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THE INDEX FORM OF A GEODESIC ON A GLUED RIEMANNIAN SPACE

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Abstract

A topological space obtained from Riemannian manifolds by identifying their isometric submanifolds is called a glued Riemannian space. In this space, we consider the variational problem with respect to arc length L of piecewise smooth curves through the identified submanifold B. The first variation formula shows that a critical point of L is a curve which is a geodesic on each Riemannian manifold and satisfies certain passage law on B. We call this curve a B-geodesic. The second variation formula for a B-geodesic is also obtained. Moreover, we study the index form and B-conjugate points for a B-geodesic in this variational problem. Especially, in a glued Riemannian space constructed from Riemannian manifolds of constant curvature, we have the passage equation which make the relation between the shape operator and the first B-conjugacy clear.

0. Introduction

In Riemannian manifolds, various results have been given on geodesics by many authors. Recently, N.Innami studied a geodesic reflecting at a boundary point of a Riemannian manifold with boundary in [4]. Let M be a Riemannian manifold with boundary B which is a union of smooth hypersurfaces. A curve on M is said to be a reflecting geodesic if it is a geodesic except at reflecting points and satisfies the reflection law. He dealt with the index form, conjugate points and so on, as in the case of a usual geodesic. Moreover, in [5], he generalized these to the case of a glued Riemannian manifold which is a space obtained from Riemannian manifolds with boundary by identifying their isometric boundary hypersurfaces.

The purpose of this paper is to generalize some of his results to the case of a glued Riemannian space, which is obtained from Riemannian manifolds M_1 and M_2 (we allow the case where $\dim M_1 \neq \dim M_2$) by identifying their isometric submanifolds B_1 and B_2. The detailed definition will be described in Section 1. We consider the variational problem with respect to arc length L of piecewise smooth curves through the identified submanifold B. The first variation formula shows that a critical point of L is a curve which is a geodesic on each Riemannian manifold and satisfies some passage law on B. We call this curve a B-geodesic. We can apply our results to the glued Riemannian manifolds as a special case where $\dim M_1 = \dim M_2 = \dim B + 1$. Moreover, examining the case where $M_1 = M_2$, $B_1 = B_2$ and an endpoint of the curve through B coincides with a starting point, we can also apply our results to the endmanifold case. For example, see [6]. Note that geodesics in this special case are normal to the submanifold, while B-geodesics are not normal to B in general.

In Section 1, for a piecewise smooth curve through a point of the identified submanifold B, we define a variation of such a curve. We give the first variation formula of arc length and show that the critical curve is a B-geodesic. The second variation formula for a B-geodesic is obtained in Section 2. Moreover we express the index form in terms of the passage endomorphism A which is defined by using the shape operators of B in M_1 and M_2. In Section 3, we consider the variation of a B-geodesic through B-geodesics and definitions of a B-Jacobi field and a B-conjugate point are given. In Section 4, we study fundamental properties of B-conjugate points and the index form. Finally, in Section 5, we consider the relations between the map A and S which is the difference of the shape operators of B in M_1 and M_2. Moreover, in a glued Riemannian space obtained from Riemannian manifolds of constant curvature K_1 and K_2, we give the passage equation which make the relation between the shape operator and the first B-conjugacy clear.

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1. Preliminaries

Let N_μ and M_λ be manifolds (possibly with boundary) for $\mu = 1, \cdots, n$ and $\lambda = 1, \cdots, m$. We allow the case where $\dim N_i \neq \dim N_j$ and $\dim M_k \neq \dim M_l$ for $i \neq j$ and $k \neq l$. A map $\bar{\varphi} : \bar{N} \to \bar{M}$ from the topological direct sum $\bar{N} := N_1 \coprod \cdots \coprod N_m$ to $\bar{M} := M_1 \coprod \cdots \coprod M_m$ is *smooth* if $\bar{\varphi}|N_\mu$ is smooth. A *tangent bundle* $T\bar{M}$ *of* $\bar{M}$ is the direct sum $T\bar{M} = TM_1 \coprod \cdots \coprod TM_m$, where TM_λ denotes the tangent bundle of M_λ. We note that a tangent bundle $T\bar{M}$ on $\bar{M}$ is not constant rank vector bundle on $\bar{M}$. We put $T_p\bar{M} := T_pM_\lambda$ for $p \in M_\lambda$. We define a map $\pi_{\bar{M}} : T\bar{M} \to \bar{M}$ by

$$\pi_{\bar{M}}(v_p) := p \quad \text{for } v_p \in T_pM_\lambda.$$

A *vector field* $\bar{V}$ *on* $\bar{M}$ is a map $\bar{V} : \bar{M} \to T\bar{M}$ such that $\pi_{\bar{M}} \circ \bar{V} = \mathrm{id}_{\bar{M}}$, where $\mathrm{id}_{\bar{M}}$ is the identity map on $\bar{M}$. If $\bar{V}|M_\lambda : M_\lambda \to TM_\lambda$ is smooth vector field on each M_λ, then $\bar{V}$ is smooth. Let I_μ be a closed interval in $\boldsymbol{R}$ which is a manifold with boundary, for $\mu = 1, \cdots, n$. A map $\bar{\alpha} : \bar{I} := I_1 \coprod \cdots \coprod I_n \to \bar{M}$ is called a *curve on* $\bar{M}$ if $\bar{\alpha}$ is smooth.

Let M_λ be a manifold (possibly with boundary) with a submanifold B_λ for $\lambda = 1,2$ and ψ a diffeomorphism from B_1 to B_2. A *glued space* $M = M_1 \cup_\psi M_2$ is defined as follows: M is the quotient topological space obtained from the topological direct sum $\bar{M} = M_1 \coprod M_2$ of M_1 and M_2 by identifying $p \in B_1$ with $\psi(p) \in B_2$. We allow the case where $B_1 = B_2 = \emptyset$, $M_1 = \emptyset$ or $M_2 = \emptyset$, where ψ is the empty map. Let $\pi : \bar{M} \to M$ be the natural projection which is defined by $\pi(p) = [p]$, where $[p]$ is the equivalence class of p. Let N_λ be a manifold with a submanifold C_λ ($\lambda = 1,2$), $\tau : C_1 \to C_2$ a diffeomorphism and $N = N_1 \cup_\tau N_2$ a glued space. A *glued smooth map* $\varphi : \bar{N} \to M$ *on* $\bar{N}$ *derived from a smooth map* $\bar{\varphi} : \bar{N} \to \bar{M}$ or, simply, *a smooth map on* N is defined by $\varphi = \pi \circ \bar{\varphi}$. We note that a glued smooth map on $\bar{N}$ is considered as a map on N which, possibly, take two values at $[p]$ ($p \in C_\lambda$). A glued smooth map φ is *continuous* if $\varphi(p) = \varphi(\tau(p))$ holds for any $p \in C_1$.

A *glued tangent bundle* TM *of* M is the glued space $TM_1 \cup_{\psi_*} TM_2$, where $\psi_* : TB_1 \to TB_2$ is the differential map of ψ. Let $\hat{\pi} : T\bar{M} \to TM$ be the natural projection which is defined by $\hat{\pi}(v) = [v]$, where $[v]$ is the equivalence class of v. For $p \in \bar{M}$, we set $T_pM := \hat{\pi}(T_p\bar{M}) = \{[v] \in TM | v \in T_p\bar{M}\}$. We define a map $\pi_M : TM \to M$ by

$$\pi_M([v_p]) := [p] \quad \text{for } v_p \in T_p\bar{M}.$$

We note that $\pi \circ \pi_{\bar{M}} = \pi_M \circ \hat{\pi}$ holds. A *glued vector field* $V : \bar{M} \to TM$ *on* $\bar{M}$ *derived from a vector field* $\bar{V}$ *on* $\bar{M}$ or, simply, *a vector field on* M is defined by $V = \hat{\pi} \circ \bar{V}$. A glued vector field V is called a *smooth glued vector field* provide V is glued smooth. If a glued vector field V on $\bar{M}$ is continuous, then we can regard it as a cross section of TM over M; that is $\pi_M \circ V = \mathrm{id}_M$. Similarly, we can define a *glued vector field* (or *vector field*) *along a curve* $\bar{\alpha} : \bar{I} := I_1 \coprod I_2 \to \bar{M}$.

Let $T_p^*\bar{M}$ be the dual vector space of $T_p\bar{M}$. We put $T^*\bar{M} = T^*M_1 \coprod T^*M_2$, where T^*M_λ is the cotangent bundle of M_λ. For $\bar{\theta}_p$ ($\in T_p^*\bar{M}$), $\bar{\omega}_q$ ($\in T_q^*\bar{M}$) $\in T^*\bar{M}$, we define an equivalence relation $\sim$ as follows: $\bar{\theta}_p \sim \bar{\omega}_q$ if and only if $\bar{\theta}_p = \bar{\omega}_q$ ($p = q$) or $\bar{\theta}_p|_{T_pB_1} = \psi^*(\bar{\omega}_q)$ ($p \in B_1$, $q = \psi(p)$) or $\bar{\omega}_q|_{T_qB_1} = \psi^*(\bar{\theta}_p)$ ($q \in B_1$, $p = \psi(q)$), where ψ^* is the dual map of ψ_*. The quotient space obtained from $T^*\bar{M}$ by this equivalence relation is denoted by T^*M. Let $\hat{\pi} : T^*\bar{M} \to T^*M$ be the natural projection, that is, $\hat{\pi}(\bar{\theta}) := [\bar{\theta}]$, where $[\bar{\theta}]$ is the equivalence class of $\bar{\theta}$. For $p \in \bar{M}$, we set $T_p^*M := \hat{\pi}(T_p^*\bar{M})$ and define a map $[\bar{\theta}] : T_pM \to \boldsymbol{R}$ by $[\bar{\theta}]([\bar{v}]) := \bar{\theta}(\bar{v})$ for $\bar{\theta} \in T_p^*\bar{M}$ and $\bar{v} \in T_p\bar{M}$. Then we can regard T_p^*M as the dual of T_pM. We put $T^{r,s}(\bar{M}) := T^{r,s}(M_1) \coprod T^{r,s}(M_2)$, where $T^{r,s}(M_\lambda)$ is the (r,s)-tensor bundle of M_λ. An (r,s)-*tensor field on* $\bar{M}$ is a cross section of $T^{r,s}(\bar{M})$. The definition of the *smoothness* of a tensor field on $\bar{M}$ is similar to that of a vector field on $\bar{M}$. Similarly, we can define the equivalence relation on $T^{r,s}(\bar{M})$ induced from those on $T\bar{M}$ and $T^*\bar{M}$, and denote the quotient space by $T^{r,s}(M)$. Let $\hat{\pi} : T^{r,s}(\bar{M}) \to T^{r,s}(M)$ be the natural projection. A *glued tensor field* T *derived from a tensor field* $\bar{T}$ *on* $\bar{M}$ is defined by

$T = \hat{\pi} \circ \bar{T}$. A glued tensor field T derived from a tensor field $\bar{T}$ on $\bar{M}$ is (*glued*) *smooth* if $\bar{T}$ is smooth.

Definition 1.1. Let (M_λ, g_λ) be a Riemannian manifold with a Riemannian submanifold B_λ for $\lambda = 1, 2$ and ψ an isometry from B_1 to B_2. Let $\bar{g}$ be the *metric on* $\bar{M}$ which is defined to be $\bar{g}_p = (g_\lambda)_p$ for $p \in M_\lambda$. A *glued Riemannian space* $(M, g) = (M_1, g_1) \cup_\psi (M_2, g_2)$ is a pair of a glued space $M = M_1 \cup_\psi M_2$ and a *glued metric* g on M derived from $\bar{g}$ which is a glued tensor field derived from the $(0, 2)$-tensor field $\bar{g}$.

We note that, for any glued smooth vector fields V and W on $\bar{M}$ derived from smooth vector fields $\bar{V}$ and $\bar{W}$ on $\bar{M}$, respectively, a map $g(V, W) : \bar{M} \to \boldsymbol{R}$ defined by

$$g(V, W)(p) := \bar{g}(\bar{V}_p, \bar{W}_p)$$

is glued smooth on $\bar{M}$ derived from a smooth map $\bar{g}(\bar{V}, \bar{W}) : \bar{M} \to \boldsymbol{R}$.

From now on, identifying B_1 with B_2 by ψ, we put $B := B_1 \cong B_2$ and $T_pB := T_pB_1 \cong T_pB_2$ for $p \in B$ and omit the symbol $[\cdot]$ of the equivalence class. In particular, $[M_\lambda] := \pi(M_\lambda)$ will be denoted by M_λ. We call a map $\alpha : [a, t_0] \coprod [t_0, b] \to M$ a *glued curve derived from a curve* $\bar{\alpha} : [a, t_0] \coprod [t_0, b] \to \bar{M}$ or, simply, *a curve on* M if $\alpha : [a, t_0] \coprod [t_0, b] \to M$ is a continuous glued smooth map derived from $\bar{\alpha}$. Let $\alpha : [a, t_0] \coprod [t_0, b] \to M$ be a glued curve derived from a curve $\bar{\alpha} : [a, t_0] \coprod [t_0, b] \to \bar{M}$. The (*glued*) *velocity vector field of* α is $\alpha' := \hat{\pi} \circ \bar{\alpha}'$. We put $\alpha'(t_0 - 0) := \hat{\pi} \circ \bar{\alpha}_1'(t_0)$ and $\alpha'(t_0 + 0) := \hat{\pi} \circ \bar{\alpha}_2'(t_0)$, where $\bar{\alpha}_1 := \bar{\alpha}|[a, t_0] : [a, t_0] \to \bar{M}$ and $\bar{\alpha}_2 := \bar{\alpha}|[t_0, b] : [t_0, b] \to \bar{M}$. We note that a glued velocity vector field is considered as a glued vector field along $\bar{\alpha}$ and not generally continuous. We call $\alpha : [a, b] \to M$ a *piecewise smooth curve on* M provided there is a partition $a = a_0 < a_1 < \cdots < a_k < a_{k+1} = b$ of $[a, b]$ such that $\alpha|[a_{i-1}, a_{i+1}] : [a_{i-1}, a_i] \coprod [a_i, a_{i+1}] \to M$ is a glued curve. We call a_j $(j = 1, \cdots, k)$ the *break*.

Remark. Let M be a smooth manifold. A usual piecewise smooth curve $\alpha : [a, b] \to M$ is considered as a piecewise smooth curve in a glued space. Moreover the fact that a usual piecewise smooth curve may have two velocity vectors $\alpha'(t_0 - 0)$ and $\alpha'(t_0 + 0)$ at the break t_0 can be naturally explained as above.

If M is a glued Riemannian space such that $(M, g) = (M_1, g_1) \cup_\psi (M_2, g_2)$, then let $\tilde{\Omega}(M_1, M_2; B) =: \tilde{\Omega}$ be the set of all piecewise smooth curves $\alpha : [a, b] \to M$ such that there is $t_0 \in (a, b)$ with $\alpha(t_0) \in B$, $\alpha([a, t_0]) \subset M_1$ and $\alpha([t_0, b]) \subset M_2$. We note that a glued Riemannian space is not a smooth manifold in general. But we can define arc length of such a curve as follows:

$$L(\alpha) := \int_a^{t_0} \|\alpha'(t)\|_1 dt + \int_{t_0}^b \|\alpha'(t)\|_2 dt,$$

where $\|\cdot\|_\lambda$ is the norm of a tangent vector to M_λ.

Definition 1.2. Let $\alpha \in \tilde{\Omega}$ be such that $\alpha(t_0) \in B$ ($t_0 \in (a,b)$). A *variation of* α *in* $\tilde{\Omega}$ is a map

$$\varphi : [a,b] \times (-\varepsilon, \varepsilon) \to M,$$

for some $\varepsilon > 0$, such that

$$\varphi_s(\cdot) := \varphi(\cdot, s) \in \tilde{\Omega}, \tag{1.1}$$

$$\varphi_0(t) = \alpha(t) \text{ for all } a \le t \le b, \tag{1.2}$$

$$\varphi(t_0(s), s) \in B, \tag{1.3}$$

where $a = a_0(s) < a_1(s) < \cdots < t_0(s) = a_j(s) < \cdots < a_k(s) < a_{k+1}(s) = b$ are the breaks of φ_s ($a_i(0) = a_i$ ($i = 1, \cdots, k$) and $t_0(0) = t_0 = a_j$). We assume that $a_i(s)$'s are smooth with respect to s.

A *fixed endpoint* variation φ of α is a variation such that

$$\varphi(a,s) = \alpha(a) \quad \text{and} \quad \varphi(b,s) = \alpha(b). \tag{1.4}$$

Let D^λ be Levi-Civita connection of Riemannian manifold M_λ ($\lambda = 1, 2$). The vector fields Y and A along α given by $Y(t) := (\partial\varphi/\partial s)(t, 0)$ and $A(t) := (D^\lambda/\partial s\ \partial\varphi/\partial s)(t, 0)$ are called *variation vector field* and *transverse acceleration vector field* of φ respectively, where $D^\lambda/\partial s := D^\lambda_{\partial/\partial s}$ and $D^\lambda/\partial t := D^\lambda_{\partial/\partial t}$. We write $X(t,s) = (\partial\varphi/\partial t)(t,s)$ ($X(t) = X(t,0) = \alpha'(t)$), $Y(t,s) = (\partial\varphi/\partial s)(t,s)$ ($Y(t) = Y(t,0)$) and $A(t,s) = (D^\lambda/\partial s\ \partial\varphi/\partial s)(t,s)$ ($A(t) = A(t,0)$). The projection from T_pM_λ to T_pB is denoted by tan.

Definition 1.3. A curve $\alpha \in \tilde{\Omega}$ such that $\alpha(t_0) \in B$ is a *geodesic through* B or a *B-geodesic* if α satisfies the following conditions:

(1.5) $\alpha|[a, t_0]$ and $\alpha|[t_0, b]$ are geodesics, that is $D^\lambda_{\alpha'}\alpha' = 0$, on M_1 and M_2, respectively,

$$\tan \alpha'(t_0 - 0) = \tan \alpha'(t_0 + 0), \tag{1.6}$$

$$g_1(\alpha'(t_0 - 0), \alpha'(t_0 - 0)) = g_2(\alpha'(t_0 + 0), \alpha'(t_0 + 0)). \tag{1.7}$$

For each $s \in (-\varepsilon, \varepsilon)$, let $L(s)$ be the length of the *longitudinal curve* $\varphi_s : t \mapsto \varphi(t,s)$. We shall find formulas for the *first* and *second variation of arclength on* φ, that is, for

$$L'(0) = \left.\frac{dL}{ds}\right|_{s=0} \quad \text{and} \quad L''(0) = \left.\frac{d^2L}{ds^2}\right|_{s=0},$$

where the latter is considered when $L'(0) = 0$.

Lemma 1.4. *Let α be an element of $\tilde{\Omega}$ such that $\alpha(t_0) \in B$. If φ is a variation of α in $\tilde{\Omega}$ with the variation vector field Y, then we have that*

$$a_i'(0)\alpha'(t_0 - 0) + Y(t_0 - 0) = a_i'(0)\alpha'(t_0 + 0) + Y(t_0 + 0). \tag{1.8}$$

In particular,

$$t_0'(0)\alpha'(t_0 - 0) + Y(t_0 - 0) = t_0'(0)\alpha'(t_0 + 0) + Y(t_0 + 0) \in T_{\alpha(t_0)}B. \tag{1.9}$$

Moreover, if $\alpha'(t_0 - 0) \notin T_{\alpha(t_0)}B$ and $\alpha'(t_0 + 0) \notin T_{\alpha(t_0)}B$, then we have that

$$t_0'(0) = -\frac{g_1(Y(t_0 - 0), \mathrm{nor}\alpha'(t_0 - 0))}{g_1(\alpha'(t_0 - 0), \mathrm{nor}\alpha'(t_0 - 0))} = -\frac{g_2(Y(t_0 + 0), \mathrm{nor}\alpha'(t_0 + 0))}{g_2(\alpha'(t_0 + 0), \mathrm{nor}\alpha'(t_0 + 0))}. \tag{1.10}$$

This lemma shows that variation vector fields are elements of the set $T_\alpha\tilde{\Omega}$ defined as below:

Definition 1.5. If $\alpha \in \tilde{\Omega}$, the set $T_\alpha\tilde{\Omega}$ consists of all vector fields Y along α which satisfy the following condition : For $i = 1, \cdots, k$, there is a real number d_i such that

$$d_i\alpha'(a_i - 0) + Y(a_i - 0) = d_i\alpha'(a_i + 0) + Y(a_i + 0), \tag{1.11}$$

and, in particular,

$$d_j\alpha'(t_0 - 0) + Y(t_0 - 0) = d_j\alpha'(t_0 + 0) + Y(t_0 + 0) \in T_{\alpha(t_0)}B. \tag{1.12}$$

We note that $\alpha' \in T_\alpha\tilde{\Omega}$ (in this case, $d_i = -1$). Conversely, given $Y \in T_\alpha\tilde{\Omega}$ we can choose a variation whose vector field is Y. In fact, we can know this claim from the following lemma.

Lemma 1.6. *If $\alpha \in \tilde{\Omega}$ and $Y \in T_\alpha\tilde{\Omega}$, then there is a variation of α whose variation vector field is Y.*

We compute the first variation formula.

Proposition 1.7 (First Variation Formula). *Let $\alpha : [a, b] \to M$ be an element of $\tilde{\Omega}$ with constant speed $c \neq 0$ such that $\alpha(t_0) \in B$. If φ is a variation of α in $\tilde{\Omega}$ with the variation vector field Y, then*

$$L'(0) = -\frac{1}{c}\left\{\int_a^{t_0} g_1(Y,\alpha'')dt + \int_{t_0}^b g_2(Y,\alpha'')dt\right\}$$
$$+\frac{1}{c}\left\{\sum_{i=1}^{j-1}\Delta_{a_i}g_1(Y,\alpha') + \sum_{i=j+1}^{k}\Delta_{a_i}g_2(Y,\alpha')\right\}$$
$$+\frac{1}{c}\{g_1(Y(t_0-0),\alpha'(t_0-0)) - g_2(Y(t_0+0),\alpha'(t_0+0))\}$$
$$+\frac{1}{c}\{g_2(Y(b),\alpha'(b)) - g_1(Y(a),\alpha'(a))\},$$

where $a_1 < \cdots < t_0 = a_j < \cdots < a_k$ *are the breaks of* α *and for* $\lambda = 1,2$

$$\Delta_{a_i}g_\lambda(Y,\alpha') = g_\lambda(Y(a_i-0),\alpha'(a_i-0)) - g_\lambda(Y(a_i+0),\alpha'(a_i+0)).$$

Lemma 1.8. *Let* $\alpha : [a,b] \to M$ *be an element of* $\tilde{\Omega}$ *with* $g_1(\alpha'(t_0-0),\alpha'(t_0-0)) = g_2(\alpha'(t_0+0),\alpha'(t_0+0))$ *such that* $\alpha(t_0) \in B$. *Then the following are equivalent:*

$$\tan\alpha'(t_0-0) = \tan\alpha'(t_0+0). \tag{1.13}$$

$$g_1(Y(t_0-0),\alpha'(t_0-0)) = g_2(Y(t_0+0),\alpha'(t_0+0)) \text{ for any } Y \in T_\alpha\tilde{\Omega}. \tag{1.14}$$

Proof. For simplicity, we put $d := d_j$, $X_\pm := \alpha'(t_0 \pm 0)$ and $Y_\pm := Y(t_0 \pm 0)$.

(1.13)⇒(1.14): If (1.13) holds, we have

$$g_2(Y_+,X_+) = g_2(Y_- + dX_- - dX_+, X_+) = g_2(Y_- + dX_-, \tan X_-) - dg_2(X_+,X_+)$$
$$= g_1(Y_- + dX_-, X_-) - dg_1(X_-,X_-) = g_1(Y_-,X_-).$$

(1.14)⇒(1.13): If (1.14) holds, we get

$$g_1(X_-,y) = g_2(X_+,y) \quad \text{for any } y \in T_{\alpha(t_0)}B.$$

Hence we have $\tan X_- = \tan X_+$. □

Corollary 1.9. *A curve* α *of* $\tilde{\Omega}$ *with constant speed* $c \neq 0$ *such that* $\alpha(t_0) \in B$ *is a B-geodesic if and only if the first variation of arc length is zero for every fixed endpoint variation of* α *in* $\tilde{\Omega}$.

2. The index form

For $\lambda = 1, 2$, let R^λ be the Riemannian curvature tensor of a Riemannian manifold M_λ defined as

$$R^\lambda(X,Y)W := D^\lambda_X D^\lambda_Y W - D^\lambda_Y D^\lambda_X W - D^\lambda{}_{[X,Y]}W,$$

for any vector field X, Y and W on M_λ, and S^λ_Z the shape operator of $B \subset M_\lambda$ defined as

$$S^\lambda_Z(V) := -\tan D^\lambda_V Z,$$

for any vector field V tangent to B and Z normal to B. Especially, if $B = \{p\}$, we have that $S^\lambda_Z = 0$ for $Z \in T_pM_\lambda$. A vector field Y along a piecewise smooth curve $\alpha : [a,b] \to M$ is a *tangent to* α if $Y = f\alpha'$ for some function f on $[a,b]$ and *perpendicular to* α if $g_\lambda(Y,\alpha') = 0$. If $\|\alpha'\|_\lambda \neq 0$, then each tangent space $T_{\alpha(t)}M_\lambda$ has a direct sum decomposition $\boldsymbol{R}\alpha' + \{\alpha'\}^\perp$. Hence each vector field Y along α has a unique expression $Y = Y^T + Y^\perp$, where Y^T is tangent to α and $Y^\perp$ is perpendicular to α, that is,

$$Y^\perp = Y - \frac{g_\lambda(Y,\alpha')}{g_\lambda(\alpha',\alpha')}\alpha'.$$

If α is a B-geodesic, then $(Y^T)' = (Y')^T$ and $(Y^\perp)' = (Y')^\perp$.

Definition 2.1. Let $q \in B$ and $v \in T_qM_\lambda$ $(\lambda = 1,2)$ is not tangent to B. A linear operator $P^v_\lambda : T_qB \oplus \mathrm{Span}\{\mathrm{nor}v\} \to T_qB$ is defined by

$$P^v_\lambda(w) := w - \frac{g_\lambda(w,\mathrm{nor}v)}{g_\lambda(v,\mathrm{nor}v)}v$$

for any $w \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}v\} (\subset T_qM_\lambda)$, where $\mathrm{nor} : T_qM_\lambda \to T_qB^\perp$ is the projection.

We note that P^v_λ is surjective, $P^v_\lambda(v) = 0$, $P^{kv}_\lambda = P^v_\lambda$ for $k \neq 0$ and if $\alpha \in \tilde{\Omega}$ and $Y \in T_\alpha\tilde{\Omega}$, then we have that

$$P_1^{\alpha'(t_0-0)}(Y(t_0-0)) = P_2^{\alpha'(t_0+0)}(Y(t_0+0)).$$

Theorem 2.2 (Second Variation Formula). *Let $\gamma : [a,b] \to M$ be a B-geodesic with constant speed $c \neq 0$ such that $\gamma(t_0) \in B$. If φ is a variation of γ in $\tilde{\Omega}$, then we have that*

$$\begin{aligned} L''(0) = \frac{1}{c}\Big\{&\int_a^{t_0}(g_1(Y^{\perp\prime},Y^{\perp\prime}) - g_1(R^1(Y,\gamma')\gamma',Y))dt \\ &+\int_{t_0}^b(g_2(Y^{\perp\prime},Y^{\perp\prime}) - g_2(R^2(Y,\gamma')\gamma',Y))dt\Big\} \\ &+\frac{1}{c}\{g_2(A(b),\gamma'(b)) - g_1(A(a),\gamma'(a)) \end{aligned}$$

$$+g_1(S^1_{nor\gamma'(t_0-0)}(P_1^{\gamma'(t_0-0)}(Y(t_0-0))), P_1^{\gamma'(t_0-0)}(Y(t_0-0)))$$
$$-g_2(S^2_{nor\gamma'(t_0+0)}(P_2^{\gamma'(t_0+0)}(Y(t_0+0))), P_2^{\gamma'(t_0+0)}(Y(t_0+0)))\},$$

where Y is the variation vector field and A is the transverse acceleration vector field of φ.

Proof. We get

$$L''(0) = \frac{1}{c}\left\{\int_a^{t_0}(g_1(Y^{\perp\prime}, Y^{\perp\prime}) - g_1(R^1(Y,\gamma')\gamma', Y))dt\right.$$
$$\left.+\int_{t_0}^b (g_2(Y^{\perp\prime}, Y^{\perp\prime}) - g_2(R^2(Y,\gamma')\gamma', Y))dt\right\}$$
$$+\frac{1}{c}\left\{g_2(A(b),\gamma'(b)) - g_1(A(a),\gamma'(a))\right.$$
$$+g_1(A(t_0-0) + 2t_0'(0)Y'(t_0-0), \gamma'(t_0-0))$$
$$-g_2(A(t_0-0) + 2t_0'(0)Y'(t_0-0), \gamma'(t_0-0))$$
$$\left.+\sum_{i=1}^{j-1}\Delta_{a_i}g_1(A + 2a_i'(0)Y', \gamma') + \sum_{i=j+1}^{k}\Delta_{a_i}g_2(A + 2a_i'(0)Y', \gamma')\right\},$$

where

$$\Delta_{a_i}g_\lambda(A + 2a_i'(0)Y', \gamma') = g_\lambda(A(a_i-0) + 2a_i'(0)Y'(a_i-0), \gamma'(a_i-0))$$
$$-g_\lambda(A(a_i+0) + 2a_i'(0)Y'(a_i+0), \gamma'(a_i+0)).$$

We show the following facts:

$$g_1(A(t_0-0) + 2t_0'(0)Y'(t_0-0), \gamma'(t_0-0))$$
$$-g_2(A(t_0-0) + 2t_0'(0)Y'(t_0-0), \gamma'(t_0-0))$$
$$= g_1(S^1_{nor\gamma'(t_0-0)}(P_1^{X(t_0-0)}(Y(t_0-0))), P_1^{X(t_0-0)}(Y(t_0-0)))$$
$$-g_2(S^2_{nor\gamma'(t_0+0)}(P_2^{X(t_0+0)}(Y(t_0+0))), P_2^{X(t_0+0)}(Y(t_0+0)))\}.$$

In fact, let $\beta : (-\delta, \delta) \to B$ be $\beta(s) := \varphi(t_0(s), s)$, then

$$\beta'(0) = t_0'(0)\gamma'(t_0-0) + Y(t_0-0) = P_1^{\gamma'(t_0-0)}(Y(t_0-0))$$
$$= t_0'(0)\gamma'(t_0+0) + Y(t_0+0) = P_2^{\gamma'(t_0+0)}(Y(t_0+0)),$$
$$D^1_{\beta'(0)}\beta' = A(t_0-0) + 2t_0'(0)Y'(t_0-0) + t_0''(0)\gamma'(t_0-0)$$

and

$$D^2_{\beta'(0)}\beta' = A(t_0+0) + 2t_0'(0)Y'(t_0+0) + t_0''(0)\gamma'(t_0+0).$$

Thus we have

$$g_1(S^1_{nor\gamma'(t_0-0)}(P_1^{\gamma'(t_0-0)}(Y(t_0-0)), P_1^{\gamma'(t_0-0)}(Y(t_0-0)))$$
$$-g_2(S^2_{nor\gamma'(t_0+0)}(P_2^{\gamma'(t_0+0)}(Y(t_0+0)), P_2^{\gamma'(t_0+0)}(Y(t_0+0)))$$

$$= g_1(S^1_{nor\gamma'(t_0-0)}(\beta'(0)),\beta'(0)) - g_2(S^2_{nor\gamma'(t_0+0)}(\beta'(0)),\beta'(0))$$
$$= g_1(D^1_{\beta'(0)}\beta', \mathrm{nor}\gamma'(t_0-0)) - g_2(D^2_{\beta'(0)}\beta', \mathrm{nor}\gamma'(t_0+0))$$
$$= g_1(A(t_0-0)+2t_0'(0)Y'(t_0-0)+t_0''(0)\gamma'(t_0-0),\gamma'(t_0-0)) - g_1(D^B_{\beta'(0)}\beta', \tan\gamma'(t_0-0))$$
$$-g_2(A(t_0+0)+2t_0'(0)Y'(t_0+0)+t_0''(0)\gamma'(t_0+0),\gamma'(t_0+0)) + g_2(D^B_{\beta'(0)}\beta', \tan\gamma'(t_0+0))$$
$$= g_1(A(t_0-0)+2t_0'(0)Y'(t_0-0),\gamma'(t_0-0)) - g_2(A(t_0+0)+2t_0'(0)Y'(t_0+0),\gamma'(t_0+0)),$$

where D^B is the Levi-Civita connection of B. This completes the proof. □

For a fixed endpoint variation, since $g_1(A(a),\gamma'(a)) = 0 = g_2(A(b),\gamma'(b))$, $L''(0)$ depends only on the variation vector field Y.

Let p and q be points of M_1 and M_2, respectively. And let $\Omega(p,q) \subset \tilde{\Omega}$ be the set of all piecewise smooth curves $\alpha : [a,b] \to M$ in $\tilde{\Omega}$ such that $\alpha(a) = p$ and $\alpha(b) = q$. A subspace $T_\alpha\Omega(p,q)$ in $T_\alpha\tilde{\Omega}$ is defined by

$$T_\alpha\Omega(p,q) := \{Y \in T_\alpha\tilde{\Omega} |\ Y(a) = 0,\ Y(b) = 0\}.$$

If $\alpha'(t_0-0)$ and $\alpha'(t_0+0)$ are not tangent to B and $Y \in T_\alpha\tilde{\Omega}$, then

$$d_Y := d_j = -\frac{g_1(Y(t_0-0), \mathrm{nor}\alpha'(t_0-0))}{g_1(\alpha'(t_0-0), \mathrm{nor}\alpha'(t_0-0))} = -\frac{g_2(Y(t_0+0), \mathrm{nor}\alpha'(t_0+0))}{g_2(\alpha'(t_0+0), \mathrm{nor}\alpha'(t_0+0))}.$$

Hence, if $Y, V \in T_\alpha\tilde{\Omega}$, then

$$d_{Y^\perp} = d_Y + \frac{g_1(Y(t_0-0),\alpha'(t_0-0))}{g_1(\alpha'(t_0-0),\alpha'(t_0-0))} = d_Y + \frac{g_2(Y(t_0+0),\alpha'(t_0+0))}{g_2(\alpha'(t_0+0),\alpha'(t_0+0))}$$

and $d_{Y+V} = d_Y + d_V$.

Lemma 2.3. *Let P_λ be a linear operator defined as definition 2.1. Then we get*

$$P_1^{\gamma'(t_0-0)}(Y(t_0-0)^\perp) = P_1^{\gamma'(t_0-0)}(Y(t_0-0)),$$

and

$$P_2^{\gamma'(t_0+0)}(Y(t_0+0)^\perp) = P_2^{\gamma'(t_0+0)}(Y(t_0+0)),$$

for all $Y \in T_\gamma\tilde{\Omega}$.

Proof. Let $X = \gamma'(t_0-0)$, $Y = Y(t_0-0)$ and $Y^\perp = Y(t_0-0)^\perp$. Then we have that

$$P_1^X(Y^\perp) = Y^\perp + d_{Y^\perp}X = \left(Y - \frac{g_1(Y,X)}{g_1(X,X)}X\right) + \left(d_Y + \frac{g_1(Y,X)}{g_1(X,X)}\right)X = P_1^X(Y). \qquad \square$$

We note that $P_1^{\gamma'(t_0-0)}(\gamma'(t_0-0)) = P_2^{\gamma'(t_0+0)}(\gamma'(t_0+0)) = 0$ and $P_1^{\gamma'(t_0-0)}(Y^T(t_0-0)) = P_2^{\gamma'(t_0+0)}(Y^T(t_0+0)) = 0$.

Definition 2.4. The *index form* I_γ of a B-geodesic γ such that $\gamma(a) = p$ and $\gamma(b) = q$ is the unique symmetric bilinear form

$$I_\gamma : T_\gamma\Omega(p,q) \times T_\gamma\Omega(p,q) \to \boldsymbol{R},$$

such that

$$I_\gamma(Y,Y) = L''(0),$$

where L is the length function of a fixed endpoint variation of γ in $\Omega(p,q)$ with variation vector field $Y \in T_\gamma\Omega(p,q)$.

Corollary 2.5. *If $\gamma \in \Omega(p,q)$ is a B-geodesic of constant speed $c \neq 0$ such that $\gamma(t_0) \in B$, then*

$$\begin{aligned}
I_\gamma(Y,W) = \frac{1}{c}\Big\{&\int_a^{t_0} g_1(Y^{\perp\prime}, W^{\perp\prime}) - g_1(R^1(Y,\gamma')\gamma', W)dt \\
&+\int_{t_0}^{b} g_2(Y^{\perp\prime}, W^{\perp\prime}) - g_2(R^2(Y,\gamma')\gamma', W)dt\Big\} \\
&+\frac{1}{c}\{g_1(S^1_{nor\gamma'(t_0-0)}(P_1^{\gamma'(t_0-0)}(Y(t_0-0))), P_1^{\gamma'(t_0-0)}(W(t_0-0))) \\
&-g_2(S^2_{nor\gamma'(t_0+0)}(P_2^{\gamma'(t_0+0)}(Y(t_0+0))), P_2^{\gamma'(t_0+0)}(W(t_0+0)))\},
\end{aligned}$$

for all $Y, W \in T_\gamma\Omega(p,q)$.

From Lemma 2.3, it follows immediately that

$$I_\gamma(Y,W) = I_\gamma(Y^\perp, W^\perp) \qquad \text{for all } Y, W \in T_\gamma\Omega(p,q).$$

Thus there is no loss of information in restricting the index form I_γ to

$$T_\gamma^\perp\Omega(p,q) := \{Y \in T_\gamma\Omega(p,q) \mid Y \perp \gamma'\}.$$

We write $I_\gamma^\perp$ for this restriction.

Integration by parts produces a new version of the formula above.

Corollary 2.6. *Let $\gamma \in \Omega(p,q)$ be a B-geodesic of constant speed $c \neq 0$ such that $\gamma(t_0) \in B$. If Y and $W \in T_\gamma\Omega(p,q)$ have breaks $a_1 < \cdots < t_0 = a_j < \cdots < a_k$, then we have that*

$$\begin{aligned}
I_\gamma(Y,W) = -\frac{1}{c}&\Big\{\int_a^{t_0} g_1(Y^{\perp\prime\prime} + R^1(Y,\gamma')\gamma', W^\perp)dt + \int_{t_0}^{b} g_2(Y^{\perp\prime\prime} + R^2(Y,\gamma')\gamma', W^\perp)dt\Big\} \\
&+\frac{1}{c}\{g_1(S^1_{nor\gamma'(t_0-0)}(P_1^{\gamma'(t_0-0)}(Y(t_0-0))) + Y^{\perp\prime}(t_0-0), P_1^{\gamma'(t_0-0)}(W(t_0-0)))
\end{aligned}$$

$$-g_2(S^2_{nor\gamma'(t_0+0)}(P_2^{\gamma'(t_0+0)}(Y(t_0+0)))+Y^{\perp\prime}(t_0+0),P_2^{\gamma'(t_0+0)}(W(t_0+0)))\}$$
$$+\frac{1}{c}\left\{\sum_{i=1}^{j-1}g_1(\Delta_{a_i}Y^{\perp\prime},W^{\perp}(a_i))+\sum_{i=j+1}^{k}g_2(\Delta_{a_i}Y^{\perp\prime},W^{\perp}(a_i))\right\}.$$

Proof. We have

$$g_1(Y^{\perp\prime}(t_0-0),W^{\perp}(t_0-0))$$
$$=g_1(Y^{\perp\prime}(t_0-0),P_1^{\gamma'(t_0-0)}(W^{\perp}(t_0-0))-d_{W^{\perp}}g_1(Y^{\perp\prime}(t_0-0),\gamma'(t_0-0))$$
$$=g_1(Y^{\perp\prime}(t_0-0),P_1^{\gamma'(t_0-0)}(W(t_0-0)).\qquad\square$$

Corollary 2.7. *Let $\gamma\in\Omega(p,q)$ be a B-geodesic of constant speed $c\neq 0$ such that $\gamma(t_0)\in B$. Then $Y\in T_\gamma^{\perp}\Omega(p,q)$ is an element of the nullspace of $I_\gamma^{\perp}$ if and only if Y satisfies the following two properties :*

(2.1) $$Y \text{ is a Jacobi vector field on } M_1 \text{ and } M_2,$$

and

(2.2) $$S^1_{nor\gamma'(t_0-0)}(P_1^{\gamma'(t_0-0)}(Y(t_0-0)))+\tan Y'(t_0-0)$$
$$=S^2_{nor\gamma'(t_0+0)}(P_2^{\gamma'(t_0+0)}(Y(t_0+0)))+\tan Y'(t_0+0).$$

Let $q\in B$, $u\in T_qM_1$ and $v\in T_qM_2$ with $\|u\|_1=\|v\|_2$, $\tan u=\tan v$ and $v\notin T_qB$. We define a linear map $A_{u,v}:T_qB\oplus\mathrm{Span}\{\mathrm{nor}v\}\to T_qB\oplus\mathrm{Span}\{\mathrm{nor}v\}$ as

$$A_{u,v}(w)=\frac{1}{\|v\|_2}\left\{(S^1_{noru}-S^2_{norv})(P_2^v(w))-\frac{g_2((S^1_{noru}-S^2_{norv})(P_2^v(w)),v)}{g_2(v,\mathrm{nor}v)}\mathrm{nor}v\right\}$$

for any $w\in T_qB\oplus\mathrm{Span}\{\mathrm{nor}v\}$. We call this map $A_{u,v}$ a *passage endomorphism.*

Lemma 2.8. *The map $A_{u,v}$ is symmetric.*

Proof. Let $w_1,w_2\in T_qB\oplus\mathrm{Span}\{\mathrm{nor}v\}$. Then we have that

$$g_2(A_{u,v}(w_1),w_2)=g_2(A_{u,v}(w_1),P_2^v(w_2)+\frac{g_2(w_2,\mathrm{nor}v)}{g_2(v,\mathrm{nor}v)}v)$$
$$=\frac{1}{\|v\|_2}\left\{g_2((S^1_{noru}-S^2_{norv})(P_2^v(w_1)),P_2^v(w_2)+\frac{g_2(w_2,\mathrm{nor}v)}{g_2(v,\mathrm{nor}v)}v)\right.$$
$$\left.-\frac{g_2((S^1_{noru}-S^2_{norv})(P_2^v(w_1)),v)}{g_2(v,\mathrm{nor}v)}g_2(\mathrm{nor}v,P_2^v(w_2)+\frac{g_2(w_2,\mathrm{nor}v)}{g_2(v,\mathrm{nor}v)}v)\right\}$$
$$=\frac{1}{\|v\|_2}\left\{g_2((S^1_{noru}-S^2_{norv})(P_2^v(w_1)),P_2^v(w_2))\right.$$

$$+g_2((S^1_{noru} - S^2_{norv})(P^v_2(w_1)), \frac{g_2(w_2, \mathrm{nor}v)}{g_2(v, \mathrm{nor}v)} v)$$

$$- \frac{g_2((S^1_{noru} - S^2_{norv})(P^v_2(w_1)), v)}{g_2(v, \mathrm{nor}v)} g_2(w_2, \mathrm{nor}v)\Big\}$$

$$= \frac{1}{\|v\|_2} g_2((S^1_{noru} - S^2_{norv})(P^v_2(w_1)), P^v_2(w_2))$$

$$= \frac{1}{\|v\|_2} g_2(P^v_2(w_1), (S^1_{noru} - S^2_{norv})(P^v_2(w_2))) = g_2(w_1, A_{u,v}(w_2)). \qquad \square$$

The following hold:

$$A_{ku,kv} = A_{u,v} \quad for\ k \neq 0, \tag{2.3}$$

$$A_{u,v}(w) \perp v, \tag{2.4}$$

$$A_{u,v}(w^{\perp}) = A_{u,v}(w) \tag{2.5}$$

where $w^{\perp} = w - \frac{g_2(w,v)}{g_2(v,v)} v$.

Corollary 2.9. *If $\gamma \in \Omega(p,q)$ is a B-geodesic of constant speed $c \neq 0$ such that $\gamma(t_0) \in B$ and $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$, then*

$$I_\gamma(Y, W) = \frac{1}{c}\Big\{\int_a^{t_0} (g_1(Y^{\perp\prime}, W^{\perp\prime}) - g_1(R^1(Y, \gamma')\gamma', W))dt$$

$$+ \int_{t_0}^{b} (g_2(Y^{\perp\prime}, W^{\perp\prime}) - g_2(R^2(Y, \gamma')\gamma', W))dt\Big\}$$

$$+ g_2(A_{\gamma'(t_0-0), \gamma'(t_0+0)}(Y(t_0 + 0)), W(t_0 + 0)),$$

for all $Y, W \in T_\gamma\Omega(p,q)$.

3. Conjugate points

Let $\gamma : [a, b] \to M$ be a B-geodesic such that $\gamma(t_0) \in B$. Consider a variation $\varphi : [a, b] \times (-\varepsilon, \varepsilon) \to M$ such that $\varphi(t, 0) = \gamma(t)$ and $\varphi_s = \varphi(\cdot, s)$ is a B-geodesic for each s and the parameters $t_0(s)$ at which the B-geodesics are through B for s. Let Y be the variation vector field. Then, we can prove the following.

Lemma 3.1.

$$Y'' + R^\lambda(Y, \gamma')\gamma' = 0 \quad on\ M_\lambda\ (\lambda = 1, 2), \tag{3.1}$$

$$(3.2) \qquad S^1_{nor\gamma'(t_0-0)}(P_1^{\gamma'(t_0-0)}(Y(t_0-0))) + \tan Y'(t_0-0)$$
$$= S^2_{nor\gamma'(t_0+0)}(P_2^{\gamma'(t_0+0)}(Y(t_0+0))) + \tan Y'(t_0+0),$$

(3.3) $g_\lambda(Y(t),\gamma'(t)) = C_1 t + C_2$ *for some constants* C_1 *and* C_2 $(\lambda = 1,2)$.

Proof. (2): Let $\beta : (-\varepsilon,\varepsilon) \to B$ be $\beta(s) = \varphi(t_0(s),s)$. And we put $Z_\pm(s) = \mathrm{nor}X(t_0(s)\pm 0, s)$. Then, we find

$$S^1_{norX(t_0-0)}(P_1^{X(t_0-0)}(Y(t_0-0))) = S^1_{Z_-(0)}(\beta'(0)) = -\tan(D^1_{\beta'}Z_-)(0)$$

and

$$S^2_{norX(t_0+0)}(P_2^{X(t_0+0)}(Y(t_0+0))) = -\tan(D^2_{\beta'}Z_+)(0).$$

Further, it holds that

$$\begin{aligned}
D^1_{\beta'}Z_- &= D^1_{\beta'}(X(t_0(s)-0,s) - \tan X(t_0(s)-0,s)) \\
&= t_0'(s)\frac{D^1X}{\partial t}(t_0(s)-0,s) + \frac{D^1X}{\partial s}(t_0(s)-0,s) \\
&\quad -D^B_{\beta'}(\tan X(t_0(s)-0,s)) - \mathrm{nor}D^1_{\beta'}(\tan X(t_0(s)-0,s)) \\
&= \frac{D^1Y}{\partial t}(t_0(s)-0,s) - D^B_{\beta'}(\tan X(t_0(s)-0,s)) - \mathrm{nor}D^1_{\beta'}(\tan X(t_0(s)-0,s)),
\end{aligned}$$

where D^B is the Levi-Civita connection of B. Hence we have that

$$(3.4) \quad S^1_{norX(t_0-0)}(P_1^{X(t_0-0)}(Y(t_0-0)) = -\tan Y'(t_0-0) + D^B_{\beta'(0)}(\tan X(t_0(s)-0,s))$$

and similarly

$$(3.5) \quad S^2_{norX(t_0+0)}(P_2^{X(t_0+0)}(Y(t_0+0)) = -\tan Y'(t_0+0) + D^B_{\beta'(0)}(\tan X(t_0(s)+0,s)).$$

□

Lemma 3.2.
If φ *is a variation through* B*-geodesics of the same constant speed, that is,* $\|\partial\varphi_s/\partial t\|_\lambda = $ *const for all* s*, then we have that*

(3.6) $g_\lambda(Y(t),\gamma'(t))$ *is constant.*

Furthermore,

$$(3.7) \qquad Y' = Y'^{\perp}.$$

Definition 3.3. Let γ be a B-geodesic such that $\gamma(t_0) \in B$. If $Y \in T_\gamma\tilde{\Omega}$ satisfies the conditions (3.1), (3.2) and

$$g_1(Y'(t_0-0), \gamma'(t_0-0)) = g_2(Y'(t_0+0), \gamma'(t_0+0)), \tag{3.8}$$

then Y is called a *B-Jacobi field along* γ.

We note that, by (3.2) and (3.8), if Y is a B-Jacobi field, then

$$\begin{aligned} &g_1(Y'(t_0-0) - \mathrm{II}^1(P(Y), \tan\gamma'(t_0-0)), \mathrm{nor}\gamma'(t_0-0)) \\ &= g_2(Y'(t_0+0) - \mathrm{II}^2(P(Y), \tan\gamma'(t_0+0)), \mathrm{nor}\gamma'(t_0+0)), \end{aligned} \tag{3.9}$$

where II^λ is the second fundamental form tensor defined by

$$\mathrm{II}^\lambda(V, W) = \mathrm{nor}D^\lambda_V W$$

for any tangent vector fields V and W to B, and $P(Y) = P_1^{\gamma'(t_0-0)}(Y(t_0-0)) = P_2^{\gamma'(t_0+0)}(Y(t_0+0))$. Let $\mathcal{J}_\gamma$ be the set of all B-Jacobi fields along γ. A B-Jacobi field Y along γ is a *perpendicular B-Jacobi field* if Y is perpendicular to γ. Let $\mathcal{J}_\gamma^\perp$ be the set of all perpendicular B-Jacobi fields along γ. A B-Jacobi field Y along γ is a *continuous B-Jacobi field* if $Y(t_0-0) = Y(t_0+0) \in T_{\gamma(t_0)}B$. Let $\mathcal{J}_\gamma^{con}$ be the set of all continuous B-Jacobi fields along γ.

Lemma 3.4. *Let γ be a B-geodesic such that $\gamma(t_0) \in B$. If Y and W are B-Jacobi fields along γ, then it holds that*

$$g_\lambda(Y(t), W'(t)) - g_\lambda(Y'(t), W(t)) \text{ is constant,} \tag{3.10}$$

and

$$g_\lambda(Y(t), \gamma'(t)) = C_1 t + C_2 \tag{3.11}$$

for some constants C_1 and C_2 ($\lambda = 1, 2$).

Proof. Let $Y(t_0 \pm 0) = Y_\pm$, $W(t_0 \pm 0) = W_\pm$, $Y'(t_0 \pm 0) = Y'_\pm$, $W'(t_0 \pm 0) = W'_\pm$ and $\gamma'(t_0 \pm 0) = X_\pm$. As usual, it is clear that

$$g_1(Y, W') - g_1(Y', W) = \text{const.} = C_1$$

and

$$g_2(Y, W') - g_2(Y', W) = \text{const.} = C_2.$$

Hence we must show that

$$g_1(Y_-, W'_-) - g_1(Y'_-, W_-) = g_2(Y_+, W'_+) - g_2(Y'_+, W_+).$$

In fact we have that, by (3.2) and (3.9),

$$
\begin{aligned}
&g_1(Y_-, W'_-) - g_1(Y'_-, W_-) - \{g_2(Y_+, W'_+) - g_2(Y'_+, W_+)\} \\
&\quad = g_1(P_1^{X_-}(Y_-) - d_Y X_-, W'_-) - g_1(Y'_-, P_1^{X_-}(W_-) - d_W X_-) \\
&\qquad -\{g_2(P_2^{X_+}(Y_+) - d_Y X_+, W'_+) - g_2(Y'_+, P_2^{X_+}(W_+) - d_W X_+)\} \\
&\quad = g_1(P_1^{X_-}(Y_-), \tan W'_- - \tan W'_+) - g_1(\tan Y'_- - \tan Y'_+, P_1^{X_-}(W_-)) \\
&\qquad -d_Y\{g_1(X_-, W'_-) - g_2(X_+, W'_+)\} + d_W\{g_1(Y'_-, X'_-) - g_2(Y'_+, X_+)\} \\
&\quad = g_1(P_1^{X_-}(Y_-), -S^1_{norX_-}(P_1^{X_-}(W_-)) + S^2_{norX_+}(P_2^{X_+}(W_+))) \\
&\qquad -g_1(-S^1_{norX_-}(P_1^{X_-}(Y_-)) + S^2_{norX_+}(P_2^{X_+}(Y_+)), P_1^{X_-}(W_-)) \\
&\quad = -g_1(P_1^{X_-}(Y_-), S^1_{norX_-}(P_1^{X_-}(W_-))) + g_2(P_2^{X_+}(Y_+), S^2_{norX_+}(P_2^{X_+}(W_+))) \\
&\qquad +g_1(S^1_{norX_-}(P_1^{X_-}(Y_-)), P_1^{X_-}(W_-))) - g_2(S^2_{norX_+}(P_2^{X_+}(Y_+)), P_2^{X_+}(W_+)) = 0.
\end{aligned}
$$

□

By Corollary 2.7 elements of the nullspace of $I_\gamma^\perp$ are perpendicular B-Jacobi fields. If Y is a B-Jacobi field, then $Y \perp \gamma \Leftrightarrow$ there exist $t_i \in [a,b]$ $(i = 1,2)$ such that $Y(t_i) \perp \gamma$ $(i = 1,2) \Leftrightarrow$ there exist $t_i \in [a,b]$ $(i = 1,2)$ such that $Y(t_1) \perp \gamma$ and $Y'(t_2) \perp \gamma$, since (3.11). Y is a B-Jacobi field if and only if Y^T and $Y^\perp$ are B-Jacobi fields. $\mathcal{J}_\gamma$, $\mathcal{J}_\gamma^\perp$ and $\mathcal{J}_\gamma^{con}$ forms real vector spaces.

Lemma 3.5. *Let γ be a B-geodesic and Y a B-Jacobi field along γ. Then Y is the variation vector field of a variation φ of γ through B-geodesics.*

Definition 3.6. Let γ be a B-geodesic with $\gamma(t_0) \in B$ and $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$, and Y a B-Jacobi field. We say that Y is *strong* if it holds

$$
(3.12) \quad S^1_{nor\gamma'(t_0-0)}(P_1^{\gamma'(t_0-0)}(Y(t_0-0))) + Y'(t_0-0) \in T_{\gamma(t_0)}B \oplus \mathrm{Span}\{\mathrm{nor}\gamma'(t_0-0)\},
$$

that is,

$$
(3.13) \quad \mathrm{nor}Y'(t_0-0) = \frac{g_1(Y'(t_0-0), \mathrm{nor}\gamma'(t_0-0))}{g_1(\gamma'(t_0-0), \mathrm{nor}\gamma'(t_0-0))}\mathrm{nor}\gamma'(t_0-0),
$$

and

$$
(3.14) \quad S^2_{nor\gamma'(t_0+0)}(P_2^{\gamma'(t_0+0)}(Y(t_0+0))) + Y'(t_0+0) \in T_{\gamma(t_0)}B \oplus \mathrm{Span}\{\mathrm{nor}\gamma'(t_0+0)\},
$$

that is,

$$
(3.15) \quad \mathrm{nor}Y'(t_0+0) = \frac{g_2(Y'(t_0+0), \mathrm{nor}\gamma'(t_0+0))}{g_2(\gamma'(t_0+0), \mathrm{nor}\gamma'(t_0+0))}\mathrm{nor}\gamma'(t_0+0).
$$

Let $\mathcal{J}_\gamma^{st}$ be the set of all strong B-Jacobi fields. $\mathcal{J}_\gamma^{st}$ forms a real vector space. If $Y \in \mathcal{J}_\gamma$, then it holds that $Y^T \in \mathcal{J}_\gamma^{st}$. We note that if $\dim M_1 = \dim M_2 = \dim B + 1$, then all B-Jacobi fields are strong.

Proposition 3.7. *Let γ be a B-geodesic with $\gamma(t_0) \in B$ and $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$, and Y a B-Jacobi field. Then there exist a strong B-Jacobi field W with $W(t_0-0) = Y(t_0-0)$, $\tan W'(t_0-0) = \tan Y'(t_0-0)$ and $g_1(W'(t_0-0), \mathrm{nor}\gamma'(t_0-0)) = g_1(Y'(t_0-0), \mathrm{nor}\gamma'(t_0-0))$, and a B-Jacobi field V with $V(t_0-0)=0$ such that $Y(t) = W(t)+V(t)$. And this decomposition is unique.*

Proof. We put

$$v_- = \mathrm{nor}Y'(t_0-0) - \frac{g_1(Y'(t_0-0), \mathrm{nor}\gamma'(t_0-0))}{g_1(\gamma'(t_0-0), \mathrm{nor}\gamma'(t_0-0))}\mathrm{nor}\gamma'(t_0-0)$$

and

$$v_+ = \mathrm{nor}Y'(t_0+0) - \frac{g_2(Y'(t_0+0), \mathrm{nor}\gamma'(t_0+0))}{g_2(\gamma'(t_0+0), \mathrm{nor}\gamma'(t_0+0))}\mathrm{nor}\gamma'(t_0-0).$$

Let V be a B-Jacobi field such that $V(t_0-0)=0$ and $V'(t_0 \pm 0) = v_\pm$. In fact V satisfies the conditions (3.2) and (3.6). We set $W = Y - V$. Then we have that

$$W(t_0-0) = Y(t_0-0),\ \tan W'(t_0-0) = \tan Y'(t_0-0)$$

and

$$\mathrm{nor}W'(t_0-0) = \mathrm{nor}Y'(t_0-0) - v_- = \frac{g_1(Y'(t_0-0), \mathrm{nor}\gamma'(t_0-0))}{g_1(\gamma'(t_0-0), \mathrm{nor}\gamma'(t_0-0))}\mathrm{nor}\gamma'(t_0-0).$$

Hence we get that

$$g_1(W'(t_0-0), \mathrm{nor}\gamma'(t_0-0)) = g_1(Y'(t_0-0), \mathrm{nor}\gamma'(t_0-0))$$

and

$$\mathrm{nor}W'(t_0-0) = \frac{g_1(W'(t_0-0), \mathrm{nor}\gamma'(t_0-0))}{g_1(\gamma'(t_0-0), \mathrm{nor}\gamma'(t_0-0))}\mathrm{nor}\gamma'(t_0-0).$$

We have

$$\mathrm{nor}W'(t_0+0) = \frac{g_2(W'(t_0+0), \mathrm{nor}\gamma'(t_0+0))}{g_2(\gamma'(t_0+0), \mathrm{nor}\gamma'(t_0+0))}\mathrm{nor}\gamma'(t_0+0)$$

in a similar way. It follows that W is strong.

We assume that there exist another decomposition $Y = W_1 + V_1$ where W_1 is a strong B-Jacobi field with $W_1(t_0-0) = Y(t_0-0)$, $\tan W_1'(t_0-0) = \tan Y'(t_0-0)$ and $g_1(W_1'(t_0-0), \mathrm{nor}\gamma'(t_0-0)) = g_1(Y'(t_0-0), \mathrm{nor}\gamma'(t_0-0))$. Then we have

$$W_1(t_0 \pm 0) = Y(t_0-0) = W(t_0-0),$$

$$\tan W_1'(t_0 \pm 0) = \tan Y'(t_0 \pm 0) = \tan W'(t_0 \pm 0)$$

and, since $g_1(W_1'(t_0 - 0), \mathrm{nor}\gamma'(t_0 - 0)) = g_1(Y'(t_0 - 0), \mathrm{nor}\gamma'(t_0 - 0)) = g_1(W'(t_0 - 0), \mathrm{nor}\gamma'(t_0 - 0))$,

$$\mathrm{nor}W_1'(t_0 - 0) = \frac{g_1(W_1'(t_0 - 0), \mathrm{nor}\gamma'(t_0 - 0))}{g_1(\gamma'(t_0 - 0), \mathrm{nor}\gamma'(t_0 - 0))}\mathrm{nor}\gamma'(t_0 - 0)$$

$$= \frac{g_1(W'(t_0 - 0), \mathrm{nor}\gamma'(t_0 - 0))}{g_1(\gamma'(t_0 - 0), \mathrm{nor}\gamma'(t_0 - 0))}\mathrm{nor}\gamma'(t_0 - 0) = \mathrm{nor}W'(t_0 - 0).$$

We get that $\mathrm{nor}W_1'(t_0 + 0) = \mathrm{nor}W'(t_0 + 0)$ in a similar way. Hence we have that $W_1 = W$. □

Proposition 3.7 gives the direct sum decomposition

$$\mathcal{J}_\gamma = \mathcal{J}_\gamma^{st} + \mathcal{J}_\gamma^{M_1, M_2}.$$

Elements of $\mathcal{J}_\gamma^{M_1, M_2}$ are called (M_1, M_2)-*Jacobi fields.* Then we have that

$$\mathcal{J}_\gamma^{M_1, M_2} = \mathcal{J}_\gamma^{M_1} + \mathcal{J}_\gamma^{M_2},$$

where $\mathcal{J}_\gamma^{M_\lambda}$ is the set of all (M_1, M_2)-Jacobi fields which is identically zero on M_μ $(\lambda \neq \mu)$. The resulting projections $\mathrm{pr}_{st} : \mathcal{J}_\gamma \to \mathcal{J}_\gamma^{st}$ and $\mathrm{pr}_{M_1, M_2} : \mathcal{J}_\gamma \to \mathcal{J}_\gamma^{M_1, M_2}$ are obviously $\boldsymbol{R}$-linear. For $Y \in \mathcal{J}_\gamma$, we put $\mathrm{pr}_{st}(Y) =: Y^{st}$.

We treat special cases of B-geodesics. Let γ be a B-geodesic with $\gamma(t_0) \in B$. If $\gamma'(t_0 + 0)$ is normal to B (thus so is $\gamma'(t_0 - 0)$), γ is called a *normal B-geodesic.* By using (3.4) and (3.5), the following assertion holds.

Proposition 3.8. *A B-Jacobi field Y along a normal B-geodesic γ is the variation vector field of a variation φ of γ through normal B-geodesics if and only if*

(3.16) $$S^1_{\gamma'(t_0 - 0)}(P_1^{\gamma'(t_0 - 0)}(Y(t_0 - 0)) + \tan Y'(t_0 - 0) = 0$$

(that is $S^1_{\gamma'(t_0 - 0)}(P_1^{\gamma'(t_0 - 0)}(Y(t_0 - 0)) + Y'(t_0 - 0)$ is normal to B),
and

(3.17) $$S^2_{\gamma'(t_0 + 0)}(P_2^{\gamma'(t_0 + 0)}(Y(t_0 + 0)) + \tan Y'(t_0 + 0) = 0$$

(that is $S^2_{\gamma'(t_0 + 0)}(P_2^{\gamma'(t_0 + 0)}(Y(t_0 + 0)) + Y'(t_0 + 0)$ is normal to B).

Remark. Let γ be a normal B-geodesic. Then any perpendicular B-Jacobi fields are continuous.

Definition 3.9. Let γ be a B-geodesic such that $\gamma(t_0) \in B$. We say that $\gamma(t_2)$ ($t_2 \in (a,b]$) is a *B-conjugate point to* $\gamma(t_1)$ ($t_1 \in [a,b)$, $t_1 < t_2$) *along* γ if there exists a B-Jacobi field Y along γ such that $Y(t_1) = 0$, $Y(t_2) = 0$ and $Y|[t_1,t_2]$ is nontrivial.

Remark. 1. Let γ be a normal B-geodesic with $\gamma(t_0) \in B$ and $\dim B = 1$. If there exist focal points of B along $\gamma|[a, t_0]$ and $\gamma|[t_0, b]$, then there exist B-conjugate points.

2. B-conjugate points in M_λ are always usual ones but the converse is not true in general. We give an example which shows this:

Example 1. Let $M = M_1 \cup_{id} M_2$ be a glued Riemannian space which consists of the following M_λ and B a submanifold of M_λ ($\lambda = 1, 2$):

$$M_1 = S^2(1) = \{(x,y,z)|x^2+y^2+z^2=1\},\ M_2 = \boldsymbol{E}^3,\ B = \{(0,-1,0)\},$$

and g_1 is a Riemannian metric induced from the natural Euclidean metric of $\boldsymbol{E}^3$ and g_2 is the natural Euclidean metric of $\boldsymbol{E}^3$. We defined a B-geodesic $\gamma : [-\pi/2, +\infty) \to M$ by

$$\gamma(t) = \begin{cases} (0, \cos t, \sin t) & on\ [-\pi/2, \pi] \\ (0, -t+\pi-1, 0) & on\ [\pi, +\infty) \end{cases}.$$

Then, $T_\gamma\tilde{\Omega}$ is the set of all vector fields Y along γ such that $Y|[a,t_0]$ and $Y|[t_0,b]$ are piecewise smooth vector fields on M_1 and M_2, respectively, and, $Y(t_0-0) = d\gamma'(t_0-0)$ and $Y(t_0+0) = d\gamma'(t_0+0)$ for some $d \in \boldsymbol{R}$. Hence, $\gamma(\pi/2)$ is a conjugate point to $\gamma(-\pi/2)$ but not a B-conjugate point.

Remark. Let γ be a B-geodesic with $\gamma(t_0) \in B$. If $\gamma(t_0)$ is a conjugate point to $\gamma(a)$ along $\gamma|[a, t_0]$, then it is also a B-conjugate point to $\gamma(a)$ along γ.

Examples of B-conjugate points: We give some examples. Let $U_1 := \partial/\partial x$, $U_2 := \partial/\partial y$ and $U_3 := \partial/\partial z$ be the natural frame field on the Euclidean space $\boldsymbol{E}^3$.

Example 2. Let $M = M_1 \cup_{id} M_2$ be a glued Riemannian space which consists of the following two surfaces in the Euclidean space $\boldsymbol{E}^3$ and B a boundary (submanifold) of M_λ ($\lambda = 1, 2$):

$$M_1 = \{(x,y,z)|x^2+y^2+z^2=1, y \geq 0\},\ M_2 = \{(x,y,z)|x^2+y^2+z^2=1, y \leq 0\},$$

$$B = \{(x,0,z)|x^2+z^2=1\},$$

and g_λ, $\lambda = 1, 2$, are Riemannian metrics induced from the natural Euclidean metric of $\boldsymbol{E}^3$. We defined a (normal) B-geodesic $\gamma : [0, \pi] \to M$ by

$$\gamma(t) = (0, \cos t, \sin t).$$

Then $Y(t) = \sin tU_1$ is a B-Jacobi field along γ. Hence $\gamma(\pi)$ is a B-conjugate point to $\gamma(0)$. If M_λ and B are replaced by the following, we get a B-geodesic which is not normal:

$$M_1 = S^2(1) \cap \{(x,y,z)|x \leq y\},\ M_2 = S^2(1) \cap \{(x,y,z)|x \geq y\},$$
$$B = S^2(1) \cap \{(x,x,z)\}.$$

In this case, $\gamma(\pi)$ is a B-conjugate point to $\gamma(0)$ as above.

Example 3. Let $M = M_1 \cup_{id} M_2$ be a glued Riemannian space which consists of the following M_λ and B a submanifold of M_λ $(\lambda = 1,2)$:

$$M_1 = S^2(1),\ M_2 = \boldsymbol{E}^3,\ B = \{(0,-1,0)\},$$

and g_1 is a Riemannian metric induced from the natural Euclidean metric of $\boldsymbol{E}^3$ and g_2 is the natural Euclidean metric of $\boldsymbol{E}^3$. We defined a B-geodesic $\gamma : [0,+\infty) \to M$ by

$$\gamma(t) = \begin{cases} (0, \cos t, \sin t) \ \textit{on}\ [0,\pi] \\ (0, -t+\pi-1, 0) \ \textit{on}\ [\pi,+\infty) \end{cases}.$$

Then,

$$Y(t) = \begin{cases} \sin tU_1 \quad \textit{on}\ [0,\pi] \\ 0 \quad \textit{on}\ [\pi,+\infty) \end{cases}$$

is a B-Jacobi field along γ. Hence, for any $t \in [\pi,+\infty)$, $\gamma(t)$ are B-conjugate points to $\gamma(0)$.

Example 4. Let $M = M_1 \cup_{id} M_2$ be a glued Riemannian space which consists of the following two surfaces in the Euclidean space $\boldsymbol{E}^3$ and B a submanifold of M_λ $(\lambda = 1,2)$:

$$M_1 = S^2(1),\ M_2 = \{(x,y,z)|x^2 + (y+2)^2 + z^2 = 1\},\ B = \{(0,-1,0)\},$$

and g_λ, $\lambda = 1,2$, are Riemannian metrics induced from the natural Euclidean metric of $\boldsymbol{E}^3$. We defined a B-geodesic $\gamma : [0,2\pi] \to M$ by

$$\gamma(t) = \begin{cases} (0, \cos t, \sin t) \ \textit{on}\ [0,\pi] \\ (0, \cos(t-\pi) - 2, \sin(t-\pi)) \ \textit{on}\ [\pi,2\pi] \end{cases}.$$

Then,

$$Y(t) = \begin{cases} \sin tU_1 \quad \textit{on}\ [0,\pi] \\ 0 \quad \textit{on}\ [\pi,2\pi] \end{cases}$$

is a B-Jacobi field along γ. Hence, for any $t \in [\pi,2\pi]$, $\gamma(t)$ are B-conjugate points to $\gamma(0)$.

Example 5. Let $M = M_1 \cup_{id} M_2$ be a glued Riemannian space which consists of the following two surfaces in the Euclidean space $\boldsymbol{E}^3$ and B a submanifold of M_λ $(\lambda = 1,2)$:

$$M_1 = S^2(1),\ M_2 = \{(x,0,z)\},\ B = \{(x,0,z)|x^2 + z^2 = 1\},$$

and g_λ, $\lambda = 1, 2$, are Riemannian metrics induced from the natural Euclidean metric of $\boldsymbol{E}^3$. We defined a B-geodesic $\gamma : [0, \pi/2 + 1] \to M$ by

$$\gamma(t) = \begin{cases} (0, \cos t, \sin t) & \textit{on } [0, \pi/2] \\ (0, 0, \pi/2 + 1 - t) & \textit{on } [\pi/2, \pi/2 + 1] \end{cases}.$$

Then

$$Y(t) = \begin{cases} \sin tU_1 & \textit{on } [0, \pi/2] \\ (1 + \pi/2 - t)U_1 & \textit{on } [\pi/2, \pi/2 + 1] \end{cases}$$

is a B-Jacobi field along γ. Hence $\gamma(\pi/2 + 1)$ is a B-conjugate point to $\gamma(0)$.

Example 6. Let $M = M_1 \cup_{id} M_2$ be a glued Riemannian space which consists of the following M_λ and B a submanifold of M_λ ($\lambda = 1, 2$):

$$M_1 = \{(x, 0, z)\},\ M_2 = \boldsymbol{E}^3,\ B = \{(x, 0, z) | x^2 + z^2 = 1\},$$

and g_1 is a Riemannian metric induced from the natural Euclidean metric of $\boldsymbol{E}^3$ and g_2 is the natural Euclidean metric of $\boldsymbol{E}^3$. For any point p of B, let γ_1 be the unit speed geodesic on M_1 from $O = (0, 0, 0)$ to p, and γ_2 the unit speed geodesic on M_2 from p to $q = (0, k, 0)$. Then, joining γ_1 and γ_2 produces a B-geodesic γ. Hence q is a B-conjugate point to O along γ.

Example 7. Let $M = M_1 \cup_{id} M_2$ be a glued Riemannian space which consists of the following M_λ and B a submanifold of M_λ ($\lambda = 1, 2$):

$$M_1 = S^2(1),\ M_2 = \boldsymbol{E}^3,\ B = \{(x, 0, z) | x^2 + z^2 = 1\},$$

and g_1 is a Riemannian metric induced from the natural Euclidean metric of $\boldsymbol{E}^3$ and g_2 is the natural Euclidean metric of $\boldsymbol{E}^3$. For any point q of B, let γ_1 be the unit speed geodesic on M_1 from $p = (0, 1, 0)$ to q, and γ_2 the unit speed geodesic on M_2 from q to $r = (0, k, 0)$. Then, joining γ_1 and γ_2 produces a B-geodesic γ. Hence r is a B-conjugate point to p along γ.

4. Fundamental properties of the index form and B-conjugate points

Let $q \in B$, $u \in T_qM_1$ and $v \in T_qM_2$ with $\|u\|_1 = \|v\|_2$, $\tan u = \tan v$ and $v \notin T_qB$. We define a linear map $Q_{u,v} : T_qB \oplus \mathrm{Span}\{\mathrm{nor}u\} \to T_qB \oplus \mathrm{Span}\{\mathrm{nor}v\}$ as

$$Q_{u,v}(w) = \left\{ w - \frac{g_1(w, \mathrm{nor}u)}{g_1(u, \mathrm{nor}u)} \mathrm{nor}u \right\} + \frac{g_1(w, \mathrm{nor}u)}{g_1(u, \mathrm{nor}u)} \mathrm{nor}v$$

for any $w \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}u\}$. The following hold:

(4.1) $$Q_{ku,kv} = Q_{u,v} \quad \text{for } k \neq 0.$$

(4.2) $$Q_{u,v}(x) = x \quad \text{for any } x \in T_qB.$$

(4.3) $$Q_{u,v}(\mathrm{nor}u) = \mathrm{nor}v.$$

(4.4) $$g_2(Q_{u,v}(w), x) = g_1(w, x)$$

for any $x \in T_qB$ and $w \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}u\}$.

(4.5) $$g_2(Q_{u,v}(w), Q_{u,v}(w)) = g_1(w, w)$$

for any $w \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}u\}$.

(4.6) $$g_2(Q_{u,v}(w), \mathrm{nor}v)g_1(w, \mathrm{nor}u) \geq 0$$

for any $w \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}u\}$. Let γ be a B-geodesic with $\gamma(t_0) \in B$ and $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$. Then we have

(4.7) $$Q_{\gamma'(t_0-0),\gamma'(t_0+0)}(\gamma'(t_0 - 0)) = \gamma'(t_0 + 0).$$

If $Y \in T_\gamma\tilde{\Omega}$, then it holds that

(4.8) $$Q_{\gamma'(t_0-0),\gamma'(t_0+0)}(Y(t_0 - 0)) = Y(t_0 + 0).$$

Remark. Let $q \in B$, $u \in T_qM_1$ and $v \in T_qM_2$ with $\|u\|_1 = \|v\|_2$, $\tan u = \tan v$ and $v \notin T_qB$. If we define a linear map $Q_{v,u} : T_qB \oplus \mathrm{Span}\{\mathrm{nor}v\} \to T_qB \oplus \mathrm{Span}\{\mathrm{nor}u\}$ as

$$Q_{v,u}(z) = \left\{ z - \frac{g_2(z, \mathrm{nor}v)}{g_2(v, \mathrm{nor}v)}\mathrm{nor}v \right\} + \frac{g_2(z, \mathrm{nor}v)}{g_2(v, \mathrm{nor}v)}\mathrm{nor}u$$

for any $z \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}v\}$. The following hold:

$$Q_{u,v} \circ Q_{v,u} = \mathrm{id}, \; Q_{v,u} \circ Q_{u,v} = \mathrm{id},$$

$$g_2(Q_{u,v}(w), z) = g_1(w, Q_{v,u}(z))$$

for $w \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}u\}$ and $z \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}v\}$.

Lemma 4.1. *Let γ be a B-geodesic such that $\gamma(t_0) \in B$ and $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$. If $Y \in T_\gamma\tilde{\Omega}$ is a B-Jacobi vector field along γ, then*

$$-\|\gamma'(t_0 + 0)\|_2 A_{\gamma'(t_0-0),\gamma'(t_0+0)}(Y(t_0 + 0))$$
$$= \tan Y'(t_0 - 0) - \tan Y'(t_0 + 0)$$

$$
\begin{aligned}
&+\frac{g_1(Y'(t_0-0),\mathrm{nor}\gamma'(t_0-0))}{g_1(\gamma'(t_0-0),\mathrm{nor}\gamma'(t_0-0))}\mathrm{nor}\gamma'(t_0+0)\\
&-\frac{g_2(Y'(t_0+0),\mathrm{nor}\gamma'(t_0+0))}{g_2(\gamma'(t_0+0),\mathrm{nor}\gamma'(t_0+0))}\mathrm{nor}\gamma'(t_0+0)\\
=\ &Q_{\gamma'(t_0-0),\gamma'(t_0+0)}(\mathrm{pr}^1(Y'(t_0-0)))-\mathrm{pr}^2(Y'(t_0+0)),
\end{aligned}
$$

where

$$\mathrm{pr}^1 : T_{\gamma(t_0)}M_1 \to T_{\gamma(t_0)}B \oplus \mathrm{Span}\{\mathrm{nor}\gamma'(t_0-0)\}$$

and

$$\mathrm{pr}^2 : T_{\gamma(t_0)}M_2 \to T_{\gamma(t_0)}B \oplus \mathrm{Span}\{\mathrm{nor}\gamma'(t_0+0)\}$$

are orthogonal projections. In particular, if Y is strong, then

$$
\begin{aligned}
&-\|\gamma'(t_0+0)\|_2 A_{\gamma'(t_0-0),\gamma'(t_0+0)}(Y(t_0+0))\\
&=Q_{\gamma'(t_0-0),\gamma'(t_0+0)}(Y'(t_0-0))-Y'(t_0+0).
\end{aligned}
$$

Proof. Let $\gamma'(t_0\pm 0)=X_\pm$, $Y(t_0\pm 0)=Y_\pm$ and $Y'(t_0\pm 0)=Y'_\pm$. Then we have that

$$
\begin{aligned}
&-\|X_+\|_2 A_{X_-,X_+}(Y_+)\\
&=-(S^1_{norX_-}-S^2_{norX_+})(P_2^{X_+}(Y_+))+\frac{g_2((S^1_{norX_-}-S^2_{norX_+})(P_2^{X_+}(Y_+)),X_+)}{g_2(X_+,\mathrm{nor}X_+)}\mathrm{nor}X_+\\
&=\tan Y'_- -\tan Y'_+ +\frac{g_2(-\tan Y'_- +\tan Y'_+,X_+)}{g_2(X_+,\mathrm{nor}X_+)}\mathrm{nor}X_+.
\end{aligned}
$$

Since $g_1(X_-,\mathrm{nor}X_-)=g_2(X_+,\mathrm{nor}X_+)$ and $g_1(Y'_-,X_-)=g_2(Y'_+,X_+)$, the first equality is true.

Moreover we have that, by (4.2) and (4.3),

$$
\begin{aligned}
&Q_{X_-,X_+}(\mathrm{pr}^1(Y'_-))-\mathrm{pr}^2(Y'_+)\\
&=\tan Y'_- +\frac{g_1(Y'_-,\mathrm{nor}X_-)}{g_1(X_-,\mathrm{nor}X_-)}Q_{X_-,X_+}(\mathrm{nor}X_-)-\left\{\tan Y'_+ +\frac{g_2(Y'_+,\mathrm{nor}X_+)}{g_2(X_+,\mathrm{nor}X_+)}\mathrm{nor}X_+\right\}\\
&=\tan Y'_- -\tan Y'_+ +\frac{g_1(Y'_-,\mathrm{nor}X_-)}{g_1(X_-,\mathrm{nor}X_-)}\mathrm{nor}X_+ -\frac{g_2(Y'_+,\mathrm{nor}X_+)}{g_2(X_+,\mathrm{nor}X_+)}\mathrm{nor}X_+.
\end{aligned}
$$

This completes the proof. □

Using Lemma 4.1 and (4.8) the following assertion can be verified.

Proposition 4.2. *Let γ be a B-geodesic such that $\gamma(t_0)\in B$ and $\gamma'(t_0+0)\notin T_{\gamma(t_0)}B$. Real vector spaces $\mathcal{J}_\gamma$, $\mathcal{J}_\gamma^{con}$, $\mathcal{J}_\gamma^\perp$ and $\mathcal{J}_\gamma^{st}$ have dimensions m_1+m_2, m_1+m_2-1, m_1+m_2-2 and $2(n+1)$, respectively, where $m_\lambda=\dim M_\lambda$ and $n=\dim B$.*

Remark. In the paper [1], the case where $M_1 = M_2$ and $B_1 = B_2$ is studied. In this case, it holds that $\dim \mathcal{J}_\gamma = 2m$, $\dim \mathcal{J}_\gamma^{con} = 2m-1$, $\dim \mathcal{J}_\gamma^\perp = 2m-2$ and $\dim\mathcal{J}_\gamma^{st} = 2(n+1)$, where $m = \dim M_1$ and $n = \dim B$. This results agree with Proposition 4.2.

In the paper [5], the author studied the case where $\dim M_1 = \dim M_2 = \dim B + 1$. In this case, it holds that $\dim \mathcal{J}_\gamma = 2(n+1)$ and $\dim \mathcal{J}_\gamma^\perp = 2n$, where $n = \dim B$. This results also agree with Proposition 4.2.

Let γ be a B-geodesic such that $\gamma(t_0)$ is not a conjugate point to $\gamma(a)$ and $\mathcal{J}_\gamma^0$ the set of all B-Jacobi field such that $Y(a) = 0$. That is $\mathcal{J}_\gamma^0 = \{Y \in \mathcal{J}_\gamma \mid Y(a) = 0\}$. Then the dimension of $\mathcal{J}_\gamma^0$ is the one of M_2.

Proposition 4.3. *Let γ be a B-geodesic. We assume that $\gamma(t_0)$ and $\gamma(b)$ are not B-conjugate points to $\gamma(a)$. Then, for any $v \in T_{\gamma(b)}M_2$, there is a unique $Y \in \mathcal{J}_\gamma^0$ with $Y(b) = v$.*

Proof. We define a map $\psi : \mathcal{J}_\gamma^0 \to T_{\gamma(b)}M_2$ by $\psi(Y) = Y(b)$. We must show that ψ is linear isomorphism. It is clear that ψ is linear. For $Y \in \mathcal{J}_\gamma^0$, we assume that $\psi(Y) = 0$. Then we have that $Y(b) = 0$ and, by the hypothesis, $Y \equiv 0$. It follows that ψ is injective. Since $\dim \mathcal{J}_\gamma^0 = \dim T_{\gamma(b)}M_2$, ψ is surjective. □

Let γ be a B-geodesic of constant speed $c \neq 0$ with $\gamma(t_0) \in B$. We set $T_\gamma\tilde{\Omega}^0 :=\{Y \in T_\gamma\tilde{\Omega} \mid Y(a) = 0\}$. Then we define the *extended index form* $I_\gamma^0 : T_\gamma\tilde{\Omega}^0 \times T_\gamma\tilde{\Omega}^0 \to \boldsymbol{R}$ by

$$\begin{aligned}
I_\gamma^0(Y,W) = \frac{1}{c}\Big\{&\int_a^{t_0}(g_1(Y^{\perp\prime},W^{\perp\prime}) - g_1(R^1(Y,\gamma')\gamma',W))dt \\
&+\int_{t_0}^b(g_2(Y^{\perp\prime},W^{\perp\prime}) - g_2(R^2(Y,\gamma')\gamma',W))dt\Big\} \\
&+\frac{1}{c}\{g_1(S^1_{nor\gamma'(t_0-0)}(P_1^{\gamma'(t_0-0)}(Y(t_0-0))), P_1^{\gamma'(t_0-0)}(W(t_0-0))) \\
&-g_2(S^2_{nor\gamma'(t_0+0)}(P_2^{\gamma'(t_0+0)}(Y(t_0+0))), P_2^{\gamma'(t_0+0)}(W(t_0+0)))\},
\end{aligned}$$

for all $Y, W \in T_\gamma\tilde{\Omega}^0$.

Let γ be a B-geodesic with $\gamma(t_0) \in B$ and $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$. We put $m_2 = \dim M_2$ and $n = \dim B$. If $\gamma(t_0)$ is not a conjugate point to $\gamma(a)$, then we note that $\dim \mathcal{J}_\gamma^{st,0} = n+1$, where $\mathcal{J}_\gamma^{st,0} = \mathcal{J}_\gamma^{st} \cap \mathcal{J}_\gamma^0$, and can take a basis of $\mathcal{J}_\gamma^0$ as follows:
Let $e_1,\cdots,e_{m_2}$ be an orthonormal basis of $T_{\gamma(t_0)}M_2$ such that $e_1,\cdots,e_n$ is an orthonormal basis of $T_{\gamma(t_0)}B$ and $e_{n+1} := \mathrm{nor}\gamma'(t_0+0)/\|\mathrm{nor}\gamma'(t_0+0)\|_2 =: e_{n+1}^+$. We put $e_{n+1}^- := \mathrm{nor}\gamma'(t_0-0)/\|\mathrm{nor}\gamma'(t_0-0)\|_1$. Then there exist $n+1$ strong B-Jacobi fields $e_1(t),\cdots,e_{n+1}(t)$ and $m_2-(n+1)$ (M_1,M_2)-Jacobi fields $e_{n+2}(t),\cdots,e_{m_2}(t)$ in $\mathcal{J}_\gamma^{M_2}$ such that

$$e_k(t_0 \pm 0) = e_k,$$

$$\tan e'_k(t_0 - 0) + S^1_{nor\gamma'(t_0-0)}(e_k) = \tan e'_k(t_0 + 0) + S^2_{nor\gamma'(t_0+0)}(e_k),$$

$$\mathrm{nor} e'_k(t_0 - 0) = g_1(e'_k(t_0 - 0), e^-_{n+1})e^-_{n+1},$$

$$\mathrm{nor} e'_k(t_0 + 0) = g_2(e'_k(t_0 + 0), e^+_{n+1})e^+_{n+1},$$

$(k = 1, \cdots, n)$;

$$e_{n+1}(t_0 \pm 0) = e^\pm_{n+1},$$

$$\tan e'_{n+1}(t_0 - 0) = \tan e'_{n+1}(t_0 + 0),$$

$$\mathrm{nor} e'_{n+1}(t_0 - 0) = g_1(e'_{n+1}(t_0 - 0), e^-_{n+1})e^-_{n+1},$$

$$\mathrm{nor} e'_{n+1}(t_0 + 0) = g_2(e'_{n+1}(t_0 + 0), e^+_{n+1})e^+_{n+1};$$

$$e_l(t) = 0 \ on [a, t_0],$$

$$e'_l(t_0 + 0) = e_l,$$

$(l = n + 2, \cdots, m_2)$.

Theorem 4.4. *Let γ be a B-geodesic such that $\gamma(t_0) \in B$ and $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$. If $\gamma(t_1)$ ($t_1 \in (t_0, b]$) is not a B-conjugate point to $\gamma(a)$ and also $\gamma(t_1)$ ($t_1 \in (a, t_0]$) is not a conjugate point to $\gamma(a)$, then, for any $Y \in T_\gamma \tilde{\Omega}^0$, there exist a unique B-Jacobi field $J \in \mathcal{J}^0_\gamma$ such that $J(b) = Y(b)$ and*

$$I^0_\gamma(J, J) \leq I^0_\gamma(Y, Y).$$

In particular, the equality holds if and only if $J^\perp = Y^\perp$.

Proof. We put $m_\lambda = \dim M_\lambda$ ($\lambda = 1, 2$) and $n = \dim B$. By Proposition 4.3, since $\gamma(b)$ is not B-conjugate point to $\gamma(a)$, there exist a unique B-Jacobi field J such that $J(b) = Y(b)$.

We can take a basis of $\mathcal{J}^0_\gamma$ as above. since γ has no B-conjugate points, $e_1(t), \cdots, e_{m_2}(t)$ are independent on $(t_0, b]$ and so are $e_1(t), \cdots, e_{n+1}(t)$ on (a, t_0). Let $\tilde{e}_{n+2}, \cdots, \tilde{e}_{m_1}$ be elements of $T_{\gamma(t_0)}M_1$ such that $e_1, \cdots, e_n, e^-_{n+1}, \tilde{e}_{n+2}, \cdots, \tilde{e}_{m_1}$ are basis of $T_{\gamma(t_0)}M_1$. Let $\tilde{e}_{n+2}(t), \cdots, \tilde{e}_{m_1}(t)$ be Jacobi fields along $\gamma|[a, t_0]$ in M_1 such that $e_1(t), \cdots, e_{n+1}(t), \tilde{e}_{n+2}(t), \cdots, \tilde{e}_{m_1}(t)$ are linearly independent on $(a, t_0]$, $\tilde{e}_l(a) = 0$ and $\tilde{e}_l(t_0 - 0) = \tilde{e}_l$ for $l = n + 2, \cdots, m_1$. For simplicity, we set $e_l(t) := \tilde{e}_l(t)$. For any $Y \in T_\gamma \tilde{\Omega}^0$, we can put

$$Y(t) = \begin{cases} \sum_{k=1}^{m_1} f_k(t) e_k(t) & on \ [a, t_0] \\ \sum_{k=1}^{m_2} f_k(t) e_k(t) & on \ [t_0, b] \end{cases},$$

where f_k ($k = 1, \cdots, n+1$) are piecewise smooth on $[a, t_0]$ and $[t_0, b]$, f_{l_1} ($l_1 = n+2, \cdots, m_1$) are piecewise smooth on $[a, t_0]$, and f_{l_2} ($l_2 = n + 2, \cdots, m_2$) are piecewise smooth on $[t_0, b]$.

Then we have that $f_k(t_0-0)=f_k(t_0+0)$ for $k=1,\cdots,n+1$ and $f_l(t_0-0)=0$ for $l=n+2,\cdots,m_1$.

We compute $I_\gamma^0(Y,Y)$. We have that, except at breaks,

$$Y'(t)=D^\lambda_{\gamma'(t)}Y(t)=\sum_{k=1}^{m_\lambda}\{f_k'(t)e_k(t)+f_k(t)e_k'(t)\}.$$

Hence we get

$$(4.9)\qquad g_\lambda(Y^{\perp\prime}(t),Y^{\perp\prime}(t))$$

$$=\Big\|\sum_{k=1}^{m_\lambda}f_k'(t)e_k^\perp(t)\Big\|_\lambda^2+2g_\lambda\Big(\sum_{k=1}^{m_\lambda}f_k'(t)e_k^\perp(t),\sum_{k=1}^{m_\lambda}f_k(t)e_k^{\perp\prime}(t)\Big)+\Big\|\sum_{k=1}^{m_\lambda}f_k(t)e_k^{\perp\prime}(t)\Big\|_\lambda^2.$$

Since $R^\lambda(e_k^\perp(t),\gamma'(t))\gamma'(t)=-e_k^{\perp\prime\prime}(t)$,

$$(4.10)\qquad g_\lambda(R^\lambda(Y(t),\gamma'(t))\gamma'(t),Y(t))=\sum_{k=1}^{m_\lambda}f_k(t)g_\lambda(R^\lambda(e_k^\perp(t),\gamma'(t))\gamma'(t),Y^\perp(t))$$

$$=-\sum_{k=1}^{m_\lambda}f_k(t)g_\lambda(e_k^{\perp\prime\prime}(t),Y^\perp(t))=-\sum_{k,l=1}^{m_\lambda}f_k(t)f_l(t)g_\lambda(e_k^{\perp\prime\prime}(t),e_l^\perp(t)).$$

We can compute

$$(4.11)\qquad \frac{d}{dt}g_\lambda\Big(\sum_{k=1}^{m_\lambda}f_k(t)e_k^\perp(t),\sum_{k=1}^{m_\lambda}f_k(t)e_k^{\perp\prime}(t)\Big)$$

$$=g_\lambda\Big(\sum_{k=1}^{m_\lambda}f_k'(t)e_k^\perp(t),\sum_{k=1}^{m_\lambda}f_k(t)e_k^{\perp\prime}(t)\Big)+g_\lambda\Big(\sum_{k=1}^{m_\lambda}f_k(t)e_k^{\perp\prime}(t),\sum_{k=1}^{m_\lambda}f_k(t)e_k^{\perp\prime}(t)\Big)$$

$$+g_\lambda\Big(\sum_{k=1}^{m_\lambda}f_k(t)e_k^\perp(t),\sum_{k=1}^{m_\lambda}f_k'(t)e_k^{\perp\prime}(t)\Big)+g_\lambda\Big(\sum_{k=1}^{m_\lambda}f_k(t)e_k^\perp(t),\sum_{k=1}^{m_\lambda}f_k(t)e_k^{\perp\prime\prime}(t)\Big).$$

By (4.9), (4.10), (4.11) and Lemma 3.4, we have that,

$$g_\lambda(Y^{\perp\prime}(t),Y^{\perp\prime}(t))-g_\lambda(R^\lambda(Y(t),\gamma'(t))\gamma'(t),Y(t))$$

$$=\Big\|\sum_{k=1}^{m_\lambda}f_k'(t)e_k^\perp(t)\Big\|_\lambda^2+\frac{d}{dt}g_\lambda\Big(\sum_{k=1}^{m_\lambda}f_k(t)e_k^\perp(t),\sum_{k=1}^{m_\lambda}f_k(t)e_k^{\perp\prime}(t)\Big)$$

$$+g_\lambda\Big(\sum_{k=1}^{m_\lambda}f_k'(t)e_k^\perp(t),\sum_{k=1}^{m_\lambda}f_k(t)e_k^{\perp\prime}(t)\Big)-g_\lambda\Big(\sum_{k=1}^{m_\lambda}f_k(t)e_k^\perp(t),\sum_{k=1}^{m_\lambda}f_k'(t)e_k^{\perp\prime}(t)\Big)$$

$$=\Big\|\sum_{k=1}^{m_\lambda}f_k'(t)e_k^\perp(t)\Big\|_\lambda^2+\frac{d}{dt}g_\lambda\Big(\sum_{k=1}^{m_\lambda}f_k(t)e_k^\perp(t),\sum_{k=1}^{m_\lambda}f_k(t)e_k^{\perp\prime}(t)\Big).$$

Since $e_k(t)\in\mathcal{J}_\gamma^0$ $(k=1,\cdots,n+1)$, we get

(4.12) $I_\gamma^0(Y,Y)$

$$= \frac{1}{c}\left\{\int_a^{t_0}\left\|\sum_{k=1}^{m_1} f_k'(t)e_k^\perp(t)\right\|_1^2 dt + \int_{t_0}^b \left\|\sum_{k=1}^{m_2} f_k'(t)e_k^\perp(t)\right\|_2^2 dt\right.$$

$$+\left[g_1(\sum_{k=1}^{m_1} f_k(t)e_k^\perp(t), \sum_{k=1}^{m_1} f_k(t)e_k^{\perp\prime}(t))\right]_a^{t_0} + \left[g_2(\sum_{k=1}^{m_2} f_k(t)e_k^\perp(t), \sum_{k=1}^{m_2} f_k(t)e_k^{\perp\prime}(t))\right]_{t_0}^b$$

$$+g_1(S^1_{nor\gamma'(t_0-0)}(P_1^{\gamma'(t_0-0)}(Y(t_0-0)), P_1^{\gamma'(t_0-0)}(Y(t_0-0)))$$

$$\left.-g_2(S^2_{nor\gamma'(t_0+0)}(P_2^{\gamma'(t_0+0)}(Y(t_0+0)), P_2^{\gamma'(t_0+0)}(Y(t_0+0)))\right\}$$

$$= \frac{1}{c}\left\{\int_a^{t_0}\left\|\sum_{k=1}^{m_1} f_k'(t)e_k^\perp(t)\right\|_1^2 dt + \int_{t_0}^b \left\|\sum_{k=1}^{m_2} f_k'(t)e_k^\perp(t)\right\|_2^2 dt\right.$$

$$+\sum_{k,l=1}^{n+1} f_k(t_0)f_l(t_0)(g_1(e_k^\perp(t_0-0), e_l^{\perp\prime}(t_0-0)) - g_2(e_k^\perp(t_0+0), e_l^{\perp\prime}(t_0+0)))$$

$$+\sum_{k,l=1}^{m_2} f_k(b)f_l(b)g_2(e_k^\perp(b), e_l^{\perp\prime}(b))$$

$$+\sum_{k,l=1}^{n+1} f_k(t_0)f_l(t_0)(g_1(S^1_{nor\gamma'(t_0-0)}(P_1^{\gamma'(t_0-0)}(e_k)), P_1^{\gamma'(t_0-0)}(e_l))$$

$$\left.-g_2(S^2_{nor\gamma'(t_0+0)}(P_2^{\gamma'(t_0+0)}(e_k)), P_2^{\gamma'(t_0+0)}(e_l)))\right\}$$

$$= \frac{1}{c}\left\{\int_a^{t_0}\left\|\sum_{k=1}^{m_1} f_k'(t)e_k^\perp(t)\right\|_1^2 dt + \int_{t_0}^b \left\|\sum_{k=1}^{m_2} f_k'(t)e_k^\perp(t)\right\|_2^2 dt\right.$$

$$+\sum_{k,l=1}^{m_2} f_k(b)f_l(b)g_2(e_k^\perp(b), e_l^{\perp\prime}(b))$$

$$+\sum_{k,l=1}^{n+1} f_k(t_0)f_l(t_0)(g_1(e_k(t_0-0), e_l'(t_0-0)) - g_2(e_k, e_l'(t_0+0))$$

$$-g_1(e_k^T(t_0-0), e_l^{T\prime}(t_0-0)) + g_2(e_k^T, e_l^{T\prime}(t_0+0))$$

$$+g_1(S^1_{nor\gamma'(t_0-0)}(P_1^{\gamma'(t_0-0)}(e_k(t_0-0))), e_l(t_0-0))$$

$$\left.-g_2(S^2_{nor\gamma'(t_0+0)}(P_2^{\gamma'(t_0+0)}(e_k)), e_l)))\right\}$$

$$= \frac{1}{c}\left\{\int_a^{t_0}\left\|\sum_{k=1}^{m_1} f_k'(t)e_k^\perp(t)\right\|_1^2 dt + \int_{t_0}^b \left\|\sum_{k=1}^{m_2} f_k'(t)e_k^\perp(t)\right\|_2^2 dt\right.$$

$$+\sum_{k,l=1}^{m_2} f_k(b)f_l(b)g_2(e_k^\perp(b), e_l^{\perp\prime}(b))$$

$$+\sum_{k,l=1}^{n+1} f_k(t_0)f_l(t_0)(g_1(S^1_{nor\gamma'(t_0-0)}(P_1^{\gamma'(t_0-0)}(e_k(t_0-0)) + e_k'(t_0-0), e_l(t_0-0))$$

$$-g_2(S^2_{nor\gamma'(t_0+0)}(P_2^{\gamma'(t_0+0)}(e_k)) + e'_k(t_0+0), e_l))\Big\}$$

$$= \frac{1}{c}\left\{\int_a^{t_0} \left\|\sum_{k=1}^{m_1} f'_k(t)e_k^{\perp}(t)\right\|_1^2 dt + \int_{t_0}^{b} \left\|\sum_{k=1}^{m_2} f'_k(t)e_k^{\perp}(t)\right\|_2^2 dt + \sum_{k,l=1}^{m_2} f_k(b)f_l(b)g_2(e_k^{\perp}(b), e_l^{\perp\prime}(b))\right\}.$$

Since J is a B-Jacobi field with $J(a)=0$ and $J(b)=Y(b)$,

$$J(t) = \sum_{k=1}^{m_2} f_k(b)e_k(t).$$

By (4.12), we get

$$I_\gamma^0(J,J) = \frac{1}{c}\sum_{k,l=1}^{m_2} f_k(b)f_l(b)g_2(e_k^{\perp}(b), e_l^{\perp\prime}(b))$$

and

$$\begin{aligned} I_\gamma^0(Y,Y) &= I_\gamma^0(J,J) + \frac{1}{c}\left\{\int_a^{t_0} \left\|\sum_{k=1}^{m_1} f'_k(t)e_k^{\perp}(t)\right\|_1^2 dt + \int_{t_0}^{b} \left\|\sum_{k=1}^{m_2} f'_k(t)e_k^{\perp}(t)\right\|_2^2 dt\right\} \\ &\geq I_\gamma^0(J,J). \end{aligned}$$

If it holds $I_\gamma^0(J,J) = I_\gamma^0(Y,Y)$, then, we have that

$$\sum_{k=1}^{m_\lambda} f'_k(t)e_k^{\perp}(t) = 0 \ (\lambda = 1,2).$$

Hence we get $Y^{\perp} = J^{\perp}$. □

Let γ be a B-geodesic of constant speed $c \neq 0$ with $\gamma(t_0) \in B$. If it holds $a \leq t_1 < t_2 \leq t_0$, we set $T_{\gamma|[t_1,t_2]}\tilde{\Omega} = \{Y \mid$ vector fields along $\gamma|[t_1,t_2]\}$. Then we define the map $\tilde{I}_{\gamma|[t_1,t_2]} : T_{\gamma|[t_1,t_2]}\tilde{\Omega} \times T_{\gamma|[t_1,t_2]}\tilde{\Omega} \to \boldsymbol{R}$ by

$$\tilde{I}_{\gamma|[t_1,t_2]}(Y,W) = \frac{1}{c}\int_{t_1}^{t_2}\{g_1(Y^{\perp\prime}, W^{\perp\prime}) - g_1(R^1(Y,\gamma')\gamma', W)\}dt,$$

for all $Y, W \in T_{\gamma|[t_1,t_2]}\tilde{\Omega}$.

Theorem 4.5. *Let γ be a B-geodesic with $\gamma(t_0) \in B$ and $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$. The following are equivalent :*

(1) *$\gamma(t_1)$ ($t_1 \in (t_0, b]$) is not a B-conjugate point to $\gamma(a)$ and also $\gamma(t_1)$ ($t_1 \in (a, t_0]$) is not a conjugate point to $\gamma(a)$.*

(2) *$I_\gamma^{\perp}$ is positive definite.*

Proof. We assume that (1) holds. By Theorem 4.4, for $Y \in T_\gamma^{\perp}\Omega(\gamma(a), \gamma(b))$, there exists a unique B-Jacobi field J with $J(b)=0$ such that

$$I_\gamma(Y,Y) \geq I_\gamma(J,J).$$

Since $\gamma(b)$ is not a B-conjugate points to $\gamma(a)$, J is a trivial B-Jacobi field. Therefore we have that $I_\gamma(Y,Y) \geq 0$, and the equality holds if and only if $Y = Y^\perp = J^\perp = 0$. This shows that $I_\gamma^\perp$ is positive definite.

Conversely, if there exists $t_1 \in (t_0, b]$ such that $\gamma(t_1)$ is a B-conjugate point to $\gamma(a)$. Then there is a nontrivial B-Jacobi field J with $J(t_1) = 0$. We define $Y \in T_\gamma^\perp\Omega(\gamma(a), \gamma(b))$ by

$$Y(t) = \begin{cases} J(t) & \text{on } [a, t_1] \\ 0 & \text{on } [t_1, b] \end{cases}.$$

By Corollary 2.7, we have that

$$I_\gamma(Y,Y) = 0$$

and $I_\gamma^\perp$ is not positive definite.

Furthermore, if there exists $t_1 \in (a, t_0]$ such that $\gamma(t_1)$ is a conjugate point to $\gamma(a)$. Then there is a nontrivial Jacobi field J along $\gamma|[a, t_1]$ with $J(a) = 0$ and $J(t_1) = 0$. We define $Y \in T_\gamma^\perp\Omega(\gamma(a), \gamma(b))$ by

$$Y(t) = \begin{cases} J(t) & \text{on } [a, t_1] \\ 0 & \text{on } [t_1, b] \end{cases}.$$

By Corollary 2.5, we have that

$$I_\gamma(Y,Y) = \tilde{I}_{\gamma|[a,t_1]}(J, J) = 0$$

and $I_\gamma^\perp$ is not positive definite. □

Corollary 4.6. *Let γ be a B-geodesic such that $\gamma(t_0) \in B$ and $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$ such that it satisfies* (1) *of Theorem 4.5. Let $\varphi : [a, b] \times (-\varepsilon, \varepsilon) \to M$ be a fixed endpoint variation of γ in $\tilde{\Omega}$. Then there exists a positive number ε' $(0 < \varepsilon' < \varepsilon)$ such that, for any $s \in (-\varepsilon', \varepsilon)$,*

$$L(s) \geq L(0).$$

Furthermore, if $\varphi_s([a, b]) \neq \gamma([a, b])$, then it holds that

$$L(s) > L(0).$$

Proof. Let $Y(\neq 0)$ be a variation vector field of φ. By Theorem 4.5, we have that

$$I_\gamma^\perp(Y, Y) > 0. \qquad \square$$

Let γ be a B-geodesic of constant speed $c \neq 0$ with $\gamma(t_0) \in B$. If it holds $t_0 < t_1 < t_2 \leq b$, we set $T_{\gamma|[t_1,t_2]}\tilde{\Omega} = \{Y \mid \text{vector fields along } \gamma|[t_1, t_2]\}$. Then we define the map $\tilde{I}_{\gamma|[t_1,t_2]} : T_{\gamma|[t_1,t_2]}\tilde{\Omega} \times T_{\gamma|[t_1,t_2]}\tilde{\Omega} \to \boldsymbol{R}$ by

$$\tilde{I}_{\gamma|[t_1,t_2]}(Y, W) = \frac{1}{c}\int_{t_1}^{t_2} \{g_2(Y^{\perp\prime}, W^{\perp\prime}) - g_2(R^2(Y, \gamma')\gamma', W)\}dt,$$

for all $Y, W \in T_{\gamma|[t_1,t_2]}\tilde{\Omega}$.

Theorem 4.7. *Let γ be a B-geodesic such that $\gamma(t_0) \in B$, $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$ and $\gamma(t)$ $(t \in (a, t_0])$ is not a conjugate point to $\gamma(a)$. If there exists $Y \in T_\gamma^\perp\Omega(\gamma(a), \gamma(b))$ such that $I_\gamma(Y, Y) < 0$, then there exists $t_1 \in (t_0, b)$ such that $\gamma(t_1)$ is a B-conjugate point to $\gamma(a)$.*

Proof. We take any $Y \in T_\gamma^\perp\Omega(\gamma(a), \gamma(b))$. If $\gamma(t_1)$ is not a B-conjugate point to $\gamma(a)$ for any $t_1 \in (t_0, b)$, then, by Theorem 4.4, there exists a nontrivial B-Jacobi field J_{t_1} along $\gamma|[a, t_1]$ such that

$$I_{\gamma|[a,t_1]}{}^0(Y, Y) \geq I_{\gamma|[a,t_1]}{}^0(J_{t_1}, J_{t_1}),$$

$J_{t_1}(a) = 0$ and $Y(t_1) = J_{t_1}(t_1)$. It is obtained that $J_{t_1} \perp \gamma'$. Since $I_{\gamma|[a,t]}{}^0$ $(t \in (t_0, b])$ is continuous on t, $J := \lim_{t\to b} J_t$ is a B-Jacobi field such that $J(b) = \lim_{t\to b} J_t(t) = Y(b) = 0$ and

$$0 = I_\gamma(J, J) \leq I_\gamma(Y, Y). \qquad \square$$

Theorem 4.8. *Let γ be a B-geodesic such that $\gamma(t_0) \in B$, $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$ and $\gamma(t_0)$ is not a conjugate point to $\gamma(a)$. If there exists $t_1 \in (a, b)$ such that $\gamma(t_1)$ is a B-conjugate point to $\gamma(a)$, then there exists $Y \in T_\gamma^\perp\Omega(\gamma(a), \gamma(b))$ such that $I_\gamma(Y, Y) < 0$.*

Proof. If $\gamma(t_1)$ is a B-conjugate point to $\gamma(a)$, then there exists a nontrivial B-Jacobi field J along $\gamma|[a, t_1]$ with $J(a) = 0$ and $J(t_1) = 0$.

In the case of $t_0 < t_1 < b$, we have that $J'(t_1) \neq 0$ by the assumption. And there is a convex neighborhood $U \ni \gamma(t_1)$ such that, for some $\varepsilon > 0$, $\gamma(t_1 - \varepsilon)$ and $\gamma(t_1 + \varepsilon)$ are contained in U and, $\gamma(t_1 + \varepsilon)$ is not conjugate point to $\gamma(t_1 - \varepsilon)$. Then there exist a Jacobi field Z along $\gamma|[t_1 - \varepsilon, t_1 + \varepsilon]$ with $Z(t_1 - \varepsilon) = J(t_1 - \varepsilon)$ and $Z(t_1 + \varepsilon) = 0$. We define $Y \in T_\gamma^\perp\Omega(\gamma(a), \gamma(b))$ by,

$$Y(t) = \begin{cases} J(t) & \text{on } [a, t_1 - \varepsilon] \\ Z(t) & \text{on } [t_1 - \varepsilon, t_1 + \varepsilon] \\ 0 & \text{on } [t_1 + \varepsilon, b] \end{cases}.$$

By Corollary 2.7, we get that

$$0 = I_{\gamma|[a,t_1]}(J, J) = I_{\gamma|[a,t_1-\varepsilon]}{}^0(J, J) + \tilde{I}_{\gamma|[t_1-\varepsilon,t_1]}(J, J).$$

Hence we obtain that

$$\begin{aligned} I_\gamma(Y, Y) &= I_{\gamma|[a,t_1-\varepsilon]}{}^0(Y, Y) + \tilde{I}_{\gamma|[t_1-\varepsilon,t_1+\varepsilon]}(Y, Y) \\ &= I_{\gamma|[a,t_1-\varepsilon]}{}^0(J, J) + \tilde{I}_{\gamma|[t_1-\varepsilon,t_1+\varepsilon]}(Z, Z) \\ &= -\tilde{I}_{\gamma|[t_1-\varepsilon,t_1]}(J, J) + \tilde{I}_{\gamma|[t_1-\varepsilon,t_1+\varepsilon]}(Z, Z). \end{aligned}$$

We set a vector field $\tilde{J}$ along $\gamma|[t_1-\varepsilon, t_1+\varepsilon]$ by

$$\tilde{J}(t) = \begin{cases} J(t) & \text{on } [t_1-\varepsilon, t_1] \\ 0 & \text{on } [t_1, t_1+\varepsilon] \end{cases}.$$

Then we have that

$$I_\gamma(Y,Y) = \tilde{I}_{\gamma|[t_1-\varepsilon,t_1+\varepsilon]}(Z,Z) - \tilde{I}_{\gamma|[t_1-\varepsilon,t_1+\varepsilon]}(\tilde{J},\tilde{J}) < 0.$$

In the case of $a < t_1 < t_0$, we can get the claim in the similar way. □

Corollary 4.9. *Let γ be a B-geodesic such that $\gamma(t_0) \in B$, $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$ and $\gamma(t_0)$ is not a conjugate point to $\gamma(a)$. If there exists $t_1 \in (a,b)$ such that $\gamma(t_1)$ is a B-conjugate point to $\gamma(a)$, then there exist a positive number ε and a fixed endpoints variation of γ in $\tilde{\Omega}$ such that $L(s) < L(0)$ for any $s \in (-\varepsilon, \varepsilon) - \{0\}$.*

Proof. If $\gamma(t_1)$ is the first B-conjugate point to $\gamma(a)$, then, by Theorem 4.8, there exists $Y \in T_\gamma^\perp\Omega(\gamma(a),\gamma(b))$ such that

$$I_\gamma(Y,Y) < 0. \qquad \square$$

Remark. In Theorem 4.8, the condition that $\gamma(t_0)$ is not a conjugate points to $\gamma(a)$ is necessary. In fact, in Example 3 and 4, for any $Y \in T_\gamma^\perp\Omega(\gamma(0),\gamma(2\pi))$, it holds that $I_\gamma(Y,Y) \geq 0$.

5. The passage equation

We show the relation between $S_{u,v} := \dfrac{1}{\|\mathrm{nor}v\|_2}(S^1_{\mathrm{nor}u} - S^2_{\mathrm{nor}v})$ and $A_{u,v}$, where $q \in B$, $u \in T_qM_1$ and $v \in T_qM_2$ with $\|u\|_1 = \|v\|_2$, $\tan u = \tan v$ and $v \notin T_qB$. We note that $S_{ku,kv} = S_{u,v}$ for $k \neq 0$. Let λ_H and $\tilde{\lambda}_H$ denote the maximal eigenvalue and the maximal absolute eigenvalue of a symmetric linear transformation H. That is

$$\lambda_H = \max\{\lambda_k \mid \text{eigenvalues of } H\},$$

and

$$\tilde{\lambda}_H = \max\{\lambda_H, \lambda_{-H}\}.$$

Lemma 5.1. *Let $q \in B$, $u \in T_qM_1$ and $v \in T_qM_2$ with $\|u\|_1 = \|v\|_2$, $\tan u = \tan v$ and $v \notin T_qB$. Then we get*

$$\frac{\|v\|_2}{\|\mathrm{nor}v\|_2}\tilde{\lambda}_{S_{u,v}} \geq \tilde{\lambda}_{A_{u,v}},$$

where $S_{u,v} := \frac{1}{\|v\|_2}(S^1_{noru} - S^2_{norv})$.

Proof. We get

$$P_2^v(w) = w - \frac{g_2(w, \mathrm{nor}v)}{g_2(v, \mathrm{nor}v)}v$$

for any $w \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}v\}$, and, as in the proof of Lemma 2.8,

$$\begin{aligned}|g_2(A_{u,v}(w), w)| &= \frac{1}{\|v\|_2}|g_2((S^1_{noru} - S^2_{norv})(P_2^v(w)), P_2^v(w))| \\ &= \frac{\|\mathrm{nor}v\|_2}{\|v\|_2}|g_2(S_{u,v}(P_2^v(w)), P_2^v(w))| \\ &\leq \frac{\|v\|_2}{\|\mathrm{nor}v\|_2}\|w\|_2^2|g_2(S_{u,v}(\frac{P_2^v(w)}{\|P_2^v(w)\|_2}), \frac{P_2^v(w)}{\|P_2^v(w)\|_2})|\end{aligned}$$

since

$$\|P_2^v(w)\|_2 \leq \frac{\|v\|_2}{\|\mathrm{nor}v\|_2}\|w\|_2.$$

This proves the inequality. □

Lemma 5.2. *Let* $q \in B$, $u \in T_qM_1$ *and* $v \in T_qM_2$ *with* $\|u\|_1 = \|v\|_2$, $\tan u = \tan v$ *and* $v \notin T_qB$. *Then the following are true.*

(1) $S_{u,v} = 0$ *if and only if* $A_{u,v} = 0$.
(2) $S_{u,v} \leq 0$ *if and only if* $A_{u,v} \leq 0$.
(3) $S_{u,v} \geq 0$ *if and only if* $A_{u,v} \geq 0$.

Proof. Let $w_1, w_2 \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}v\}$. We have that

$$g_2(A_{u,v}(w_1), w_2) = \frac{\|\mathrm{nor}v\|_2}{\|v\|_2}g_2(S_{u,v}(P_2^v(w_1)), P_2^v(w_2)).$$

Since P_2^v is surjective, the statements are clear. □

Proposition 5.3. *Let* $q \in B$, $u \in T_qM_1$ *and* $v \in T_qM_2$ *with* $\|u\|_1 = \|v\|_2$, $\tan u = \tan v$ *and* $v \notin T_qB$. *Then the following are true.*

(1) *If* $S_{u,v} \leq 0$, *then* $\mathrm{tr}A_{u,v} \geq \frac{\|v\|_2}{\|\mathrm{nor}v\|_2}\mathrm{tr}S_{u,v}$ *and* $\frac{\|v\|_2}{\|\mathrm{nor}v\|_2}\lambda_{S_{u,v}} \leq \lambda_{A_{u,v}}$.

(2) *If* $S_{u,v} \geq 0$, *then* $\mathrm{tr}A_{u,v} \leq \frac{\|v\|_2}{\|\mathrm{nor}v\|_2}\mathrm{tr}S_{u,v}$ *and* $\frac{\|v\|_2}{\|\mathrm{nor}v\|_2}\lambda_{S_{u,v}} \geq \lambda_{A_{u,v}}$.

(3) *If* $S_{u,v} = \lambda I$, *then* $\mathrm{tr}A_{u,v} \geq \frac{\|v\|_2}{\|\mathrm{nor}v\|_2}\lambda\left\{1 + (n-1)\frac{\|\mathrm{nor}v\|_2^2}{\|v\|_2^2}\right\}$.

Where n *is the dimension of* B.

Proof. In order to prove this lemma, we extend $S_{u,v}$ linearly on $T_qB \oplus \mathrm{Span}\{\mathrm{nor}v\}$ by putting $S_{u,v}(\mathrm{nor}v) = 0$. The trace of $S_{u,v}$ do not change. Take an orthonormal basis $\{e_k\}$

such that $e_1, \cdots e_n \in T_qB$ are eigenvectors of S with eigenvectors $\lambda_1, \cdots, \lambda_n$, respectively, and $e_{n+1} = \mathrm{nor}\dot{v}/\|\mathrm{nor}v\|_2$. Then we get

$$\begin{aligned}\mathrm{tr}A_{u,v} &= \sum_{k=1}^{n+1} g_2(A_{u,v}(e_k), e_k) = \frac{\|\mathrm{nor}v\|_2}{\|v\|_2}\sum_{k=1}^{n+1} g_2(S_{u,v}(P_2^v(e_k)), P_2^v(e_k)) \\ &= \frac{\|\mathrm{nor}v\|_2}{\|v\|_2}\left\{\sum_{k=1}^{n} g_2(S_{u,v}(e_k), e_k) + \frac{1}{g_2(v, \mathrm{nor}v)} g_2(S_{u,v}(v), v)\right\} \\ &= \frac{\|\mathrm{nor}v\|_2}{\|v\|_2}\left\{\sum_{k=1}^{n} \lambda_k + \frac{1}{\|\mathrm{nor}v\|_2^2} g_2(S_{u,v}(v), v)\right\}.\end{aligned}$$

Since $S_{u,v}(v) = \sum_{k=1}^{n} \lambda_k g_2(v, e_k)e_k$, we have that

$$\mathrm{tr}A_{u,v} = \frac{\|v\|_2}{\|\mathrm{nor}v\|_2}\sum_{k=1}^{n} \lambda_k \left\{\frac{\|\mathrm{nor}v\|_2^2}{\|v\|_2^2} + g_2(\frac{v}{\|v\|_2}, e_k)^2\right\}.$$

Since $\|\mathrm{nor}v\|_2^2 + g_2(v, e_k)^2 \leq \|v\|_2^2$, we see that

$$\frac{\|\mathrm{nor}v\|_2^2}{\|v\|_2^2} + g_2(\frac{v}{\|v\|_2}, e_k)^2 \leq 1$$

for each k. Hence we have that

$$\mathrm{tr}A_{u,v} \geq \frac{\|v\|_2}{\|\mathrm{nor}v\|_2}\mathrm{tr}S_{u,v} \qquad \text{if } S_{u,v} \leq 0,$$

$$\mathrm{tr}A_{u,v} \leq \frac{\|v\|_2}{\|\mathrm{nor}v\|_2}\mathrm{tr}S_{u,v} \qquad \text{if } S_{u,v} \geq 0,$$

and

$$\mathrm{tr}A_{u,v} = \frac{\lambda\|v\|_2}{\|\mathrm{nor}v\|_2}\left\{1 + (n-1)\frac{\|\mathrm{nor}v\|_2^2}{\|v\|_2^2}\right\} \qquad \text{if } S_{u,v} = \lambda I.$$

This completes the proof of the statement (3) and the first parts of (1) and (2). Others are clear from the proof of Lemma 5.1 and the statement of Lemma 5.2. □

Let γ be a B-geodesic in $\tilde{\Omega}$ with $\gamma(t_0) \in B$ and $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$. We put $\dim M_\lambda = m_\lambda$ ($\lambda = 1, 2$) and $\dim B = n$. Let $e_1^- := \gamma'(t_0-0)/\|\gamma'(t_0-0)\|_1, e_2^-, \cdots, e_{m_1}^-$ be an orthonormal basis of $T_{\gamma(t_0)}M_1$ such that $e_1^-, \cdots, e_{n+1}^-$ is an orthonormal basis of $T_{\gamma(t_0)}B \oplus \mathrm{Span}\{\mathrm{nor}\gamma'(t_0-0)\}$. We take $e_k^+ := Q_{\gamma'(t_0-0),\gamma'(t_0+0)}(e_k^-)$ ($k = 1, \cdots, n+1$) and $e_{n+2}^+, \cdots, e_{m_2}^+ \in T_{\gamma(t_0)}M_2$ such that $e_1^+, \cdots, e_{m_2}^+$ is an orthonormal basis of $T_{\gamma(t_0)}M_2$. Let $e_1(t), \cdots, e_{n+1}(t)$ be parallel vector fields along γ such that $e_k(t_0 \pm 0) = e_k^\pm$ ($k = 1, \cdots, n+1$). Furthermore let $e_{n+2}^-(t), \cdots, e_{m_1}^-(t)$ be parallel vector fields along $\gamma|[a, t_0]$ such that $e_{l_1}^-(t_0-0) = e_{l_1}^-$ ($l_1 = n+2, \cdots, m_1$) and $e_{n+2}^+(t), \cdots, e_{m_2}^+(t)$ parallel vector fields along $\gamma|[t_0, b]$ such that $e_{l_2}^-(t_0-0) = e_{l_2}^-$ ($l_2 = n+2, \cdots, m_2$). For simplicity, we set $e_l(t) := e_l^\pm(t)$ ($l = n+2, \cdots, m_\lambda$).

If $Y(t) = \sum_{k=1}^{m_\lambda} Y_k(t)e_k(t)$ is a B-Jacobi field, then we have $Y^{st}(t) = \sum_{k=1}^{n+1} Y_k(t)e_k(t)$. Let $R^{st}(t)$ be the matrix representation of $R^\lambda(\cdot, \gamma'(t))\gamma'(t)$ with respect to the basis $\{e_2(t), \cdots, e_{n+1}(t)\}$, A the one of $A_{\gamma'(t_0-0),\gamma'(t_0+0)}$ with respect to the basis $\{e_2(t), \cdots, e_{n+1}(t)\}$ and $\hat{Y}^{st}(t) ={}^t (Y_2(t), \cdots, Y_{n+2}(t))$, where ^{t} is a transposition. Then we have that

(5.1) $$(\hat{Y}^{st})''(t) + R^{st}(t)\hat{Y}^{st}(t) = 0 \text{ for } t \neq t_0.$$

(5.2) $$\hat{Y}^{st}(t) \text{ is continuous,}$$

(5.3) $$(\hat{Y}^{st})'(t_0 - 0) - (\hat{Y}^{st})'(t_0 + 0) = -\|\gamma'(t_0 + 0)\|_2 A\hat{Y}^{st}(t_0).$$

We put $\hat{Y}_{M_1}(t) ={}^t (Y_{n+2}(t), \cdots, Y_{m_1}(t))$ on $[a, t_0]$ and $\hat{Y}_{M_2}(t) ={}^t (Y_{n+2}(t), \cdots, Y_{m_2}(t))$ on $[t_0, b]$. Let $R_{M_\lambda}(t)$ be the matrix representation of $R^\lambda(\cdot, \gamma'(t))\gamma'(t)$ with respect to the basis $\{e_{n+2}(t), \cdots, e_{m_\lambda}(t)\}$. Then we have that

(5.4) $$\hat{Y}''_{M_\lambda}(t) + R_{M_\lambda}(t)\hat{Y}_{M_\lambda}(t) = 0,$$

(5.5) $$\hat{Y}_{M_1}(t_0 - 0) = 0 \text{ and } \hat{Y}_{M_2}(t_0 + 0) = 0.$$

In particular, if $\gamma(t_0)$ is not conjugate point to $\gamma(a)$ and $\hat{Y}_{M_1}(a) = 0$, then we have that

(5.6) $$\hat{Y}_{M_1}(t) = 0 \text{ on } [a, t_0].$$

We define the function $\rho_K : [a, b] \to \boldsymbol{R}$ and $f_K : [a, b] \to \boldsymbol{R}$ by

$$\rho_K(t) = \begin{cases} ct & \text{if } K = 0 \\ \dfrac{1}{\sqrt{K}} \tan c\sqrt{K}t & \text{if } K > 0 \\ \dfrac{1}{\sqrt{-K}} \text{tanh} c\sqrt{-K}t & \text{if } K < 0 \end{cases}$$

and

$$f_K(t) = \begin{cases} t & \text{if } K = 0 \\ \dfrac{1}{c\sqrt{K}} \sin c\sqrt{K}t & \text{if } K > 0 \\ \dfrac{1}{c\sqrt{-K}} \text{sinh} c\sqrt{-K}t & \text{if } K < 0 \end{cases},$$

respectively. We prove the passage equation.

Proposition 5.4 (The passage equation). *Let M_1 and M_2 be Riemannian manifolds of constant curvature K_1 and K_2, respectively. Let $\gamma : [a,b] \to M$ be a B-geodesic with $\gamma(t_0) \in B$, $\|\gamma'(t_0+0)\|_2 = c$ and $\gamma'(t_0+0) \notin T_{\gamma'(t_0)}B$. If $\gamma(b)$ is the first B-conjugate point to $\gamma(a)$, then we get that $f_{K_1}(t-a) > 0$ for $t \in (a,t_0]$ and*

$$\lambda_{-A} = \frac{1}{\rho_{K_1}(t_0-a)} + \frac{1}{\rho_{K_2}(b-t_0)} \qquad \textit{when } f_{K_2}(b-t_0) > 0.$$

Proof. The matrix strong B-Jacobi field D_B along γ with $D^{st}(a) = 0$ and $(D^{st})'(a) = I$ is written

$$D^{st}(t) = f_{K_1}(t-a)I \tag{5.7}$$

for $t \in [a, t_0]$, where I is the identity map. By the assumption, we have that

$$f_{K_1}(t-a) > 0 \text{ for } t \in (a, t_0].$$

From (5.7), we get

$$D^{st}(t) = \{f'_{K_1}(t_0-a)f_{K_2}(t-t_0) + f_{K_1}(t_0-a)f'_{K_2}(t-t_0)\}I + cf_{K_1}(t_0-a)f_{K_2}(t-t_0)A \tag{5.8}$$

for $t \in [t_0, b]$.

If there is a matrix (M_1, M_2)-Jacobi field D_{M_2} along $\gamma|[t_0, b]$, then D_{M_2} with $D_{M_2}(t_0) = 0$ and $D'_{M_2}(t_0) = I$ is written

$$D_{M_2}(t) = f_{K_2}(t-t_0)I$$

for $t \in [t_0, b]$. By the assumption, we have that

$$f_{K_2}(t-t_0) > 0 \text{ for } t \in (t_0, b).$$

Then D^{st} is symmetric from (5.7) and (5.8), $\det D^{st}(b) = 0$ and $D^{st}(b) = 0$ since $\gamma(b)$ is the first B-conjugate point to $\gamma(a)$. We see that

$$-A \leq \left\{ \frac{f'_{K_1}(t_0-a)}{cf_{K_1}(t_0-a)} + \frac{f'_{K_2}(b-t_0)}{cf_{K_2}(b-t_0)} \right\} I$$

and $\lambda_{-A} = \dfrac{1}{\rho_{K_1}(t_0-a)} + \dfrac{1}{\rho_{K_2}(b-t_0)}$ if $f_{K_2}(b-t_0) > 0$. □

Remark. If $K_1 \leq 0$, then it holds that $f_{K_1}(t-a) > 0$ for $t \in (a, t_0]$. In the case of $K_1 > 0$, we have that $f_{K_1}(t-a) > 0$ for $t \in (a, t_0]$ if $c\sqrt{K_1}(t_0-a) < \pi$.

If $K_2 \leq 0$, then it holds that $f_{K_2}(b-t_0) > 0$. In the case of $K_2 > 0$, we have that $f_{K_2}(b-t_0) > 0$ if $c\sqrt{K_2}(b-t_0) < \pi$.

REFERENCES

[1] N.Abe and M.Takiguchi, Geodesics reflecting on a pseudo-Riemannian submanifold, SUT J. Math., **34**, No.2, (1998), 139-168.

[2] T.Hasegawa, The index theorem of geodesics on a Riemannian manifold with boundary, Kodai Math. J., **1** (1978), 285-288.

[3] T.Hasegawa, On the position of a conjugate point of a reflected geodesic in E^2 and E^3, Yokohama Math. J., **32** (1984), 233-237.

[4] N.Innami, Integral formulas for polyhedral and spherical billiards, J. Math. Soc. Japan, **50**, No.2, (1998), 339-357.

[5] N.Innami, Jacobi vector fields along geodesics in glued Riemannian manifolds. (preprint)

[6] B.O'Neill, *Semi-Riemannian Geometry with Application to Relativity*, Academic Press (1983).

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