le \varphi_{2,\mu}(v) + |\lambda|\,(b+\varepsilon) + (5\mu/12)\left(f'(z), \left(\nabla^2 z - \nabla^2 v\right)^2\right)$$

$$
\begin{aligned}
&- (1/2) \left|\nabla^2 z - \nabla^2 v\right|^2 + (5\mu/12) \left(f'(v) - f'(z), \left(\nabla^2 v\right)^2\right) \\
&- (\mu/6) \left|\nabla^3 z - \nabla^3 v\right|^2 + (\mu/12) \left|\lambda\right| a \left[- \left|\nabla^3 z\right|^2 + 5 \left(f'(z), \left(\nabla^2 z\right)^2\right)\right].
\end{aligned}
$$

From this inequality we obtain the estimate

(6.18)

$$
\begin{aligned}
(1 - |\lambda|\, a)\, \varphi_{2,\mu}(z) \le \varphi_{2,\mu}(v) + |\lambda|\,(b + \varepsilon) + (1/12)\, |\lambda|\, a\mu \left[- \left|\nabla^3 z\right| + 5 \left(f'(z), \left(\nabla^2 z\right)^2\right)\right] \\
+ (5/12)\lambda\mu \left(\int_0^1 f''\left(\theta v + (1-\theta) z\right) d\theta \left(\nabla^2 v\right)^2, A_\mu z + B_\mu w\right).
\end{aligned}
$$

By (6.13), the last term on the right-hand side is estimated as

$$
\begin{aligned}
\mu \left|\left(\int_0^1 f''\left(\theta v + (1-\theta) z\right) d\theta \left(\nabla^2 v\right)^2, A_\mu z + B_\mu w\right)\right| \\
\le M_2 \beta_2^2 \mu \left(\left|\nabla^3 v\right| + N_0\right) \le M_2 \beta_2^2 \mu^{1/2} \left(\beta_2 + \mu^{1/2} N_0\right),
\end{aligned}
$$

and the second last term is estimated as

$$
\begin{aligned}
5 \left(f'(z), \left(\nabla^2 z\right)^2\right) &\le 5 M_1 \left|\nabla^2 z\right|_{L^4}^2 \\
&\le 5\sqrt{M_1} \left|\nabla^3 z\right|^{1/2} \left|\nabla^2 z\right|^{3/2} \\
&\le (1/2) \left(4 \left|\nabla^3 z\right| + 25/2 M_1^2 \left|\nabla^2 z\right|^3\right) \\
&\le \left|\nabla^3 z\right|^2 + 1 + 25/4 M_1^2 \left|\nabla^2 z\right|^3.
\end{aligned}
$$

Combining these estimates, we obtain the desired inequality (6.13). This shows that $\Gamma_{\lambda,\mu} v \in K_{\lambda\mu}$. The conclusion of this theorem is obtained in a way similar to that of Theorem 4.1, noting that the last inequality in (6.6) follows from Lemma 6.2 and the identity (5.9). □

By virtue of Theorems 2.1 and 6.1, one obtains a regularity result for the groups $G_\mu = \{G_\mu(t)\,; t \in \mathbb{R}\}$.

Theorem 6.2. *Let $\mu \in (0,1)$ and $G_\mu = \{G_\mu(t)\,; t \in \mathbb{R}\}$ the nonlinear group of locally Lipschitzian operators obtained in Theorem 5.1. In addition to the properties stated in Theorem 5.1, the following statements are valid:*

(i) *$G_\mu(\cdot)v \in \bigcap_{m=0}^{3} \mathscr{C}^m(\mathbb{R}; H^{3-m})$ for $v \in H^3$ and $G_\mu(\cdot)v \in \bigcap_{m=0}^{4} \mathscr{C}^m(\mathbb{R}; H^{4-m})$ for $v \in H^4$. If in particular $v \in H^4$, then $u(t,x) = [G_\mu(t)v](x)$ satisfies equation (5.1) pointwise on $\mathbb{R} \times \mathbb{R}$.*

(ii) *For $v \in H^2$ the exponential formula*

$$
G_\mu(t)\, v = H^2\text{-}\lim_{n\to\infty} \left(I - (t/n)\left(A_\mu + B_\mu\right)\right)^{-n} v
$$

holds for $t \in \mathbb{R}$ and the convergence is uniform on bounded subintervals of $\mathbb{R}$.

(iii) *For* $\alpha_0,\alpha_1,\alpha_2 \geq 0$ *and* $\tau > 0$, *there exist numbers* $\hat{a} = \hat{a}(\alpha_0,\alpha_1) > 0$, $\hat{b} = \hat{b}(\alpha_0,\alpha_1) > 0$, $\hat{\omega}_0 = \hat{\omega}_0(\alpha_0,\alpha_1,\alpha_2,\tau)$ *and* $\mu_0 = \mu_0(\alpha_0,\alpha_1,\alpha_2,\tau)$ *such that*

(iii.1) $\varphi_{2,\mu}(G_\mu(t)v) \leq e^{\hat{a}|t|}\left(\varphi_{2,\mu}(v) + \hat{b}|t|\right)$,

for $\mu \in (0,\mu_0)$, $t \in [-\tau,\tau]$ *and* $v \in H^3$ *with* $\varphi_{k,\mu}(v) \leq \alpha_k$, $k = 0,1,2$.

(iii.2) $\varphi_3(G_\mu(t)v) \leq e^{\hat{\omega}_0|t|}\varphi_3(v)$,

for $\mu \in (0,\mu_0)$, $t \in [-\tau,\tau]$ *and* $v \in H^3$ *with* $\varphi_{k,\mu}(v) \leq \alpha_k$, $k = 0,1,2$

(iii.3) $|G_\mu(t)v - G_\mu(t)w|_\mu \leq e^{\hat{\omega}_0|t|}|v-w|_\mu$,

for $\mu \in (0,\mu_0)$, $t \in [-\tau,\tau]$ *and* $v \in H^3$ *with* $\varphi_{k,\mu}(v) \leq \alpha_k$, $k = 0,1,2$ *and* $\varphi_{k,\mu}(w) \leq \alpha_k$, $k = 0,1,2$.

We are now in a position to state the convergence theorem

Theorem 6.3. *The following statements hold:*

(i) $(I - \lambda(A+B))^{-1}v = H^2\text{-}\lim_{n\to\infty}(I - \lambda(A_\mu + B_\mu))^{-1}v$

for $v \in H^3$ *and* $\lambda \in \mathbb{R}$ *with* $|\lambda| < \min\left\{\lambda_0(|v|_3,\varepsilon), \hat{\lambda}_0(|v|_3,\varepsilon)\right\}$, *where* $\varepsilon > 0$, $\lambda_0 = \lambda_0(|v|_3,\varepsilon)$ *is the number given in Theorem 2.1 and* $\hat{\lambda}_0 = \hat{\lambda}_0(|v|_3,\varepsilon)$ *is the number given in Theorem 6.1.*

(ii) *If* $v \in H^2$, $v_\mu \in H^3$, $v_\mu \to v$ *in* H^2 *as* $\mu \to 0$ *and* $\mu|\nabla^3 v_\mu|^2 \leq M$ *as* $\mu \to 0$ *for some* $M \geq 0$, *then*

$$G(t)v = H^1\text{-}\lim_{\mu\to 0} G_\mu(t)v_\mu \qquad \textit{for } t \in \mathbb{R}$$

and the convergence is uniform on bounded subintervals of $\mathbb{R}$. *If in particular* $v \in H^3$, *then*

$$G(t)v = H^1\text{-}\lim_{\mu\to 0} G_\mu(t)v \qquad \textit{for } t \in \mathbb{R}$$

and the convergence is uniform on bounded subinterval of $\mathbb{R}$.

Proof. (i) Let $v \in H^3$, $\varepsilon > 0$ and let $\lambda \in \mathbb{R}$ be such that $|\lambda| < \min\left\{\lambda_0, \hat{\lambda}_0\right\}$. If we write $v_\lambda = (I - \lambda(A+B))^{-1}v$ and $v_{\lambda,\mu} = (I - \lambda(A_\mu + B_\mu))^{-1}v$, then v_λ satisfies (4.1) and $v_{\lambda,\mu}$ makes sense and satisfies (6.5) for $\mu > 0$ sufficiently small. It is obvious that $\varphi_{k,\mu}(v_\lambda) \to \varphi_k(v_\lambda)$ as $\mu \downarrow 0_+$, for $k = 0,1,2$. Therefore, we see from Lemma 6.2 that

$$\begin{aligned}
(6.19)\qquad &|((A_\mu + B_\mu)v_{\lambda,\mu} - (A_\mu + B_\mu)v_\lambda, v_{\lambda,\mu} - v_\lambda) \\
&\quad + \mu(\nabla(A_\mu + B_\mu)v_{\lambda,\mu} - \nabla(A_\mu + B_\mu)v_\lambda, \nabla(v_{\lambda,\mu} - v_\lambda))| \\
&= |(B_\mu v_{\lambda,\mu} - B_\mu v_\lambda, v_{\lambda,\mu} - v_\lambda) + \mu(\nabla B_\mu v_{\lambda,\mu} - \nabla B_\mu v_\lambda, \nabla(v_{\lambda,\mu} - v_\lambda))| \\
&\leq \hat{\omega}_0|v_{\lambda,\mu} - v_\lambda|^2.
\end{aligned}$$

An easy computation yields

$$(6.20)\qquad \lambda\left((A_\mu + B_\mu)v_\lambda - (A+B)v_\lambda, v_{\lambda,\mu} - v_\lambda - \mu\nabla^2(v_{\lambda,\mu} - v_\lambda)\right)$$

$$
\begin{aligned}
&= \lambda\left(\left(A_\mu + B_\mu\right) v_\lambda - \left(A_\mu + B_\mu\right) v_{\lambda,\mu}, v_{\lambda,\mu} - v_\lambda\right) \\
&\quad + \mu\left(\nabla\left(A_\mu + B_\mu\right) v_\lambda - \nabla\left(A_\mu + B_\mu\right) v_{\lambda,\mu}, \nabla\left(v_{\lambda,\mu} - v_\lambda\right)\right) \\
&\quad + \left|v_{\lambda,\mu} - v_\lambda\right|^2 + \mu\left|\nabla\left(v_{\lambda,\mu} - v_\lambda\right)\right|^2,
\end{aligned}
$$

combining (6.19) and (6.20), we obtain

$$
\begin{aligned}
(6.21) \qquad & \left(1 - |\lambda|\,\hat{\omega}_0\right)\left|v_{\lambda,\mu} - v_\lambda\right|^2 \\
& \leq |\lambda|\left|\left(A_\mu + B_\mu\right) v_\lambda - (A+B) v_\lambda\right|\left|v_{\lambda,\mu} - v_\lambda - \mu\nabla^2\left(v_{\lambda,\mu} - v_\lambda\right)\right|.
\end{aligned}
$$

Since $\left(A_\mu + B_\mu\right) v_\lambda \rightarrow (A+B) v_\lambda$ in L^2 as $\mu \downarrow 0$ and $\overline{\lim_{\mu\downarrow 0}}\left|v_{\lambda,\mu}\right|_3 < \infty$, it follows that $v_{\lambda,\mu} \rightarrow v_\lambda$ in L^2 as $\mu \downarrow 0$. Noting that $|\nabla w| \leq |w|^{1/2}\left|\nabla^2 w\right|^{1/2}$ and $\left|\nabla^2 w\right| \leq |w|^{1/3}\left|\nabla^3 w\right|^{2/3}$ for $w \in H^3$, we conclude that $v_{\lambda,\mu} \rightarrow v_\lambda$ in H^2 as $\mu \downarrow 0_+$. Thus assertion (i) is obtained.

We next prove (ii). If $v \in H^3$, it is easy to see that $G_\mu(t) v$ converges to $G(t) v$ in H^2 as $\mu \downarrow 0$.

If $v \in H^2$, we construct $\{v_\mu\} \subset H^3$, $v_\mu \rightarrow v$ as $\mu \rightarrow 0$ and $\mu\left|\nabla^3 v_\mu\right|^2 \leq \overline{M}$ for some $\overline{M} \geq 0$. Let $\{v_\lambda\}$ be any sequence in H^3 such that $v_\lambda \rightarrow v$ as $\lambda \rightarrow 0$ and $\lambda\left|\nabla^3 v_\lambda\right|^2 \leq \overline{M}_1$ for some $M_1 \geq 0$. Then

$$
\begin{aligned}
\left|G_\mu(t) v_\mu - G(t) v\right|_\mu &\leq \left|G(t) v - G(t) v_\lambda\right|_\mu + \left|G(t) v_\lambda - G_\mu(t) v_\lambda\right|_\mu \\
&\quad + \left|G_\mu(t) v_\lambda - G_\mu(t) v_\mu\right|_\mu \\
&\leq \left|G(t) v - G(t) v_\lambda\right|_1 + \left|G(t) v_\lambda - G_\mu(t) v_\lambda\right|_1 \\
&\quad + e^{\hat{\omega}_0|t|}\left|v_\lambda - v_\mu\right|_\mu
\end{aligned}
$$

provided that $\varphi_{k,\mu}(v_\lambda) \leq \alpha_k$, $\varphi_{k,\mu}(v_\mu) \leq \alpha_k$ for $k = 0, 1, 2$.

From the above relation it is seen that $G_\mu(t) v_\mu \rightarrow G(t) v$ in L^2 as $\mu \rightarrow 0$. We also have

$$
\begin{aligned}
&\varphi_{0,\mu}\left(G_\mu(t) v_\mu\right) \leq \varphi_{0,\mu}\left(v_\mu\right), \\
&\varphi_{1,\mu}\left(G_\mu(t) v_\mu\right) \leq \varphi_{1,\mu}\left(v_\mu\right), \\
&\varphi_{2,\mu}\left(G_\mu(t) v_\mu\right) \leq e^{\hat{a}|t|}\left(\varphi_{2,\mu}\left(v_\mu\right) + \hat{b}\,|t|\right),
\end{aligned}
$$

and so $\varphi_{k,\mu}\left(G_\mu(t) v_\mu\right) \leq \gamma_k$, for some $\gamma_k \in \mathbb{R}$, $k = 0, 1, 2$ and $t \in [-\tau, \tau]$, $\tau > 0$. In view of this, one finds $\beta > 0$ such that $\left|\nabla^2 G_\mu(t) v_\mu\right| \leq \beta$ for each $\mu > 0$ and $t \in [-\tau, \tau]$. Since

$$
\left|\nabla\left(G_\mu(t) v_\mu - G(t) v\right)\right|^2 \leq \left|\nabla^2 G_\mu(t) v_\mu - \nabla^2 G(t) v\right|\left|G_\mu(t) v_\mu - G(t) v\right|,
$$

it follows that $G_\mu(t) v_\mu \rightarrow G(t) v$ in H^1 as $\mu \rightarrow 0$. This complete the proof. □

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