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L^2 THEORY FOR THE OPERATOR $\Delta+(k\times x)\cdot\nabla$ IN EXTERIOR DOMAINS

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ABSTRACT. In exterior domains of $\mathbb{R}^3$, we consider the differential operator $\Delta+(k\times x)\cdot\nabla$ with Dirichlet boundary condition, where k stands for the angular velocity of a rotating obstacle. We show, among others, a certain smoothing property together with estimates near $t=0$ of the generated semigroup (it is not an analytic one) in the space L^2. The result is not trivial because the coefficient $k\times x$ is unbounded at infinity. The proof is mainly based on a cut-off technique. The equation $\partial_t u=\Delta u+(k\times x)\cdot\nabla u$ can be taken as a model problem for a linearized form of the Navier-Stokes equations in a domain exterior to a rotating obstacle. This paper is a step toward an analysis of the Navier-Stokes flow in such a domain.
Key words and phrases: differential operators with unbounded coefficients, exterior domains, semigroups, smoothing effects.

1. Introduction and statement of main results

Let $\mathcal{O}\subset\mathbb{R}^3$ be a compact obstacle which is bounded by a smooth surface Γ. In the exterior domain $\Omega=\mathbb{R}^3\setminus\mathcal{O}$ we consider the initial boundary value problem

$$(1.1)\qquad \begin{cases} \partial_t u=\Delta u+(k\times x)\cdot\nabla u, & x\in\Omega,\ t>0,\\ u(x,t)=0, & x\in\Gamma,\ t>0,\\ u(x,t)\to 0, & |x|\to\infty,\ t>0,\\ u(x,0)=a(x), & x\in\Omega, \end{cases}$$

where $k=(0,0,1)^T$, so that $k\times x=(-x_2,x_1,0)^T$. The aim of the present paper is to establish some fundamental properties for the differential operator $\Delta+(k\times x)\cdot\nabla$ in exterior domains. It is proved that the operator with homogeneous Dirichlet boundary condition generates a semigroup having a certain smoothing property and enjoys an elliptic regularity estimate in the space L^2.

Let us explain the motivation of this study. Assume that the exterior domain Ω is occupied by a viscous incompressible fluid and that the obstacle $\mathcal{O}$ is rotating about the x_3-axis with angular velocity k. We then consider the fluid motion governed by the Navier-Stokes equation in the domain $\Omega(t)=\{O(t)x;\ x\in\Omega\}$, where

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$$O(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Unless the obstacle $\mathcal{O}$ is axisymmetric, the domain $\Omega(t)$ actually varies with time t. In general it is not so easy to treat directly initial boundary value problems in exterior domains with moving boundaries. It is reasonable to reduce our problem to an equivalent one in the fixed domain Ω by using the coordinate system attached to the rotating obstacle together with an appropriate transformation of unknown functions. Borchers [3] has first constructed the Navier-Stokes flow as a weak solution to the reduced problem. Chen and Miyakawa [4] have also discussed the existence and some decay properties of weak solutions to the related Cauchy problem. One of the important problems is now to find a unique strong solution. To this end, we have to carry out the analysis of a linearized form of the reduced Navier-Stokes equation, which is given by (see [3, 4, 8])

$$(1.2) \qquad \begin{cases} \partial_t u = \Delta u + (k \times x) \cdot \nabla u - k \times u - \nabla p, & x \in \Omega,\ t > 0, \\ \nabla \cdot u = 0, & x \in \Omega,\ t \geq 0, \\ u(x,t) = 0, & x \in \Gamma,\ t > 0, \\ u(x,t) \to 0, & |x| \to \infty,\ t > 0, \\ u(x,0) = a(x), & x \in \Omega, \end{cases}$$

where u and p denote, respectively, unknown velocity vector field and pressure of the fluid. The coefficient $k \times x$ of the convection term is understood as the rigid motion rotating about the x_3-axis. Since this unbounded coefficient is a significant feature of the problem (1.2), we consider (1.1) as a model problem for (1.2).

For the study of semilinear problems associated with (1.1) (we keep the Navier-Stokes equations in mind), the following properties for the operator $\Delta + (k \times x) \cdot \nabla$ play crucial roles: the generation of a semigroup, its smoothing property, an elliptic regularity estimate and an embedding property for the domains of fractional powers. In this paper we discuss them in the framework of L^2 theory. As already mentioned, the essential difficulty is the growth at infinity of the coefficient of $(k \times x) \cdot \nabla$, which cannot be treated as a perturbation of the Laplace operator. It is also impossible to apply the technique of Agmon [1] (see also Tanabe [15, Chapter 3]) on the generation of analytic semigroups. In fact, as shown in Proposition 3.8, the related semigroup for the Cauchy problem in $\mathbb{R}^3$ is never analytic on $L^2(\mathbb{R}^3)$; besides, it is not a differentiable semigroup in the sense of Pazy [12, Chapter 2]. This tells us that the operator $(k \times x) \cdot \nabla$ does not conserve some properties of the Laplace operator. Nevertheless, as clarified in this paper, the generated semigroup possesses a certain smoothing property in the space L^2.

Concerning differential operators with unbounded coefficients in another function space over $\mathbb{R}^N$, we find the work of DaPrato and Lunardi [5]. They have studied the Ornstein-Uhlenbeck operator $\Lambda f = \mathrm{Tr}\,[QD^2 f] + (Bx) \cdot \nabla f$ in $\mathbb{R}^N$, where Q is a symmetric positive definite matrix and B is a nonzero matrix. Such a class of operators covers $\Delta + (k \times x) \cdot \nabla$. For the generated semigroup $S(t)$, they have shown the following result: $S(t)$ is not an analytic semigroup on $UCB(\mathbb{R}^N)$, the space of the uniformly continuous and bounded functions in $\mathbb{R}^N$, but it enjoys the remarkable smoothing properties $D^j S(t)f \in UCB(\mathbb{R}^N)$ for all $f \in UCB(\mathbb{R}^N), j \in \mathbb{N}$ and $t > 0$ with $\sup_{x\in\mathbb{R}^N} |D^j[S(t)f](x)| = O\left(t^{-j/2}\right)$ as $t \to 0$ for $j = 1, 2$. They have also derived Schauder type estimates for the operator Λ.

Although their results are interesting, it is difficult to apply directly them to the Navier-Stokes equations. So, we here independently develop the L^2 theory for the operator $\Delta + (k\times x)\cdot\nabla$ in exterior domains as well as in $\mathbb{R}^3$ because, as is well known, L^2 is the standard function space in the Navier-Stokes theory.

We define the linear operator $\mathcal{L}_0 : L^2(\Omega) \to L^2(\Omega)$ by

$$\begin{cases} D(\mathcal{L}_0) = \left\{u \in H^2(\Omega)\cap H^1_0(\Omega); (k\times x)\cdot\nabla u \in L^2(\Omega)\right\}, \\ \mathcal{L}_0 = -\Delta - (k\times x)\cdot\nabla. \end{cases} \tag{1.3}$$

Here, for integer $m \geq 0$, $H^m(\Omega) = W^{m,2}(\Omega)$, and $W^{m,p}(\Omega)$ is the usual L^p Sobolev space. By $H^m_0(\Omega)$ we denote the completion of $C^\infty_0(\Omega)$, the class of smooth functions with compact supports, in $H^m(\Omega)$. Since the operator $\mathcal{L}_0$ has an unbounded coefficient, elliptic regularity estimates as given in Agmon, Douglis and Nirenberg [2] are not clear. To derive such an a priori estimate in $L^2(\Omega)$ is also our task (Theorem 2). Therefore, at present, it is not so easy to show the closedness of $\mathcal{L}_0$ directly. But it is verified that $\mathcal{L}_0$ is a closable operator with dense domain in $L^2(\Omega)$ (section 2). We thus define the operator $\mathcal{L} : L^2(\Omega) \to L^2(\Omega)$ by

$$\mathcal{L} = \overline{\mathcal{L}_0} \quad \text{(the closure of } \mathcal{L}_0). \tag{1.4}$$

We start with the result on the generation of a semigroup.

Theorem 1. *The operator $\mathcal{L}$ is m-accretive, so that $-\mathcal{L}$ generates a (C_0) semigroup $\left\{e^{-t\mathcal{L}}\right\}_{t\geq 0}$ of contractions on $L^2(\Omega)$.*

We next give an L^2 a priori estimate for the operator $\mathcal{L}$ and clarify the domain $D(\mathcal{L})$.

Theorem 2. *For each $u \in D(\mathcal{L})$, we have*

$$u \in H^2(\Omega)\cap H^1_0(\Omega), \qquad (k\times x)\cdot\nabla u \in L^2(\Omega).$$

There is a constant $C > 0$ such that

$$\|u\|_{H^2(\Omega)} + \|(k\times x)\cdot\nabla u\|_{L^2(\Omega)} \leq C\|(1+\mathcal{L})u\|_{L^2(\Omega)}, \tag{1.5}$$

for all $u \in D(\mathcal{L})$. As a result, $D(\mathcal{L}) = D(\mathcal{L}_0)$ and, therefore, $\mathcal{L} = \mathcal{L}_0$.

By Theorem 1 fractional powers $\mathcal{L}^\alpha, 0 < \alpha < 1$, of $\mathcal{L}$ are well defined as closed operators in $L^2(\Omega)$. An embedding property of Sobolev type for the domains of $\mathcal{L}^\alpha$ is given by the following theorem.

Theorem 3. *Let $m = 0, 1$. Assume that $2 < p < \infty$ if $m = 0$, and that $2 \leq p \leq 6$ if $m = 1$. Put $\alpha = 3(1/2 - 1/p)/2 + m/2$. Then there is a constant $C = C(m, p) > 0$ such that $D(\mathcal{L}^\alpha) \subset W^{m,p}(\Omega)$ with estimate*

$$\|u\|_{W^{m,p}(\Omega)} \leq C\|(1 + \mathcal{L})^\alpha u\|_{L^2(\Omega)}, \tag{1.6}$$

for all $u \in D(\mathcal{L}^\alpha)$.

We finally present the result on the smoothing effect for the semigroup $\{e^{-t\mathcal{L}}\}_{t\geq 0}$ obtained in Theorem 1. Let A be the operator from $L^2(\Omega)$ into itself defined by

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega), \qquad A = -\Delta, \tag{1.7}$$

which is closed on account of the well known L^2 estimate ([2, 6])

$$\|u\|_{H^2(\Omega)} \leq C\|(1 + A)u\|_{L^2(\Omega)}, \qquad u \in D(A). \tag{1.8}$$

By Theorem 2 we have $D(\mathcal{L}) \subset D(A)$. The operator A is nonnegative selfadjoint so that its fractional powers A^α are well defined. The following theorem asserts that $e^{-t\mathcal{L}}a$ is in $D(A)$ for all $t > 0$ without any regularity assumptions on a, and that $e^{-t\mathcal{L}}a$ is in $D(\mathcal{L})$ for all $t > 0$ under the additional assumption $(k \times x) \cdot \nabla a \in H^{-\infty}(\Omega) \equiv \bigcup_{m\geq 0} H^{-m}(\Omega)$, where $H^{-m}(\Omega)$ is the dual space of $H_0^m(\Omega)$.

Theorem 4. *(i) Suppose that $a \in L^2(\Omega)$. Then $e^{-t\mathcal{L}}a \in D(A)$ for all $t > 0$. Let $0 < \alpha \leq 1$. Then there is a constant $C = C(\alpha) > 0$ such that*

$$\left\|A^\alpha e^{-t\mathcal{L}}a\right\|_{L^2(\Omega)} \leq Ct^{-\alpha}\|a\|_{L^2(\Omega)}, \tag{1.9}$$

for $0 < t < 1$.

(ii) Suppose that $a \in L^2(\Omega)$ and that $(k \times x) \cdot \nabla a \in H^{-m}(\Omega)$ for some integer $m \geq 0$. Then $e^{-t\mathcal{L}}a \in D(\mathcal{L})$ for all $t > 0$, and

$$\mathcal{L}e^{-t\mathcal{L}}a \in C\left((0, \infty); L^2(\Omega)\right), \qquad e^{-t\mathcal{L}}a \in C^1\left((0, \infty); L^2(\Omega)\right), \tag{1.10}$$

with

$$\frac{d}{dt}e^{-t\mathcal{L}}a + \mathcal{L}e^{-t\mathcal{L}}a = 0, \qquad t > 0, \tag{1.11}$$

in $L^2(\Omega)$. Let $0 < \alpha \leq 1$. Then there is a constant $C = C(m, \alpha) > 0$ such that

(1.12)
$$\left\|\mathcal{L}^\alpha e^{-t\mathcal{L}}a\right\|_{L^2(\Omega)} \leq C\left[t^{-\alpha}\|a\|_{L^2(\Omega)} + t^{-m\alpha/2}\left\{\|(k \times x) \cdot \nabla a\|_{H^{-m}(\Omega)} + \|a\|_{L^2(\Omega)}\right\}\right],$$

for $0<t<1$.

(iii) Suppose that $a \in D(\mathcal{L}^\delta)$ for some $0<\delta<1$ and that $(k\times x)\cdot\nabla a \in H^{-m}(\Omega)$ for some integer $m \geq 0$. Let $\delta < \alpha \leq 1$. Then there is a constant $C = C(\delta, m, \alpha) > 0$ such that

(1.13)
$$\left\|\mathcal{L}^\alpha e^{-t\mathcal{L}} a\right\|_{L^2(\Omega)} \leq C\left[t^{-\alpha+\delta}\|a\|_{D(\mathcal{L}^\delta)} + t^{-\frac{m(\alpha-\delta)}{2(1-\delta)}}\left\{\|(k\times x)\cdot\nabla a\|_{H^{-m}(\Omega)} + \|a\|_{D(\mathcal{L}^\delta)}\right\}\right],$$

for $0<t<1$.

(iv) Let $0<\alpha\leq 1$. Then

$$\left\|A^\alpha e^{-t\mathcal{L}} a\right\|_{L^2(\Omega)} = o\left(t^{-\alpha}\right) \qquad \text{as} \quad t \to 0, \tag{1.14}$$

for all $a \in L^2(\Omega)$, and

$$\left\|\mathcal{L}^\alpha e^{-t\mathcal{L}} a\right\|_{L^2(\Omega)} = o\left(t^{-\alpha}\right) \qquad \text{as} \quad t \to 0, \tag{1.15}$$

for all $a \in L^2(\Omega)$ satisfying $(k\times x)\cdot\nabla a \in H^{-1}(\Omega)$.

By the known method in nonhomogeneous problems (Pazy [12, Chapter 4]), the result (ii) of Theorem 4 implies that the function

$$u(t) = \int_0^t e^{-(t-s)\mathcal{L}} F(s)ds,$$

is of the same class as (1.10) and satisfies

$$\frac{du}{dt} + \mathcal{L}u = F(t), \qquad 0<t<T; \quad u(0)=0,$$

in $L^2(\Omega)$ if the forcing term fulfills

$$F \in C^\sigma\left(I; L^2(\Omega)\right) \cap L^1\left(0,T; L^2(\Omega)\right),$$
$$(k\times x)\cdot\nabla F \in C^\sigma\left(I; H^{-2}(\Omega)\right) \cap L^1\left(0,T; H^{-2}(\Omega)\right),$$

with some Hölder exponent $\sigma \in (0,1)$ for every compact interval $I \subset (0,T)$. Estimates (1.9), (1.12) and (1.13) near $t=0$ together with (1.6) are useful in proving the local existence of the strong solution to the semilinear problem associated with (1.1).

To overcome the difficulty caused by the lack of boundedness of the coefficient of $\mathcal{L}$, we employ the method based on a cut-off procedure, which is for instance similar to [9, 14]. In the proof of Theorem 1, besides the accretivity (section 2), we construct the resolvent $(\lambda+\mathcal{L})^{-1}$ for Re $\lambda > 0$ by using resolvents for $\mathbb{R}^3$ and for a bounded domain near the boundary Γ (section 5). It is noted that the surjectivity of $\lambda + \mathcal{L}$ does not follow from the simple consideration of the adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$ since $D(\mathcal{L}^*)$ as well as $D(\mathcal{L})$ is not clear before the proof of Theorem 2. In the proof of Theorems 2 and 3, a key step is that $u \in D(\mathcal{L})$ implies $\Delta u \in L^2(\Omega)$. We prove such a regularity property for a unique solution to the boundary value problem (section 5). In the proof of Theorem 4, we employ the cut-off procedure once again for the initial boundary value problem with the aid of some smoothing properties of solutions in $\mathbb{R}^3$ and in bounded domains (section 7). For the problem in $\mathbb{R}^3$ we make full use of the Fourier transform together with an explicit formula of the solution (section 3), while the operator $\Delta+(k\times x)\cdot\nabla$ in bounded domains generates an analytic semigroup (section 4).

In order to construct a unique local solution of the Navier-Stokes equations in the exterior of a rotating obstacle, the above theorems for the associated operator are necessary. Most part of the method developed here combined with Bogovskiĭ's technique for the recovery of the solenoidal condition in cut-off procedures can be applied to the problem (1.2); this will be discussed elsewhere [7, 8].

The content of this paper is as follows. In section 2 we show the accretivity of $\mathcal{L}$. In sections 3 and 4 we respectively carry out the analysis of the operator $\Delta + (k \times x) \cdot \nabla$ in $\mathbb{R}^3$ and in bounded domains. Section 5 is concerned with the construction of the resolvent $(\lambda + \mathcal{L})^{-1}$ to prove Theorem 1. In section 6 we derive L^2 a priori estimates to prove Theorem 2 together with Theorem 3. The final section is devoted to the investigation of the smoothing effect for the semigroup $e^{-t\mathcal{L}}$ to prove Theorem 4.

Notation. Besides the symbols which have been already introduced, we adopt the following notation. For a domain G in $\mathbb{R}^3$, we denote the norm of $L^2(G)$ by $\|\cdot\|_G = \|\cdot\|_{L^2(G)}$. For simplicity, we use the abbreviation $\|\cdot\|$ for $\|\cdot\|_\Omega$, where $\Omega = \mathbb{R}^3\setminus\mathcal{O}$ is the given exterior domain. The scalar product on $L^2(\Omega)$ and some duality pairings are denoted by $(\cdot,\cdot)$. We set $\mathbb{C}_+ = \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda > 0\}$, that is, the right half complex plane.

2. Accretivity

In this section we prove that the operator $\mathcal{L}$ defined by (1.4) is accretive and has the dense domain in $L^2(\Omega)$. It follows from the relation $D(\mathcal{L}) \supset D(\mathcal{L}_0) \supset C_0^\infty(\Omega)$ that $D(\mathcal{L})$ is dense in $L^2(\Omega)$. So, it is sufficient to show the accretivity of the operator $\mathcal{L}_0$. In fact, this implies that $\mathcal{L}_0$ is closable (so that $\mathcal{L} = \overline{\mathcal{L}_0}$ is well defined), and that $\mathcal{L}$ is also accretive (Tanabe [15, Chapter 2]).

First of all, since we easily observe the following lemma without using the accretivity, we give its proof.

Lemma 2.1. *The operator $\mathcal{L}_0$ defined by (1.3) is closable in $L^2(\Omega)$.*

Proof. Suppose that $u_j \in D(\mathcal{L}_0), \|u_j\| \to 0, \|\mathcal{L}_0 u_j - v\| \to 0$ as $j \to \infty$. For every $\varphi \in C_0^\infty(\Omega)$, an integration by parts together with $\nabla\cdot(k\times x) = 0$ yields

$$(\mathcal{L}_0 u_j, \varphi) = (u_j, -\Delta\varphi + (k\times x)\cdot\nabla\varphi) \to 0 \qquad \text{as} \quad j\to\infty.$$

We thus have $(v,\varphi) = 0$, which implies $v = 0$. □

Due to the special structure of $(k \times x)\cdot\nabla$, we obtain the following lemma, which just asserts that the operator $\mathcal{L}_0$ is accretive in $L^2(\Omega)$.

Lemma 2.2. *For each $u \in D(\mathcal{L}_0)$, we have*

$$Re\,(\mathcal{L}_0 u, u) = \|\nabla u\|^2. \tag{2.1}$$

Proof. Let $u \in D(\mathcal{L}_0)$. An integration by parts yields the following equality for large $R > 0$:

$$\int_{\Omega_R} [(k \times x) \cdot \nabla u]\overline{u} dx = - \int_{\Omega_R} u[(k \times x) \cdot \nabla \overline{u}] dx,$$

where $\Omega_R = \{x \in \Omega; |x| < R\}$. Here, note that the integral on Γ vanishes because of $u \in H_0^1(\Omega)$, and that the integral on $|x| = R$ does because of $x \cdot (k \times x) = 0$. Since $(k \times x) \cdot \nabla u \in L^2(\Omega)$, letting $R \to \infty$ in the above equality implies $((k \times x) \cdot \nabla u, u) = -(u, (k \times x) \cdot \nabla u)$, so that Re $((k \times x) \cdot \nabla u, u) = 0$. We thus obtain (2.1). □

Corollary 2.3. *For each $u \in D(\mathcal{L})$, we have $u \in H_0^1(\Omega)$ with the relation (2.1), so that $\mathcal{L}$ is an accretive operator in $L^2(\Omega)$. As a result, the operator $\lambda + \mathcal{L}$ has a continuous inverse for each $\lambda \in \mathbb{C}_+$.*

Proof. For $u \in D(L)$, we take $u_j \in D(\mathcal{L}_0)$ so that $\|u_j - u\|_{D(\mathcal{L})} \to 0$ as $j \to \infty$. Then, by (2.1) the sequence $\{u_j\}$ is a Cauchy one in $H^1(\Omega)$ so that $\|u_j - u\|_{H^1(\Omega)} \to 0$ as $j \to \infty$. This implies $u \in H_0^1(\Omega)$ with (2.1). It follows from (2.1) for $u \in D(\mathcal{L})$ that

$$\|(\lambda + \mathcal{L})u\| \|u\| \geq \text{Re } \lambda \|u\|^2 + \|\nabla u\|^2, \qquad u \in D(\mathcal{L}), \tag{2.2}$$

which implies

$$\|(\lambda + \mathcal{L})u\| \geq \text{Re } \lambda \|u\|, \qquad u \in D(\mathcal{L}). \tag{2.3}$$

Therefore, we obtain the latter assertion. □

Remark 2.1. By (2.3) the operator $1 + \mathcal{L}$ has a closed range on account of the closedness of $\mathcal{L}$. Consider the adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$, which is of the form $-\Delta + (k \times x) \cdot \nabla$. If $1 + \mathcal{L}^*$ is injective, then $1 + \mathcal{L}$ is surjective so that $\mathcal{L}$ is an m-accretive operator. However, the same argument as above implies (2.3) with $\mathcal{L}$ replaced by $\mathcal{L}^*$ for only u belonging to the completion of $D(\mathcal{L}_0)$ under the graph norm of $\mathcal{L}^*$. Since $D(\mathcal{L}^*)$ is not clear (at present) and may be larger, this consideration is not sufficient for the proof of the m-accretivity of $\mathcal{L}$. The surjectivity of $1 + \mathcal{L}$ will be proved directly through the corresponding boundary value problem in section 5.

3. The operator $\Delta + (k \times x) \cdot \nabla$ in $\mathbb{R}^3$

Let $\mathcal{L}_{e,0}$ be the operator from $L^2(\mathbb{R}^3)$ into itself defined by

$$\begin{cases} D(\mathcal{L}_{e,0}) = \{u \in H^2(\mathbb{R}^3); (k \times x) \cdot \nabla u \in L^2(\mathbb{R}^3)\}, \\ \mathcal{L}_{e,0} = -\Delta - (k \times x) \cdot \nabla. \end{cases}$$

Then the same way as in Lemma 2.2 implies that $\mathcal{L}_{e,0}$ is a densely defined accretive operator in $L^2(\mathbb{R}^3)$. Thus the operator $\mathcal{L}_{e,0}$ is closable so that we can define the operator $\mathcal{L}_e : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ by

$$\mathcal{L}_e = \overline{\mathcal{L}_{e,0}} \text{ (the closure of } \mathcal{L}_{e,0}).$$

It is evident that the operator $\mathcal{L}_e$ has the same property as in Corollary 2.3. Therefore, for each $\lambda \in \mathbb{C}_+$ and $f \in L^2(\mathbb{R}^3)$, we have the uniqueness of solutions in $D(\mathcal{L}_e)$ for the equation

$$\lambda u + \mathcal{L}_e u = f \quad \text{in} \quad L^2(\mathbb{R}^3), \tag{3.1}$$

which corresponds to the spectral steady problem

$$\lambda u - \Delta u - (k \times x) \cdot \nabla u = f, \qquad x \in \mathbb{R}^3. \tag{3.2}$$

The first purpose of this section is to prove the following proposition.

Proposition 3.1. *The resolvent set $\rho(-\mathcal{L}_e)$ of the operator $-\mathcal{L}_e$ contains the right half plane $\mathbb{C}_+$. As a result, the operator $\mathcal{L}_e$ is m-accretive, so that $-\mathcal{L}_e$ generates a (C_0) semigroup $\{e^{-t\mathcal{L}_e}\}_{t\geq 0}$ of contractions on $L^2(\mathbb{R}^3)$.*

For the proof of Proposition 3.1, we have to establish the existence of the solution to (3.1) for every $f \in L^2(\mathbb{R}^3)$. The equation (3.2) is formally deduced from the Laplace transform of

$$\partial_t v = \Delta v + (k \times x) \cdot \nabla v, \qquad x \in \mathbb{R}^3, t > 0, \tag{3.3}$$

$$v(x, 0) = f(x), \qquad x \in \mathbb{R}^3, \tag{3.4}$$

with respect to the time variable. By a direct calculation (cf. the reduction of (1.2) in [8]) the function

$$v(x,t) = [U(t)f](x) = \left[e^{t\Delta} f\right](O(t)x), \qquad x \in \mathbb{R}^3, t > 0, \tag{3.5}$$

satisfies the initial value problem (3.3) and (3.4) with $f \in L^2(\mathbb{R}^3)$, where

$$\left[e^{t\Delta} f\right](x) = (4\pi t)^{-3/2} \int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4t}} f(y)dy. \tag{3.6}$$

It is easy to see that

$$\|U(t)f\|_{\mathbb{R}^3} = \left\| \left[e^{t\Delta} f\right](O(t)\ \cdot) \right\|_{\mathbb{R}^3} = \|e^{t\Delta} f\|_{\mathbb{R}^3} \leq \|f\|_{\mathbb{R}^3}, \qquad t > 0. \tag{3.7}$$

Set $U(0) = I$. It is possible to show that the family $\{U(t)\}_{t\geq 0}$ is a (C_0) semigroup on $L^2(\mathbb{R}^3)$. But, at present, the coincidence between the domain of its infinitesimal generator and the domain $D(\mathcal{L}_e)$ of $\mathcal{L}_e$ defined above is not clear (this will be clarified later). The solvability for (3.1), however, can be established by using $U(t)$.

Lemma 3.2. *For each $\lambda \in \mathbb{C}_+$ and $f \in L^2(\mathbb{R}^3)$, there exists a unique solution $u \in D(\mathcal{L}_e)$ to (3.1). This solution possesses the regularity $\Delta u \in L^2(\mathbb{R}^3)$ with estimate*

$$\|\Delta u\|_{\mathbb{R}^3} \leq \|f\|_{\mathbb{R}^3}, \tag{3.8}$$

and is of class

$$u \in H^2(\mathbb{R}^3), \qquad (k \times x) \cdot \nabla u \in L^2(\mathbb{R}^3). \tag{3.9}$$

Proof. The uniqueness of solutions in $D(\mathcal{L}_e)$ can be deduced in the same way as in the proof of Corollary 2.3 (as already mentioned). For given $\lambda \in \mathbb{C}_+$ and $f \in L^2(\mathbb{R}^3)$, we may expect that the solution to (3.1) is given by

$$u(x,\lambda) = \int_0^\infty e^{-\lambda t}[U(t)f](x)dt, \tag{3.10}$$

which is in $L^2(\mathbb{R}^3)$ because of (3.7). We have to prove that $u(\lambda) \equiv u(\cdot,\lambda) \in D(\mathcal{L}_e)$. We first show that $\Delta u(\lambda) \in L^2(\mathbb{R}^3)$ with estimate (3.8). Consider the Fourier transform of $u(\lambda)$ with respect to the space variable:

$$\begin{aligned} \mathcal{F}[u(\lambda)](\xi) \equiv \widehat{u}(\xi,\lambda) &= (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-i\,x\cdot\xi}\, u(x,\lambda)dx \\ &= \int_0^\infty e^{-\lambda t}\,\mathcal{F}\left[U(t)f\right](\xi)dt, \end{aligned} \tag{3.11}$$

where $i = \sqrt{-1}$. For $v(t) = U(t)f \in L^2(\mathbb{R}^3)$, we have $(k \times x) \cdot \nabla v \in \mathcal{S}'(\mathbb{R}^3)$, the class of temperate distributions. We thus take the Fourier transform of the initial value problem (3.3) and (3.4). One can verify

$$\mathcal{F}[(k \times x) \cdot \nabla_x v] = (k \times \xi) \cdot \nabla_\xi \widehat{v}, \tag{3.12}$$

so that

$$\mathcal{F}[U(t)f](\xi) \equiv \widehat{v}(\xi,t) = e^{-|\xi|^2 t}\widehat{f}(O(t)\xi). \tag{3.13}$$

Therefore, by (3.11) and (3.13) the Fubini and the Plancherel theorems imply

$$\int_{\mathbb{R}^3} |\xi|^4 |\widehat{u}(\xi,\lambda)|^2 d\xi$$

$$
\begin{aligned}
&= \int_{\mathbf{R}^3} |\xi|^4 \left| \int_0^\infty e^{-(\lambda+|\xi|^2)t} \widehat{f}(O(t)\xi)dt \right|^2 d\xi \\
&\leq \int_{\mathbf{R}^3} |\xi|^4 \left(\int_0^\infty e^{-(\mathrm{Re}\,\lambda+|\xi|^2)t} dt \right) \left(\int_0^\infty e^{-(\mathrm{Re}\,\lambda+|\xi|^2)t} \left|\widehat{f}(O(t)\xi)\right|^2 dt \right) d\xi \\
&= \int_0^\infty \int_{\mathbf{R}^3} \frac{|\xi|^4}{\mathrm{Re}\,\lambda + |\xi|^2} e^{-(\mathrm{Re}\,\lambda+|\xi|^2)t} \left|\widehat{f}(\xi)\right|^2 d\xi dt \\
&= \int_{\mathbf{R}^3} \frac{|\xi|^4}{(\mathrm{Re}\,\lambda + |\xi|^2)^2} \left|\widehat{f}(\xi)\right|^2 d\xi \\
&\leq \left\|\widehat{f}\right\|^2_{\mathbf{R}^3_\xi} = \|f\|^2_{\mathbf{R}^3_x}.
\end{aligned}
$$

This yields $\Delta u(\lambda) \in L^2(\mathbf{R}^3)$ with estimate

$$\|\Delta u(\lambda)\|_{\mathbf{R}^3_x} = \|\mathcal{F}[\Delta u(\lambda)]\|_{\mathbf{R}^3_\xi} \leq \|f\|_{\mathbf{R}^3_x}.$$

From this it follows that $u \in H^2(\mathbf{R}^3)$ and that $(k \times x) \cdot \nabla u = \lambda u - \Delta u - f \in L^2(\mathbf{R}^3)$. Hence, $u \in D(\mathcal{L}_{e,0}) \subset D(\mathcal{L}_e)$, so that we can consider $\mathcal{L}_e u$. By (3.3)–(3.5), (3.7) and (3.10) we obtain

$$\mathcal{L}_e u(\lambda) = -\int_0^\infty e^{-\lambda t} \partial_t [U(t)f] dt = f - \lambda u(\lambda),$$

in $L^2(\mathbf{R}^3)$. Thus, $u(\lambda)$ is actually the solution to (3.1). We have completed the proof. □

Proof of Proposition 3.1. Lemma 3.2 asserts that the operator $\lambda + \mathcal{L}_e$ is bijective in $L^2(\mathbf{R}^3)$ for every $\lambda \in \mathbf{C}_+$. This implies $\mathbf{C}_+ \subset \rho(-\mathcal{L}_e)$ so that $\mathcal{L}_e$ is an m-accretive operator. By the Lumer-Phillips theorem (Pazy [12, Chapter 1]) the operator $-\mathcal{L}_e$ generates a (C_0) semigroup $\{e^{-t\mathcal{L}_e}\}_{t \geq 0}$ of contractions. □

We here mention that the semigroup $\{e^{-t\mathcal{L}_e}\}_{t \geq 0}$ exactly coincides with the family $\{U(t)\}_{t \geq 0}$ given by (3.5). In fact, since the unique solution to (3.1) is given by (3.10), the resolvent is of the form

$$(\lambda + \mathcal{L}_e)^{-1} = \int_0^\infty e^{-\lambda t} U(t) dt, \qquad \lambda \in \mathbf{C}_+. \tag{3.14}$$

On the other hand, we have

$$(\lambda + \mathcal{L}_e)^{-1} = \int_0^\infty e^{-\lambda t} e^{-t\mathcal{L}_e} dt, \qquad \lambda \in \mathbf{C}_+,$$

by the standard theory of semigroups ([12, 15]). Therefore, for all f and φ in $L^2(\mathbf{R}^3)$ we get

$$\int_0^\infty e^{-\lambda t} (U(t)f, \varphi) dt = \int_0^\infty e^{-\lambda t} (e^{-t\mathcal{L}_e} f, \varphi) dt.$$

It follows from the uniqueness of the inverse Laplace transform in the class of continuous functions that $(U(t)f,\varphi) = (e^{-t\mathcal{L}_e}f,\varphi)$ for all $\varphi \in L^2(\mathbb{R}^3)$. We thus obtain $U(t)f = e^{-t\mathcal{L}_e}f$ for all $f \in L^2(\mathbb{R}^3)$ and $t > 0$.

Besides

$$\|(\lambda+\mathcal{L}_e)^{-1}f\|_{\mathbb{R}^3} \leq \frac{1}{\mathrm{Re}\,\lambda}\|f\|_{\mathbb{R}^3}, \tag{3.15}$$

we give some estimates for the derivatives of the resolvent.

Lemma 3.3. *It holds that*

$$\|\nabla(\lambda+\mathcal{L}_e)^{-1}f\|_{\mathbb{R}^3} \leq \frac{1}{(\mathrm{Re}\,\lambda)^{1/2}}\|f\|_{\mathbb{R}^3}, \tag{3.16}$$

$$\|D^2(\lambda+\mathcal{L}_e)^{-1}f\|_{\mathbb{R}^3} \leq \|f\|_{\mathbb{R}^3}, \tag{3.17}$$

for all $\lambda \in \mathbb{C}_+$ and $f \in L^2(\mathbb{R}^3)$.

Proof. Set $u = (\lambda+\mathcal{L}_e)^{-1}f$. Then the relation $\|D^2u\|_{\mathbb{R}^3} = \|\Delta u\|_{\mathbb{R}^3}$ (by using the Riesz transform) and (3.8) give (3.17). By $\|\nabla u\|^2_{\mathbb{R}^3} \leq \|\Delta u\|_{\mathbb{R}^3}\|u\|_{\mathbb{R}^3} \leq \|f\|_{\mathbb{R}^3}\|u\|_{\mathbb{R}^3}$, estimate (3.15) implies (3.16). □

The following proposition presents an L^2 a priori estimate for the operator $\mathcal{L}_e$. This is the entire space version of Theorem 2.

Proposition 3.4. *For each $u \in D(\mathcal{L}_e)$, we have*

$$u \in H^2(\mathbb{R}^3), \qquad (k\times x)\cdot\nabla u \in L^2(\mathbb{R}^3),$$

with estimates

$$\|\Delta u\|_{\mathbb{R}^3} \leq \|\mathcal{L}_e u\|_{\mathbb{R}^3}, \tag{3.18}$$

$$\|(k\times x)\cdot\nabla u\|_{\mathbb{R}^3} \leq 2\|\mathcal{L}_e u\|_{\mathbb{R}^3}. \tag{3.19}$$

There is a constant $C > 0$ such that

$$\|u\|_{H^2(\mathbb{R}^3)} + \|(k\times x)\cdot\nabla u\|_{\mathbb{R}^3} \leq C\,\|(1+\mathcal{L}_e)u\|_{\mathbb{R}^3}, \tag{3.20}$$

for all $u \in D(\mathcal{L}_e)$. As a result, $D(\mathcal{L}_e) = D(\mathcal{L}_{e,0})$ and, therefore, $\mathcal{L}_e = \mathcal{L}_{e,0}$.

Proof. For $u \in D(\mathcal{L}_e)$ and $\lambda > 0$, we set $f = (\lambda+\mathcal{L}_e)u \in L^2(\mathbb{R}^3)$. Then, by Lemma 3.2 the unique solution u to (3.1) with such f is of class (3.9) so that $D(\mathcal{L}_e) = D(\mathcal{L}_{e,0})$. Further, by (3.8) we have

$$\|\Delta u\|_{\mathbf{R}^3} \le \|f\|_{\mathbf{R}^3} = \|(\lambda + \mathcal{L}_e)u\|_{\mathbf{R}^3}.$$

Letting $\lambda \to 0$ yields (3.18). From this it follows that

$$\|(k \times x) \cdot \nabla u\|_{\mathbf{R}^3} = \|\mathcal{L}_e u + \Delta u\|_{\mathbf{R}^3} \le 2\|\mathcal{L}_e u\|_{\mathbf{R}^3},$$

which is just (3.19). We thus obtain

$$\begin{aligned} \|u\|_{H^2(\mathbf{R}^3)} + \|(k \times x) \cdot \nabla u\|_{\mathbf{R}^3} &\le C\left(\|\Delta u\|_{\mathbf{R}^3} + \|u\|_{\mathbf{R}^3}\right) + 2\|\mathcal{L}_e u\|_{\mathbf{R}^3} \\ &\le C\left(\|\mathcal{L}_e u\|_{\mathbf{R}^3} + \|u\|_{\mathbf{R}^3}\right), \end{aligned}$$

for $u \in D(\mathcal{L}_e)$. Since $1 + \mathcal{L}_e$ has a bounded inverse, the above estimate gives (3.20). □

We will derive the smoothing effect for the semigroup $\{U(t)\}_{t \ge 0}$ given by (3.5) (and Proposition 3.1).

Proposition 3.5. *Suppose that $f \in L^2(\mathbf{R}^3)$. Then $U(t)f \in H^n(\mathbf{R}^3)$ for all $t > 0$ and every integer $n \ge 1$. Given an integer $n \ge 1$, let $0 \le s \le n$. If $f \in H^s(\mathbf{R}^3) \equiv (1 - \Delta)^{-s/2}\left[L^2(\mathbf{R}^3)\right]$, then there is a constant $C = C(n-s) > 0$ such that*

$$\|D^n U(t) f\|_{\mathbf{R}^3} \le C\, t^{-(n-s)/2} \|f\|_{H^s(\mathbf{R}^3)}, \tag{3.21}$$

for $t > 0$, where D^n denotes each of derivatives of n-th order.

Proof. Since we have

$$\int_{\mathbf{R}^3} |\xi|^{2n} e^{-2|\xi|^2 t} \left|\widehat{f}(O(t)\xi)\right|^2 d\xi \le C\, t^{-(n-s)} \left\| |\xi|^s \widehat{f} \right\|^2_{\mathbf{R}^3_\xi},$$

for $0 \le s \le n$, the formula (3.13) gives $U(t)f \in H^n(\mathbf{R}^3)$ with estimate (3.21). □

Proposition 3.6. *Suppose that $f \in L^2(\mathbf{R}^3)$ and that $(k \times x) \cdot \nabla f \in H^{-m}(\mathbf{R}^3)$ for some integer $m \ge 0$, where $H^{-m}(\mathbf{R}^3)$ denotes the dual space of $H^m(\mathbf{R}^3)$. Then $(k \times x) \cdot \nabla U(t) f \in H^n(\mathbf{R}^3)$ for all $t > 0$ and every integer $n \ge 0$ with estimate*

$$\|(k \times x) \cdot \nabla U(t) f\|_{H^n(\mathbf{R}^3)} \le C(t \wedge 1)^{-(m+n)/2} \|(k \times x) \cdot \nabla f\|_{H^{-m}(\mathbf{R}^3)}, \tag{3.22}$$

for $t > 0$, where $t \wedge 1 = \min\{t, 1\}$ and $C = C(m+n) > 0$.

Proof. Set $g = (k \times x) \cdot \nabla f$ and $v(t) = U(t)f$. By (3.12), (3.13) and $(k \times \xi) \cdot \xi = 0$, we obtain

$$\mathcal{F}[(k \times x) \cdot \nabla_x v](\xi) = (k \times \xi) \cdot \nabla_\xi \widehat{v}(\xi, t) = e^{-|\xi|^2 t} (k \times \xi) \cdot \nabla_\xi \left[\widehat{f}(O(t)\xi)\right].$$

Since $\widehat{g} = (k \times \xi) \cdot \nabla_\xi \widehat{f}$ and since $k \times (O(t)\xi) = O(t)(k \times \xi)$, we have

$$\widehat{g}(O(t)\xi) = [k \times (O(t)\xi)] \cdot \left(\left[\nabla \widehat{f}\right](O(t)\xi)\right) = (k \times \xi) \cdot \nabla_\xi \left[\widehat{f}(O(t)\xi)\right].$$

Combining the above equalities yields

$$\mathcal{F}[(k \times x) \cdot \nabla v](\xi) = e^{-|\xi|^2 t}\, \widehat{g}(O(t)\xi), \tag{3.23}$$

in $\mathcal{S}'(\mathbb{R}^3)$. Hence, it follows from

$$\begin{aligned}
&\int_{\mathbb{R}^3} (1+|\xi|^2)^n e^{-2|\xi|^2 t}\, |\widehat{g}(O(t)\xi)|^2 \, d\xi \\
&\leq \left(\max\left\{ \frac{m+n}{2t},\, 1 \right\} \right)^{m+n} \left\| (1+|\xi|^2)^{-m/2} \widehat{g} \right\|^2_{\mathbb{R}^3_\xi} \\
&= \begin{cases} \|g\|^2_{\mathbb{R}^3} & \text{if} \quad m = n = 0, \\ \left[\left(\frac{2t}{m+n} \right) \wedge 1 \right]^{-(m+n)} \|g\|^2_{H^{-m}(\mathbb{R}^3)} & \text{if} \quad m+n > 0, \end{cases}
\end{aligned}$$

that $(k \times x) \cdot \nabla v \in H^n(\mathbb{R}^3)$ with estimate (3.22). □

By (3.23) it turns out that $U(t)$ is not an analytic semigroup on $L^2(\mathbb{R}^3)$ although it possesses the smoothing effect (3.21). This will be proved in Proposition 3.8.

The next proposition is concerned with the existence of a strong solution to the Cauchy problem.

Proposition 3.7. *Suppose that $f \in L^2(\mathbb{R}^3)$ and that $(k \times x) \cdot \nabla f \in H^{-\infty}(\mathbb{R}^3) \equiv \bigcup_{m \geq 0} H^{-m}(\mathbb{R}^3)$. Then $U(t)f \in D(\mathcal{L}_e)$ for all $t > 0$, and*

$$\mathcal{L}_e U(\cdot) f \in C\left((0,\infty); H^s(\mathbb{R}^3)\right), \qquad U(\cdot) f \in C^1\left((0,\infty); H^s(\mathbb{R}^3)\right),$$

with

$$\frac{d}{dt} U(t) f + \mathcal{L}_e U(t) f = 0, \qquad t > 0, \tag{3.24}$$

in $H^s(\mathbb{R}^3)$ for every $s \geq 0$.

Proof. Set $g = (k \times x) \cdot \nabla f$ and $v(t) = U(t)f$. By Propositions 3.4, 3.5 and 3.6 we have $v(t) \in D(\mathcal{L}_e)$ and $\mathcal{L}_e v(t) \in H^s(\mathbb{R}^3)$ for all $t > 0$ and every $s \geq 0$. By (3.13) and (3.23) we make a change of variable $\eta = O(t/2)\xi$ to obtain

$$\begin{aligned}
\widehat{(\mathcal{L}_e v)}(\xi, t+h) &= |\eta|^2 e^{-|\eta|^2 (t+h)} \widehat{f}(O(t/2+h)\eta) - e^{-|\eta|^2 (t+h)} \widehat{g}(O(t/2+h)\eta) \\
&= e^{-|\eta|^2 t/2}\, \widehat{(\mathcal{L}_e v)}(\eta, t/2+h).
\end{aligned}$$

Using this relation, we get

$$\|\mathcal{L}_e v(t+h) - \mathcal{L}_e v(t)\|^2_{H^s(\mathbb{R}^3)}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} (1+|\xi|^2)^s \left|\widehat{(\mathcal{L}_e v)}(\xi, t+h) - \widehat{(\mathcal{L}_e v)}(\xi, t)\right|^2 d\xi \\
&= \int_{\mathbb{R}^3} (1+|\eta|^2)^s e^{-|\eta|^2 t} \left|\widehat{(\mathcal{L}_e v)}(\eta, t/2+h) - \widehat{(\mathcal{L}_e v)}(\eta, t/2)\right|^2 d\eta \\
&\le \left(\max\left\{\frac{s}{t}, 1\right\}\right)^s \|\mathcal{L}_e v(t/2+h) - \mathcal{L}_e v(t/2)\|^2_{\mathbb{R}^3} \\
&= C_s (t\wedge 1)^{-s} \|U(t/4+h)\mathcal{L}_e v(t/4) - U(t/4)\mathcal{L}_e v(t/4)\|^2_{\mathbb{R}^3}.
\end{aligned}
$$

Hence, $\mathcal{L}_e v \in C\left((0,\infty); H^s(\mathbb{R}^3)\right)$. The same reasoning as above gives

$$
\begin{aligned}
&\left\| \frac{v(t+h) - v(t)}{h} + \mathcal{L}_e v(t) \right\|^2_{H^s(\mathbb{R}^3)} \\
&\le C_s (t\wedge 1)^{-s} \left\| \frac{v(t/2+h) - v(t/2)}{h} + \mathcal{L}_e v(t/2) \right\|^2_{\mathbb{R}^3} \\
&= C_s (t\wedge 1)^{-s} \left\| \frac{U(t/4+h)v(t/4) - U(t/4)v(t/4)}{h} + \mathcal{L}_e U(t/4) v(t/4) \right\|^2_{\mathbb{R}^3}.
\end{aligned}
$$

Since $v(t/4) \in D(\mathcal{L}_e)$, letting $h \to 0$ implies that $v \in C^1\left((0,\infty); H^s(\mathbb{R}^3)\right)$ with (3.24). □

Remark 3.1. It seems to be difficult to show further regularity properties of the semigroup with respect to the time variable under the assumptions of Proposition 3.7. If, in addition, $(k\times x)\cdot\nabla\left[(k\times x)\cdot\nabla f\right] \in H^{-\infty}(\mathbb{R}^3)$, then we obtain $U(\cdot)f \in C^2\left((0,\infty); H^s(\mathbb{R}^3)\right)$ for every $s \ge 0$; we do not enter into the detail.

We finally prove the following proposition.

Proposition 3.8. *The semigroup $\{U(t)\}_{t\ge 0}$ on $L^2(\mathbb{R}^3)$ given by (3.5) (and Proposition 3.1) is not analytic.*

Proof. Suppose that $\{U(t)\}_{t\ge0}$ is an analytic semigroup. Then for all $f \in L^2(\mathbb{R}^3)$ and $t > 0$, we have $U(t)f \in D(\mathcal{L}_e)$ (see [12, 15]); in particular, $(k\times x)\cdot\nabla U(t)f \in L^2(\mathbb{R}^3)$ by Proposition 3.4. Therefore, for $g = (k\times x)\cdot\nabla f \in \mathcal{S}'(\mathbb{R}^3)$ with $\widehat{g} = (k\times\xi)\cdot\nabla_\xi \widehat{f}$, it follows from (3.23) that

$$
e^{-|\xi|^2 t}\, \widehat{g} \in L^2(\mathbb{R}^3_\xi), \qquad \forall\, f \in L^2(\mathbb{R}^3_x),\ \forall\, t > 0. \tag{3.25}
$$

We fix $\varphi \in \mathcal{S}(\mathbb{R}^3_\xi)$, the class of rapidly decreasing functions, so that $\varphi(\xi) = 1$ for $|\xi| \le 2$. Let $\ell = (1,0,0)^T$ and consider the function f given by

$$
f = \mathcal{F}^{-1}\widehat{f} \in L^2(\mathbb{R}^3_x), \qquad \widehat{f}(\xi) = \frac{\varphi(\xi-\ell)}{|\xi-\ell|} \in L^2(\mathbb{R}^3_\xi),
$$

where $\mathcal{F}^{-1}$ is the inverse of the Fourier transform (3.11). Then

$$
\nabla_\xi \widehat{f}(\xi) = \frac{(\nabla\varphi)(\xi-\ell)}{|\xi-\ell|} - \frac{(\xi-\ell)\,\varphi(\xi-\ell)}{|\xi-\ell|^3} \in L^1(\mathbb{R}^3_\xi),
$$

so that

$$\widehat{g}(\xi) = \frac{-\xi_2}{|\xi - \ell|^3} \qquad \text{if} \quad \xi \in B \equiv \{\xi \in \mathbb{R}^3; |\xi - \ell| < 1\}.$$

We thus observe

$$\int_{\mathbb{R}^3} e^{-2|\xi|^2 t} |\widehat{g}(\xi)|^2 \, d\xi \geq \int_B e^{-2|\xi|^2 t} \frac{\xi_2^2}{|\xi - \ell|^6} \, d\xi = \infty,$$

for each $t > 0$, which is a contradiction to (3.25). □

The proof of Proposition 3.8 shows also that $\{U(t)\}_{t \geq 0}$ is not a differentiable semigroup in the sense of Pazy [12, Chapter 2].

Remark 3.2. Lunardi and Vespri [11] have discussed generation of (C_0) semigroups in $L^p(\mathbb{R}^N)$ by elliptic operators of the type

$$\Lambda f = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(q_{ij}(x) \frac{\partial f}{\partial x_j} \right) + \sum_{i=1}^{N} \left[\frac{\partial}{\partial x_i} (a_i(x) f) + b_i(x) \frac{\partial f}{\partial x_i} \right].$$

The coefficients $a_i(x)$ and $b_i(x)$ are allowed to be unbounded, and thus the operator $\Delta + (k \times x) \cdot \nabla$ is covered. But the proof of Proposition 3.1 is easier than that of [11], in which more delicate argument is necessary because of the lack of continuity of $q_{ij}(x)$.

Remark 3.3. DaPrato and Lunardi [5] have studied nice smoothing properties of the semigroup $S(t)$ (it is neither strongly continuous nor analytic) in $UCB(\mathbb{R}^N)$ generated by the Ornstein-Uhlenbeck operator

$$\Lambda f = \mathrm{Tr}\left[QD^2 f\right] + (Bx) \cdot \nabla f,$$

as mentioned in section 1. It has been also pointed out in [5, Section 2] that $S(t)$ is an analytic semigroup in suitable weighted L^p spaces under either of the following assumptions: (i) $Q^{-1}B$ is symmetric and Det $B \neq 0$, (ii) all the eigenvalues of B have negative real part. But, clearly, the operator $\Delta + (k \times x) \cdot \nabla$ is not covered.

4. The operator $\Delta + (k \times x) \cdot \nabla$ in bounded domains

Let D be a bounded domain with smooth boundary. Define the operators $\mathcal{L}_b$ and A_b from $L^2(D)$ into itself by

$$\begin{cases} D(\mathcal{L}_b) = D(A_b) = H^2(D) \cap H_0^1(D), \\ \mathcal{L}_b = A_b - (k \times x) \cdot \nabla, \\ A_b = -\Delta. \end{cases}$$

In the case of bounded domains, it is possible to deal with $\mathcal{L}_b$ as A_b plus its perturbation. For $\delta > 0$ and $r \geq 0$, set

$$\Sigma_\delta(r) = \left\{ \lambda \in \mathbb{C}; |\arg \lambda| < \frac{\pi}{2} + \delta, |\lambda| > r \right\}, \qquad \Sigma_\delta = \Sigma_\delta(0) \cup \{0\}.$$

We begin with the following lemma.

Lemma 4.1. *The operator $-\mathcal{L}_b$ is closed, and generates an analytic (C_0) semigroup $\{e^{-t\mathcal{L}_b}\}_{t\geq 0}$ on $L^2(D)$.*

Proof. We use the standard method of perturbation. For any $\eta > 0$, there is a constant $C_\eta > 0$ such that

$$
\begin{aligned}
(4.1)\qquad \|(k\times x)\cdot\nabla u\|_D &\leq \left(\sup_{x\in D}|x|\right)\|\nabla u\|_D = \left(\sup_{x\in D}|x|\right)\left\|A_b^{1/2}u\right\|_D \\
&\leq C\|A_b u\|_D^{1/2}\|u\|_D^{1/2} \leq \eta\|A_b u\|_D + C_\eta\|u\|_D.
\end{aligned}
$$

Thus the closedness of A_b implies that of $\mathcal{L}_b$. Fix $\varepsilon\in(0,\pi/2)$ arbitrarily. It is well known (see, for instance, [12]) that there is a constant $K_\varepsilon > 0$ such that

$$
\|(\lambda + A_b)^{-1}f\|_D \leq \frac{K_\varepsilon}{1+|\lambda|}\|f\|_D, \qquad \lambda\in\Sigma_{\pi/2-\varepsilon}, f\in L^2(D).
$$

This together with (4.1) enables us to take a constant $r_\varepsilon > 0$ so that

$$
\|(k\times x)\cdot\nabla(\lambda + A_b)^{-1}\|_{L^2(D)\to L^2(D)} \leq \frac{1}{2}, \qquad \lambda\in\Sigma_{\pi/2-\varepsilon}(r_\varepsilon).
$$

Then, by the Neumann series we obtain $\Sigma_{\pi/2-\varepsilon}(r_\varepsilon)\subset\rho(-\mathcal{L}_b)$ with estimate

$$
(4.2)\quad \|(\lambda+\mathcal{L}_b)^{-1}f\|_D = \left\|(\lambda + A_b)^{-1}\left[1-(k\times x)\cdot\nabla(\lambda+A_b)^{-1}\right]^{-1}f\right\|_D \leq \frac{2K_\varepsilon}{|\lambda|}\|f\|_D,
$$

for all $\lambda\in\Sigma_{\pi/2-\varepsilon}(r_\varepsilon)$ and $f\in L^2(D)$. We thus obtain the assertion by the standard theory of analytic semigroups ([15]). □

We next refine the result of Lemma 4.1.

Proposition 4.2. *(i) There are constants $\delta > 0$ and $M > 0$ so that the resolvent set $\rho(-\mathcal{L}_b)$ of the operator $-\mathcal{L}_b$ contains Σ_δ and*

$$
(4.3)\qquad \|(\lambda+\mathcal{L}_b)^{-1}f\|_D \leq \frac{M}{1+|\lambda|}\|f\|_D,
$$

for all $\lambda\in\Sigma_\delta$ and $f\in L^2(D)$. As a result, the analytic semigroup $e^{-t\mathcal{L}_b}$ obtained in Lemma 4.1 is uniformly bounded.

(ii) There is a constant $M > 0$ so that

$$
(4.4)\qquad \|\nabla(\lambda+\mathcal{L}_b)^{-1}f\|_D \leq \frac{M}{(1+|\lambda|)^{1/2}}\|f\|_D,
$$

$$
(4.5)\qquad \|D^2(\lambda+\mathcal{L}_b)^{-1}f\|_D \leq M\|f\|_D,
$$

for all $\lambda \in \Sigma_\delta$ *and* $f \in L^2(D)$.

(iii) There is a constant $C > 0$ *so that*

$$\|u\|_{H^2(D)} \le C\|\mathcal{L}_b u\|_D, \tag{4.6}$$

for all $u \in D(\mathcal{L}_b)$.

Proof. By the same calculation as in section 2 the operator $\mathcal{L}_b$ is also accretive in $L^2(D)$. Moreover, by the Poincaré inequality we get

$$\|(\lambda + \mathcal{L}_b)u\|_D \|u\|_D \ge \text{Re}\,\lambda \|u\|_D^2 + \|\nabla u\|_D^2 \ge (\text{Re}\lambda + \mu)\|u\|_D^2, \tag{4.7}$$

for $u \in D(\mathcal{L}_b)$, where $\mu > 0$ is the least eigenvalue of A_b. On account of the closedness of $\mathcal{L}_b$, the operator $\lambda + \mathcal{L}_b$ for $\text{Re}\,\lambda > -\mu$ has both a closed range and a continuous inverse. Consider the adjoint operator $\mathcal{L}_b^* = A_b + (k \times x) \cdot \nabla$ with domain $D(\mathcal{L}_b^*) = H^2(D) \cap H_0^1(D)$ (cf. Tanabe [15, Chapter 3]). Since $\mathcal{L}_b^*$ also satisfies (4.7) so that $\lambda + \mathcal{L}_b$ is surjective for $\text{Re}\,\lambda > -\mu$, we obtain $\{\lambda \in \mathbb{C}; \text{Re}\,\lambda > -\mu\} \subset \rho(-\mathcal{L}_b)$ with estimate

$$\|(\lambda + \mathcal{L}_b)^{-1} f\|_D \le \frac{1}{\text{Re}\lambda + \mu}\|f\|_D, \qquad \text{Re}\,\lambda > -\mu,\ f \in L^2(D).$$

This together with (4.2) gives (i). We next show (iii). By (4.1) and (4.3) with $\lambda = 0$ we have

$$\|A_b u\|_D \le \|\mathcal{L}_b u\|_D + \frac{1}{2}\|A_b u\|_D + C\|u\|_D \le C\|\mathcal{L}_b u\|_D + \frac{1}{2}\|A_b u\|_D, \tag{4.8}$$

which together with the well known L^2 estimate $\|u\|_{H^2(D)} \le C\|A_b u\|_D$ ([2, 6]) implies (4.6). We finally show (ii). Let $\lambda \in \Sigma_\delta$ and $f \in L^2(D)$. Then it follows from (4.3) and (4.6) that $\|(\lambda + \mathcal{L}_b)^{-1} f\|_{H^2(D)} \le C\|f\|_D$, which yields (4.5). Since

$$\begin{aligned} \|\nabla(\lambda + \mathcal{L}_b)^{-1} f\|_D &= \left\|A_b^{1/2}(\lambda + \mathcal{L}_b)^{-1} f\right\|_D \\ &\le \|A_b(\lambda + \mathcal{L}_b)^{-1} f\|_D^{1/2} \|(\lambda + \mathcal{L}_b)^{-1} f\|_D^{1/2}, \end{aligned}$$

(4.3) and (4.5) imply (4.4). We have completed the proof. □

By (i) of Proposition 4.2 fractional powers $\mathcal{L}_b^\alpha$ of $\mathcal{L}_b$ are well defined.

Proposition 4.3. *For each* $0 \le \alpha \le 1$, *it holds that* $D(\mathcal{L}_b^\alpha) = D(A_b^\alpha)$ *with equivalent norms. In particular,* $D(\mathcal{L}_b^{1/2}) = H_0^1(D)$ *with equivalent norms.*

Proof. Let $u \in H^2(D) \cap H_0^1(D)$. By (4.8) we have obtained

$$\|A_b u\|_D \le C\|\mathcal{L}_b u\|_D. \tag{4.9}$$

On the other hand, it follows from (4.1) that

$$\|(k\times x)\cdot\nabla u\|_D \le \frac{1}{3}\left(\|\mathcal{L}_b u\|_D + \|(k\times x)\cdot\nabla u\|_D\right) + C\|u\|_D,$$

so that

$$\|(k\times x)\cdot\nabla u\|_D \le \frac{1}{2}\|\mathcal{L}_b u\|_D + C\|u\|_D.$$

This together with $\|\mathcal{L}_b u\|_D \le \|A_b u\|_D + \|(k\times x)\cdot\nabla u\|_D$ implies that

$$\|\mathcal{L}_b u\|_D \le C\|A_b u\|_D. \tag{4.10}$$

In the proof of (i) of Proposition 4.2, it is proved that $\mathcal{L}_b$ is an m-accretive operator. Since A_b is nonnegative selfadjoint, it is also m-accretive. Applying the Heinz-Kato inequality [10, Section 3] (see also Tanabe [15, Chapter 2]) to estimates (4.9) and (4.10), we obtain the assertion. □

5. Construction of the resolvent, proof of Theorem 1

In this section we construct the resolvent $(\lambda+\mathcal{L})^{-1}$ concretely to prove the following proposition.

Proposition 5.1. *The resolvent set $\rho(-\mathcal{L})$ of the operator $-\mathcal{L}$ contains the right half plane $\mathbb{C}_+$.*

We first observe that Proposition 5.1 immediately implies Theorem 1.

Proof of Theorem 1. By Proposition 5.1 together with Corollary 2.3, the operator $\mathcal{L}$ is m-accretive in $L^2(\Omega)$. The assertion thus follows from the Lumer-Phillips theorem ([12]). □

We will prove Proposition 5.1. In what follows we assume $\lambda\in\mathbb{C}_+$ throughout. Given $f\in L^2(\Omega)$, we have already established in Corollary 2.3 the uniqueness of solutions in $D(\mathcal{L})$ for the equation

$$\lambda u + \mathcal{L}u = f \qquad \text{in} \quad L^2(\Omega). \tag{5.1}$$

Since $D(\mathcal{L})\subset H_0^1(\Omega)$ (Corollary 2.3), this equation corresponds to the following boundary value problem with spectral parameter:

$$\lambda u - \Delta u - (k\times x)\cdot\nabla u = f, \qquad x\in\Omega, \tag{5.2}$$

$$u = 0, \qquad x\in\Gamma. \tag{5.3}$$

We intend to construct the solution to (5.2) and (5.3) with the aid of the results in sections 3 and 4 via a cut-off procedure. We fix $b>0$ such that $\mathcal{O}\equiv\mathbb{R}^3\setminus\Omega\subset\{x\in\mathbb{R}^3; |x|<b\}$, and set $D=\{x\in\Omega; |x|<b+3\}$. For given $f\in L^2(\Omega)$, define the functions $f_e\in L^2(\mathbb{R}^3)$ and $f_b\in L^2(D)$ by

$$f_e(x) = \begin{cases} f(x) & (x \in \Omega), \\ 0 & (x \in \mathcal{O}), \end{cases} \qquad f_b(x) = f(x) \qquad (x \in D). \tag{5.4}$$

Let $\varphi \in C^\infty(\mathbb{R}^3)$ be a fixed function satisfying

$$0 \le \varphi \le 1, \qquad \varphi(x) = \begin{cases} 0 & (|x| \le b+1), \\ 1 & (|x| \ge b+2). \end{cases} \tag{5.5}$$

We set

$$u_e = \mathcal{R}_e(\lambda)f = (\lambda + \mathcal{L}_e)^{-1} f_e, \qquad u_b = \mathcal{R}_b(\lambda)f = (\lambda + \mathcal{L}_b)^{-1} f_b, \tag{5.6}$$

and

$$w = \mathcal{R}(\lambda)f = \varphi \mathcal{R}_e(\lambda)f + (1-\varphi)\mathcal{R}_b(\lambda)f \in L^2(\Omega). \tag{5.7}$$

Since u_e and u_b are, respectively, solutions to

$$\lambda u_e - \Delta u_e - (k \times x) \cdot \nabla u_e = f_e, \qquad x \in \mathbb{R}^3, \tag{5.8}$$

$$\begin{cases} \lambda u_b - \Delta u_b - (k \times x) \cdot \nabla u_b = f_b, & x \in D, \\ u_b = 0, & x \in \Gamma \cup \{x; |x| = b+3\}, \end{cases} \tag{5.9}$$

the function w should satisfy

$$\lambda w - \Delta w - (k \times x) \cdot \nabla w = f + T(\lambda) f, \qquad x \in \Omega, \tag{5.10}$$

as well as the boundary condition (5.3) on Γ, where the remaining term is given by

$$T(\lambda)f = -2\nabla\varphi \cdot \nabla(u_e - u_b) - [\Delta\varphi + (k \times x)\cdot\nabla\varphi]\,(u_e - u_b). \tag{5.11}$$

We first deduce the regularity of the function $w = \mathcal{R}(\lambda)f$.

Lemma 5.2. *For each $\lambda \in \mathbb{C}_+$ and $f \in L^2(\Omega)$, the function $w = \mathcal{R}(\lambda)f$ defined by (5.7) is of class*

$$w \in H^2(\Omega) \cap H^1_0(\Omega), \qquad (k \times x) \cdot \nabla w \in L^2(\Omega),$$

with estimate $\|\Delta w\| = \|\Delta \mathcal{R}(\lambda) f\| \le C_\lambda \|f\|$, *where the constant* $C_\lambda > 0$ *is independent of* $f \in L^2(\Omega)$.

Proof. We first show that $\Delta w \in L^2(\Omega)$. By (5.7) with (5.6) we have

$$\Delta w = \varphi \Delta u_e + (1-\varphi)\Delta u_b + 2\nabla\varphi \cdot \nabla(u_e - u_b) + (\Delta\varphi)(u_e - u_b).$$

It follows from (3.8) and (4.5) that

$$\|\varphi \Delta u_e\| \le \|\Delta \mathcal{R}_e(\lambda) f\|_{\mathbf{R}^3} \le \|f_e\|_{\mathbf{R}^3} = \|f\|, \tag{5.12}$$

$$\|(1-\varphi)\Delta u_b\| \le \|\Delta \mathcal{R}_b(\lambda) f\|_D \le C\|f_b\|_D \le C\|f\|. \tag{5.13}$$

Also, by (3.15), (3.16), (4.3) and (4.4) we get

$$\|\nabla\varphi \cdot \nabla(u_e - u_b)\| \le C\left\{\frac{1}{(\mathrm{Re}\,\lambda)^{1/2}} + \frac{1}{(1+|\lambda|)^{1/2}}\right\}\|f\|, \tag{5.14}$$

$$\|(\Delta\varphi)(u_e - u_b)\| \le C\left\{\frac{1}{\mathrm{Re}\,\lambda} + \frac{1}{1+|\lambda|}\right\}\|f\|. \tag{5.15}$$

Collecting (5.12)–(5.15), we obtain $\Delta w \in L^2(\Omega)$. Since w satisfies the boundary condition (5.3), the standard elliptic regularity theory ([2]) for the Laplace operator implies that $w \in H^2(\Omega) \cap H_0^1(\Omega)$. Furthermore, by (5.14) and (5.15) we have $T(\lambda) f \in L^2(\Omega)$. We so get $(k \times x) \cdot \nabla w = \lambda w - \Delta w - f - T(\lambda) f \in L^2(\Omega)$. This completes the proof. □

The following lemma plays an important role for the proof of Proposition 5.1.

Lemma 5.3. *Suppose that* $\lambda \in \mathbf{C}_+$. *The operator* $1 + T(\lambda)$ *has a bounded inverse in* $L^2(\Omega)$, *where* $T(\lambda)$ *is given by (5.11).*

Proof. Since the support of $T(\lambda) f$ is a compact set in D, Lemma 3.3 and Proposition 4.2 imply that $T(\lambda)$ is a bounded operator from $L^2(\Omega)$ into $H_0^1(D)$ for each $\lambda \in \mathbf{C}_+$. The operator $T(\lambda)$ is thus compact in $L^2(\Omega)$. Hence, by the Fredholm alternative it is sufficient to show the injectivity of the operator $1 + T(\lambda)$. To do so, we employ the argument of Iwashita [9, Section 3]. Suppose that $f \in L^2(\Omega)$ fulfills $[1 + T(\lambda)] f = 0$. By (5.10) the function w defined by (5.7) with such f satisfies $\lambda w - \Delta w - (k \times x) \cdot \nabla w = 0$ in Ω as well as (5.3) on Γ. Since $w \in D(\mathcal{L}_0) \subset D(\mathcal{L})$ by Lemma 5.2, it follows from Corollary 2.3 that $w = 0$ in $L^2(\Omega)$. Therefore, for almost all x satisfying $|x| \ge b + 2$, we have $u_e(x) = [\mathcal{R}_e(\lambda) f](x) = 0$, which yields $f(x) = 0$ by (5.8). Similarly, for almost all $x \in \Omega$ satisfying $|x| \le b + 1$, we have $u_b(x) = [\mathcal{R}_b(\lambda) f](x) = 0$, and thus $f(x) = 0$ by (5.9). So the support of f is contained in $\{x; b + 1 \le |x| \le b + 2\}$. We now consider the Dirichlet problem in the ball $B = \{x; |x| < b + 3\}$:

$$\begin{cases} \lambda u - \Delta u - (k \times x) \cdot \nabla u = f, & x \in B, \\ u = 0, & |x| = b + 3. \end{cases} \tag{5.16}$$

Define $\widetilde{u}_b$ on B by $\widetilde{u}_b(x) = u_b(x)$ (if $x \in D$) and by $\widetilde{u}_b(x) = 0$ (if $x \in \mathcal{O} \equiv B \setminus D$). We denote the restriction of u_e on B by the same symbol. Then both $\widetilde{u}_b$ and u_e are solutions to (5.16), and belong to $H^2(B) \cap H_0^1(B)$. By the same argument as in section 2, the solution to (5.16) in the class $H^2(B) \cap H_0^1(B)$ is unique for each $\lambda \in \mathbb{C}_+$, so that $\widetilde{u}_b = u_e$ in $L^2(B)$. Therefore, $u_b = \varphi u_e + (1-\varphi)u_b = w = 0$ in $L^2(D)$. By (5.9) again, $f(x) = 0$ for almost all $x \in D$. After all, we have obtained that $f = 0$ in $L^2(\Omega)$. Thus the operator $1 + T(\lambda)$ is injective in $L^2(\Omega)$. □

Proof of Proposition 5.1. For each $\lambda \in \mathbb{C}_+$ and $f \in L^2(\Omega)$, we set $u = \mathcal{R}(\lambda)\,[1+T(\lambda)]^{-1} f$. It then follows from Lemmas 5.2 and 5.3 that the function u is of class

$$u \in H^2(\Omega) \cap H_0^1(\Omega), \qquad (k \times x) \cdot \nabla u \in L^2(\Omega),$$

with estimate

$$\|\Delta u\| \le C_\lambda \|[1+T(\lambda)]^{-1} f\| \le M_\lambda \|f\|, \tag{5.17}$$

where the constant $M_\lambda > 0$ is independent of $f \in L^2(\Omega)$. So, $u \in D(\mathcal{L}_0) \subset D(\mathcal{L})$ and it is actually the solution to (5.1). The uniqueness of solutions in $D(\mathcal{L})$ has been already proved in Corollary 2.3. As a result, the operator $\lambda + \mathcal{L}$ is bijective for all $\lambda \in \mathbb{C}_+$. We thus obtain $\mathbb{C}_+ \subset \rho(-\mathcal{L})$ with the representation of the resolvent $(\lambda + \mathcal{L})^{-1} = \mathcal{R}(\lambda)[1+T(\lambda)]^{-1}$. □

6. L^2 estimates, proof of Theorem 2 and Theorem 3

This section clarifies the domains of the operators $\mathcal{L}$ and $\mathcal{L}^\alpha$ together with some estimates to prove Theorem 2 and Theorem 3.

Proof of Theorem 2. Given $u \in D(\mathcal{L})$, we set $f = (1+\mathcal{L})u \in L^2(\Omega)$. By the proof of Proposition 5.1 the unique solution u to (5.1) with such f should belong to $D(\mathcal{L}_0)$. We thus obtain $D(\mathcal{L}) = D(\mathcal{L}_0)$. From this it follows that $D(\mathcal{L}) \subset D(A)$. By (5.17) with $\lambda = 1$ we have also

$$\|Au\| \le M_1 \|f\| = M_1 \|(1+\mathcal{L})u\|, \tag{6.1}$$

for all $u \in D(\mathcal{L})$. By (6.1) we get

$$\|(k \times x) \cdot \nabla u\| = \|u + Au - f\| \le (1 + M_1)\|(1+\mathcal{L})u\| + \|u\|. \tag{6.2}$$

Collecting (1.8), (6.1) and (6.2) leads us to

$$\|u\|_{H^2(\Omega)} + \|(k \times x) \cdot \nabla u\| \le C\|(1+\mathcal{L})u\| + C\|u\|.$$

Since $1 + \mathcal{L}$ has a bounded inverse, the above estimate gives (1.5). □

Proof of Theorem 3. We make use of the relation $D(\mathcal{L}) \subset D(A)$ with estimate (6.1). Since both $1 + \mathcal{L}$ and A are m-accretive operators in $L^2(\Omega)$ by Theorem 1, we can apply the Heinz-Kato inequality ([10, 15]). We then get $D(\mathcal{L}^\alpha) = D((1+\mathcal{L})^\alpha) \subset D(A^\alpha)$ for $0 \le \alpha \le 1$ with estimate

$$\|A^\alpha u\| \le e^{\pi\sqrt{\alpha(1-\alpha)}} M_1^\alpha \|(1+\mathcal{L})^\alpha u\|, \tag{6.3}$$

for $u \in D(\mathcal{L}^\alpha)$. Let m, p and α be the exponents as in Theorem 3. Then, combining the well known embedding estimate

$$\|u\|_{W^{m,p}(\Omega)} \le C(m,p)(\|A^\alpha u\| + \|u\|), \qquad u \in D(A^\alpha),$$

with (6.3), we obtain (1.6). □

Remark 6.1. It is possible to give another proof of Theorem 3 without using the operator A. Due to the m-accretivity of $\mathcal{L}$, its purely imaginary powers $\mathcal{L}^{is}$ are bounded operators for all $s \in \mathbb{R}$ with estimate (Kato [10, Section 2], Prüss and Sohr [13, Example 2]): $\|\mathcal{L}^{is}\|_{L^2(\Omega)\to L^2(\Omega)} \le e^{\frac{\pi}{2}|s|}$. An important consequence of the uniform boundedness of $\mathcal{L}^{is}$ for $|s| \le 1$ is the coincidence between $D(\mathcal{L}^\alpha)$ and the complex interpolation space $[L^2(\Omega), D(\mathcal{L})]_\alpha$ for each $0 \le \alpha \le 1$ (see, for instance, Triebel [16]). Since we have already known the relation $D(\mathcal{L}) \subset H^2(\Omega)$ with estimate (1.5) in Theorem 2, it follows from the reiteration theorem, an embedding property and the interpolation of L^p Sobolev spaces ([16]) that

$$\begin{aligned} D(\mathcal{L}^\alpha) &= [L^2(\Omega), D(\mathcal{L})]_\alpha \subset [L^2(\Omega), H^2(\Omega)]_\alpha = \left[L^2(\Omega), [L^2(\Omega), H^2(\Omega)]_{1/2}\right]_{2\alpha} \\ &= [L^2(\Omega), H^1(\Omega)]_{2\alpha} \subset [L^2(\Omega), L^6(\Omega)]_{2\alpha} = L^p(\Omega), \end{aligned}$$

for $0 \le \alpha \le 1/2, 2 \le p \le 6$ and $1/p = 1/2 - 2\alpha/3$. Similarly, for $1/2 \le \alpha \le 1, 2 \le p \le 6$ and $1/p = 1/2 - (2\alpha - 1)/3$, we get

$$D(\mathcal{L}^\alpha) \subset \left[[L^2(\Omega), H^2(\Omega)]_{1/2}, H^2(\Omega)\right]_{2\alpha-1} \subset [H^1(\Omega), W^{1,6}(\Omega)]_{2\alpha-1} \subset W^{1,p}(\Omega).$$

We thus obtain (1.6) for $m = 1$. It remains to show the case $m = 0, 1/2 < \alpha < 3/4, 6 < p < \infty$. But, in this case, we take $2 < q < 3$ satisfying $1/p = 1/q - 1/3$ and utilize (1.6) with $m = 1$ to get $D(\mathcal{L}^\alpha) \subset W^{1,q}(\Omega) \subset L^p(\Omega)$.

7. Smoothing effect, proof of Theorem 4

In this section we derive some smoothing properties of the semigroup $\{e^{-t\mathcal{L}}\}_{t\ge 0}$ obtained in Theorem 1. Consider the Cauchy problem for the evolution equation

$$\frac{du}{dt} + \mathcal{L}u = 0, \qquad t > 0; \quad u(0) = a, \tag{7.1}$$

in $L^2(\Omega)$, which corresponds to the initial boundary value problem (1.1). By Theorem 1 this problem has a strong solution $u(t) = e^{-t\mathcal{L}}a$, provided that $a \in D(\mathcal{L})$. We here intend to prove that $e^{-t\mathcal{L}}a$ is the strong solution to (7.1) under an weaker assumption.

Let the constant $b > 0$ and the bounded domain D be the same as in section 5. We also fix two functions $\eta, \zeta \in C^\infty(\mathbb{R}^3)$ so that

$$0 \le \eta \le 1, \qquad \eta(x) = \begin{cases} 0 & (|x| \le b), \\ 1 & (|x| \ge b+1), \end{cases}$$

$$0 \le \zeta \le 1, \qquad \zeta(x) = \begin{cases} 1 & (|x| \le b+2), \\ 0 & (|x| \ge b+3). \end{cases}$$

Given $a \in L^2(\Omega)$, we define the functions $\underline{a_e} \in L^2(\mathbb{R}^3)$ and $\underline{a_b} \in L^2(D)$ by

$$\underline{a_e}(x) = \begin{cases} \eta(x)a(x) & (x \in \Omega), \\ 0 & (x \in \mathcal{O}), \end{cases} \qquad \underline{a_b}(x) = \zeta(x)a(x) \quad (x \in D), \tag{7.2}$$

which are different from (5.4). The reason why we here introduce the functions η and ζ is Lemma 7.2 below. In what follows, the symbols $\underline{[\,\cdot\,]_e}$ and $\underline{[\,\cdot\,]_b}$ are respectively adopted for other functions. Let $U(t) = e^{-t\mathcal{L}_e}$ be the semigroup given by (3.5) (and Proposition 3.1), and $V(t) = e^{-t\mathcal{L}_b}$ the analytic semigroup obtained in section 4. We set

$$v_e(t) = U(t)\underline{a_e}, \qquad v_b(t) = V(t)\underline{a_b}. \tag{7.3}$$

Making use of the fixed function $\varphi \in C^\infty(\mathbb{R}^3)$ satisfying (5.5), we define the function $v(t) \in L^2(\Omega)$ by

$$v(t) = \varphi v_e(t) + (1-\varphi)v_b(t), \tag{7.4}$$

which satisfies $\|v(t)\| \le 2\|a\|$ for $t \ge 0$ and

$$\|v(t) - a\| \le \|v_e(t) - \underline{a_e}\|_{\mathbb{R}^3} + \|v_b(t) - \underline{a_b}\|_D \to 0 \quad \text{as} \quad t \to 0. \tag{7.5}$$

We have the relations

$$\Delta v = \varphi\Delta v_e + (1-\varphi)\Delta v_b + 2\nabla\varphi \cdot \nabla(v_e - v_b) + (\Delta\varphi)(v_e - v_b), \tag{7.6}$$

$$(k \times x)\cdot\nabla v = \varphi[(k\times x)\cdot\nabla v_e] + (1-\varphi)[(k\times x)\cdot\nabla v_b] + [(k\times x)\cdot\nabla\varphi](v_e - v_b), \tag{7.7}$$

so that

$$\partial_t v = \Delta v + (k\times x)\cdot\nabla v + \Phi[a], \qquad x \in \Omega,\ t > 0, \tag{7.8}$$

as well as the boundary condition $v = 0$ on Γ, where the remaining term is given by

$$\Phi[a](t) = -2\nabla\varphi\cdot\nabla(v_e - v_b) - [\Delta\varphi + (k\times x)\cdot\nabla\varphi](v_e - v_b). \tag{7.9}$$

We first deduce the regularity of the function $v(t)$ defined by (7.4) and justify the equation (7.8) in $L^2(\Omega)$.

Lemma 7.1. *Suppose that $a \in L^2(\Omega)$. Then $v(t) \in D(A)$ for all $t > 0$. If, in addition,*

$$(k \times x) \cdot \nabla a \in H^{-m}(\Omega), \qquad \exists \text{ integer } m \geq 0, \tag{7.10}$$

then $v(t) \in D(\mathcal{L})$ for all $t > 0$, and

$$\mathcal{L}v \in C\left((0,\infty); L^2(\Omega)\right), \qquad v \in C^1\left((0,\infty); L^2(\Omega)\right), \tag{7.11}$$

with

$$\frac{dv}{dt} + \mathcal{L}v = \Phi[a], \qquad t > 0; \quad v(0) = a, \tag{7.12}$$

in $L^2(\Omega)$.

Proof. By (7.6) we make use of Proposition 3.5 for $U(t)$ together with the analyticity of $V(t)$ to obtain $\Delta v(t) \in L^2(\Omega)$ for $t > 0$. Since $v = 0$ on Γ, we get $v(t) \in D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ ([2]). Let us assume (7.10). From this it follows that $(k \times x) \cdot \nabla \underline{a_e} \in H^{-m}(\mathbf{R}^3)$ since

$$\begin{aligned} &\left|\left((k \times x) \cdot \nabla \underline{a_e}, \psi\right)\right| \\ &\leq \|(k \times x) \cdot \nabla a\|_{H^{-m}(\Omega)} \|\eta\psi\|_{H^m(\Omega)} + \|a\| \; \|[(k \times x) \cdot \nabla \eta]\, \psi\| \\ &\leq C\left(\|(k \times x) \cdot \nabla a\|_{H^{-m}(\Omega)} + \|a\|\right) \|\psi\|_{H^m(\mathbf{R}^3)}, \end{aligned} \tag{7.13}$$

for all $\psi \in C_0^\infty(\mathbf{R}^3)$. So, Proposition 3.6 implies that $(k \times x) \cdot \nabla v_e(t) \in L^2(\mathbf{R}^3)$. And, therefore, (7.7) gives $(k \times x) \cdot \nabla v(t) \in L^2(\Omega)$. We thus obtain $v(t) \in D(\mathcal{L})$ for $t > 0$. By (7.6) and (7.7) we have

$$\begin{aligned} &\|\mathcal{L}v(t+h) - \mathcal{L}v(t)\| \\ &\leq \|\mathcal{L}_e v_{e,h}(t)\|_{\mathbf{R}^3} + C\|\nabla v_{e,h}(t)\|_{\mathbf{R}^3} + C\|v_{e,h}(t)\|_{\mathbf{R}^3} \\ &\quad + \|\mathcal{L}_b v_{b,h}(t)\|_D + C\|\nabla v_{b,h}(t)\|_D + C\|v_{b,h}(t)\|_D, \end{aligned}$$

and

$$\begin{aligned} &\left\|\frac{v(t+h) - v(t)}{h} + \mathcal{L}v(t) - \Phi[a](t)\right\| \\ &\leq \left\|\frac{v_{e,h}(t)}{h} + \mathcal{L}_e v_e(t)\right\|_{\mathbf{R}^3} + \left\|\frac{v_{b,h}(t)}{h} + \mathcal{L}_b v_b(t)\right\|_D, \end{aligned}$$

where $v_{e,h}(t) = v_e(t+h) - v_e(t)$, $v_{b,h}(t) = v_b(t+h) - v_b(t)$. Hence, it follows from Proposition 3.7 for $U(t)$ together with the analyticity of $V(t)$ that $\mathcal{L}v(t)$ is continuous in

$L^2(\Omega)$ and that $v(t)$ is differentiable in $L^2(\Omega)$ with the equation of (7.12). Since $\Phi[a](t)$ as well as $\mathcal{L}v(t)$ is continuous, $v(t)$ is of class C^1. The initial condition of (7.12) is satisfied in $L^2(\Omega)$ on account of (7.5). This completes the proof. □

Using Lemma 7.1, we consider $(d/ds)\{e^{-(t-s)\mathcal{L}}v(s)\}$ for $0 < s < t$ to obtain

$$e^{-t\mathcal{L}}a = v(t) - w(t), \qquad t > 0, \tag{7.14}$$

in $L^2(\Omega)$ under the condition (7.10), where

$$w(t) = \int_0^t e^{-(t-s)\mathcal{L}}\,\Phi[a](s)ds. \tag{7.15}$$

By Proposition 3.5 for $U(t)$ together with the analyticity of $V(t)$, it is possible to see that $\Phi[a](t) \in D(\mathcal{L}) \subset D(A)$ for all $t > 0$ with estimates

$$\|\Phi[a](t)\| \le C(t \wedge 1)^{-1/2}\|a\|, \tag{7.16}$$

$$\|A\Phi[a](t)\| \le C(t \wedge 1)^{-3/2}\|a\|, \tag{7.17}$$

$$\|(k \times x) \cdot \nabla\Phi[a](t)\| \le C(t \wedge 1)^{-1}\|a\|, \tag{7.18}$$

where $t \wedge 1 = \min\{t, 1\}$ (see Lemma 7.3 below). The function $w(t)$ is well defined by (7.15) in $L^2(\Omega)$ on account of (7.16). However, (7.17) and (7.18) are too singular near $t = 0$ to prove $w(t) \in D(\mathcal{L})$ directly. In order to control the behavior of $\mathcal{L}\Phi[a](t)$, we impose the further regularity on a as the first step (Lemma 7.5). We begin with the following lemma.

Lemma 7.2. *Suppose that $a \in D(A^\delta)$ with $0 \le \delta < 1$. Then the functions $\underline{a_e}$ and $\underline{a_b}$ defined by (7.2) respectively satisfy*

$$\underline{a_e} \in H^{2\delta}(\mathbb{R}^3), \qquad \underline{a_b} \in D(A_b^\delta).$$

There is a constant $C = C(\delta) > 0$ such that

$$\|\underline{a_e}\|_{H^{2\delta}(\mathbb{R}^3)} \le C\|a\|_{D(A^\delta)}, \tag{7.19}$$

$$\|\underline{a_b}\|_{D(A_b^\delta)} \le C\|a\|_{D(A^\delta)}, \tag{7.20}$$

for all $a \in D(A^\delta)$.

Proof. For a domain G in $\mathbb{R}^3$, we define X_G to be the set of all bounded continuous functions f from the strip $S = \{z \in \mathbb{C}; 0 \le \text{Re } z \le 1\}$ to the space $L^2(G)$ so that:

(7.21) f is analytic on the open strip $S^\circ = \{z \in \mathbb{C}; 0 < \text{Re } z < 1\}$,

(7.22) $$f(z) \in H^2(G) \cap H_0^1(G) \quad \text{if} \quad \text{Re } z = 1,$$

(7.23) $$\|f\|_{X_G} \equiv \max\left\{\sup_{s\in\mathbf{R}} \|f(is)\|_G,\ \sup_{s\in\mathbf{R}} \|f(1+is)\|_{H^2(G)}\right\} < \infty.$$

Then X_G is a Banach space equipped with norm $\|\cdot\|_{X_G}$ and complex interpolation spaces between L^2 Sobolev spaces over the domain $G(=\Omega, \mathbf{R}^3, D)$ are defined in terms of X_G ([16]). For each $a \in D(A^\delta) = [L^2(\Omega), D(A)]_\delta$ and for every $\varepsilon > 0$, there is a function $f_\varepsilon \in X_\Omega$ such that $f_\varepsilon(\delta) = a$ with

(7.24) $$\|f_\varepsilon\|_{X_\Omega} < \|a\|_{[L^2(\Omega),D(A)]_\delta} + \varepsilon \leq C\|a\|_{D(A^\delta)} + \varepsilon.$$

Bearing $\underline{[f_\varepsilon(\delta)]_e} = \underline{a_e}$ and $\underline{[f_\varepsilon(\delta)]_b} = \underline{a_b}$ in mind, we will see that

(7.25) $$\underline{[f_\varepsilon(\cdot)]_e} \in X_{\mathbf{R}^3}, \qquad \underline{[f_\varepsilon(\cdot)]_b} \in X_D.$$

The analyticity of f_ε implies that both

$$S^\circ \ni z \mapsto \int_{\mathbf{R}^3} \underline{[f_\varepsilon(z)]_e}(x)\overline{g(x)}dx = \int_\Omega [f_\varepsilon(z)]\,(x)\eta(x)\overline{g(x)}dx,$$
$$S^\circ \ni z \mapsto \int_D \underline{[f_\varepsilon(z)]_b}(x)\overline{h(x)}dx = \int_\Omega [f_\varepsilon(z)]\,(x)\zeta(x)\overline{h(x)}dx,$$

are analytic for all $g \in L^2(\mathbf{R}^3)$ and $h \in L^2(D)$ since $\eta g, \zeta h \in L^2(\Omega)$. We thus get (7.21). Other conditions (7.22), (7.23), boundedness and continuity are easily verified from the definition (7.2) of the symbols $\underline{[\,\cdot\,]_e}$ and $\underline{[\,\cdot\,]_b}$, so that we obtain (7.25) with estimates

(7.26) $$\left\|\underline{[f_\varepsilon(\cdot)]_e}\right\|_{X_{\mathbf{R}^3}} \leq C\|f_\varepsilon\|_{X_\Omega}, \qquad \left\|\underline{[f_\varepsilon(\cdot)]_b}\right\|_{X_D} \leq C\|f_\varepsilon\|_{X_\Omega},$$

where $C > 0$ is independent of $\varepsilon > 0$. Hence,

$$\underline{a_e} = \underline{[f_\varepsilon(\delta)]_e} \in [L^2(\mathbf{R}^3), H^2(\mathbf{R}^3)]_\delta = H^{2\delta}(\mathbf{R}^3),$$
$$\underline{a_b} = \underline{[f_\varepsilon(\delta)]_b} \in [L^2(D), D(A_b)]_\delta = D(A_b^\delta).$$

Furthermore, it follows from (7.24) and (7.26) that

$$\|\underline{a_e}\|_{H^{2\delta}(\mathbf{R}^3)} \leq C\left\|\underline{[f_\varepsilon(\cdot)]_e}\right\|_{X_{\mathbf{R}^3}} < C\left(\|a\|_{D(A^\delta)} + \varepsilon\right),$$
$$\|\underline{a_b}\|_{D(A_b^\delta)} \leq C\left\|\underline{[f_\varepsilon(\cdot)]_b}\right\|_{X_D} < C\left(\|a\|_{D(A^\delta)} + \varepsilon\right).$$

Since $\varepsilon > 0$ is arbitrary, we obtain (7.19) and (7.20). □

By using Lemma 7.2 we derive the estimates of $\Phi[a](t)$ near $t = 0$ for $a \in D(A^\delta)$.

Lemma 7.3. *Suppose that $a \in D(A^\delta)$ with $0 \le \delta < 1$. Then the function $\Phi[a]$ defined by (7.9) enjoys $\Phi[a](t) \in D(\mathcal{L}) \subset D(A)$ for all $t > 0$. There is a constant $C = C(\delta) > 0$ such that*

$$\|\Phi[a](t)\| \le C(t \wedge 1)^{-(1/2-\delta)_+}\|a\|_{D(A^\delta)}, \tag{7.27}$$

$$\|A\Phi[a](t)\| \le C(t \wedge 1)^{-3/2+\delta}\|a\|_{D(A^\delta)}, \tag{7.28}$$

$$\|(k \times x)\cdot\nabla\Phi[a](t)\| \le C(t \wedge 1)^{-1+\delta}\|a\|_{D(A^\delta)}, \tag{7.29}$$

for all $t > 0$ and $a \in D(A^\delta)$, where $t \wedge 1 = \min\{t, 1\}$ and $(\cdot)_+ = \max\{\,\cdot, 0\}$.

Proof. Since we are concerned with the behavior of $\Phi[a](t)$ near $t = 0$, we have only to investigate the highest order term $[\Delta + (k \times x)\cdot\nabla]\nabla(v_e - v_b)$ for the proof of (7.28) and (7.29). By Lemma 7.2 we have $\underline{a_e} \in H^{2\delta}(\mathbb{R}^3)$. It thus follows from (3.21) together with (7.19) that

$$\|\Delta\,[\nabla v_e(t)]\|_{\mathbb{R}^3} \le Ct^{-3/2+\delta}\,\|\underline{a_e}\|_{H^{2\delta}(\mathbb{R}^3)} \le Ct^{-3/2+\delta}\|a\|_{D(A^\delta)}. \tag{7.30}$$

Similarly, we get

$$\|\nabla\varphi\cdot\{(k \times x)\cdot\nabla[\nabla v_e(t)]\}\| \le C\|D^2 v_e(t)\|_{\mathbb{R}^3} \le Ct^{-1+\delta}\|a\|_{D(A^\delta)}. \tag{7.31}$$

We next consider

$$[\Delta + (k \times x)\cdot\nabla]\nabla v_b = \nabla\,[\Delta v_b + (k \times x)\cdot\nabla v_b] + k \times [\nabla v_b].$$

By Proposition 4.3 we have

$$\begin{aligned}\|[\Delta + (k \times x)\cdot\nabla]\nabla v_b(t)\|_D &\le \left\|\nabla\,[\mathcal{L}_b V(t)\underline{a_b}\,]\right\|_D + \|\nabla V(t)\underline{a_b}\|_D \\ &\le C\left\|\mathcal{L}_b^{3/2}V(t)\underline{a_b}\right\|_D + C\left\|\mathcal{L}_b^{1/2}V(t)\underline{a_b}\right\|_D.\end{aligned}$$

We have only to investigate the first term of the right hand side. Since $\underline{a_b} \in D(A_b^\delta)$ by Lemma 7.2 and since $V(t)$ is a bounded analytic semigroup, it follows from Proposition 4.3 and (7.20) that

$$\left\|\mathcal{L}_b^{3/2}V(t)\underline{a_b}\right\|_D \le Ct^{-3/2+\delta}\,\|\mathcal{L}_b^\delta\underline{a_b}\|_D \le Ct^{-3/2+\delta}\,\|\underline{a_b}\|_{D(A_b^\delta)} \le Ct^{-3/2+\delta}\|a\|_{D(A^\delta)}.$$

Therefore, we obtain

$$(7.32) \qquad \|[\Delta + (k\times x)\cdot\nabla]\nabla v_b(t)\|_D \le C(t\wedge 1)^{-3/2+\delta}\|a\|_{D(A^\delta)}.$$

By (4.6), Proposition 4.3 and (7.20) we have

$$(7.33) \qquad \|(k\times x)\cdot\nabla\,[\nabla v_b(t)]\|_D \le C\left\|\mathcal{L}_b V(t)\underline{a_b}\right\|_D \le Ct^{-1+\delta}\|a\|_{D(A^\delta)}.$$

Combining (7.32) with (7.33) yields

$$(7.34) \qquad \|\Delta\,[\nabla v_b(t)]\|_D \le C(t\wedge 1)^{-3/2+\delta}\|a\|_{D(A^\delta)}.$$

We gather (7.30), (7.31), (7.33) and (7.34) to obtain (7.28) and (7.29). The proof of (7.27) is similar, and is omitted. Since the support of $\Phi[a](t)$ is compact, estimates (7.27)–(7.29) imply that $\Phi[a](t)\in D(\mathcal{L})$. □

In Lemma 7.1 the regularity properties of $v(t)$ have been already deduced. The following lemma gives some estimates of $v(t)$ near $t=0$ for $a\in D(A^\delta)$.

Lemma 7.4. *Suppose that $a\in D(A^\delta)$ with $0\le\delta<1$. Then the function $v(t)$ defined by (7.4) satisfies*

$$(7.35) \qquad \|Av(t)\| \le C(t\wedge 1)^{-1+\delta}\|a\|_{D(A^\delta)},$$

with a constant $C=C(\delta)>0$ independent of $t>0$. In addition, let us assume (7.10). Then there are constants $C=C(\delta)>0$ and $C'=C'(m)>0$ such that

$$(7.36) \qquad \begin{aligned} &\|(k\times x)\cdot\nabla v(t)\| \\ &\le Ct^{-(1/2-\delta)_+}\|a\|_{D(A^\delta)} + C'\,(t\wedge 1)^{-m/2}\left(\|(k\times x)\cdot\nabla a\|_{H^{-m}(\Omega)} + \|a\|\right), \end{aligned}$$

for $t>0$.

Proof. As in (7.31) and (7.33), we have

$$(7.37) \qquad \|\Delta v_e(t)\|_{\mathbf{R}^3} \le Ct^{-1+\delta}\|a\|_{D(A^\delta)}, \quad \|\nabla v_e(t)\|_{\mathbf{R}^3} \le Ct^{-(1/2-\delta)_+}\|a\|_{D(A^\delta)},$$

$$(7.38) \qquad \|\Delta v_b(t)\|_D \le Ct^{-1+\delta}\|a\|_{D(A^\delta)}, \quad \|\nabla v_b(t)\|_D \le Ct^{-(1/2-\delta)_+}\|a\|_{D(A^\delta)}.$$

In view of (7.6), estimates (7.37) and (7.38) imply (7.35). By (7.7), (3.22), (7.13) and (7.38) we get

$$\begin{aligned} &\|(k\times x)\cdot\nabla v(t)\| \\ &\le C(t\wedge 1)^{-m/2}\left\|(k\times x)\cdot\nabla\underline{a_e}\right\|_{H^{-m}(\mathbf{R}^3)} + Ct^{-(1/2-\delta)_+}\|a\|_{D(A^\delta)} + C\|a\| \\ &\le C(t\wedge 1)^{-m/2}\left(\|(k\times x)\cdot\nabla a\|_{H^{-m}(\Omega)} + \|a\|\right) + Ct^{-(1/2-\delta)_+}\|a\|_{D(A^\delta)}, \end{aligned}$$

which gives (7.36). □

In order to derive the smoothing effect for the semigroup $e^{-t\mathcal{L}}$, we assume that a is rather smooth as the first step.

Lemma 7.5. *Suppose that $a \in D(A^{\delta})$ for some $1/2 < \delta < 1$ as well as (7.10). Then $e^{-t\mathcal{L}}a \in D(\mathcal{L}) \subset D(A)$ for all $t > 0$. For each $T > 0$, there is a constant $C_T = C_T(\delta) > 0$ such that*

$$\left\|Ae^{-t\mathcal{L}}a\right\| \le C_T\, t^{-1+\delta}\|a\|_{D(A^{\delta})}, \tag{7.39}$$

$$\left\|\mathcal{L}e^{-t\mathcal{L}}a\right\| \le C_T\, t^{-1+\delta}\|a\|_{D(A^{\delta})} + C'\,(t\wedge 1)^{-m/2}\left(\|(k\times x)\cdot\nabla a\|_{H^{-m}(\Omega)} + \|a\|\right), \tag{7.40}$$

for $0 < t < T$, where $C' = C'(m) > 0$ is the constant in (7.36).

Proof. By (7.27)–(7.29) with $\delta > 1/2$ we have $\Phi[a] \in L^1(0,T;D(\mathcal{L}))$ for all $T > 0$. On account of the closedness of $\mathcal{L}$, the function $w(t)$ defined by (7.15) belongs to $D(\mathcal{L})$ for all $t > 0$ with

$$\mathcal{L}w(t) = \int_0^t e^{-(t-s)\mathcal{L}}\mathcal{L}\Phi[a](s)ds,$$

so that (7.27)–(7.29) give

$$\|\mathcal{L}w(t)\| \le C_T\|a\|_{D(A^{\delta})}, \qquad \|w(t)\| \le CT\|a\|_{D(A^{\delta})}, \tag{7.41}$$

for $0 < t < T$. Also, (6.1) and (7.41) lead us to

$$\|Aw(t)\| \le M_1\|(1+\mathcal{L})w(t)\| \le C_T\|a\|_{D(A^{\delta})}.$$

By (7.14) the above estimate combined with (7.35) implies that $e^{-t\mathcal{L}}a \in D(A)$ for $t > 0$ with (7.39). Likewise, it follows from (7.35), (7.36) and (7.41) that $e^{-t\mathcal{L}}a \in D(\mathcal{L})$ for $t > 0$ with (7.40). This completes the proof. □

We next assume that a is a little smooth and show the same results as Lemma 7.5.

Lemma 7.6. *Suppose that $a \in D(A^{\delta})$ for some $0 < \delta \le 1/2$ as well as (7.10). Then the assertions of Lemma 7.5 hold true.*

Proof. Since $\Phi[a](t) \in D(\mathcal{L})$ by Lemma 7.3, we can apply Lemma 7.5 with a replaced by $\Phi[a](t)$. Indeed, by (7.40) with $m = 0$ we get

$$\begin{aligned}&\left\|\mathcal{L}e^{-(t-s)\mathcal{L}}\Phi[a](s)\right\| \\ &\le C_T\,(t-s)^{-(1-\delta)/2}\|\Phi[a](s)\|_{D(A^{(1+\delta)/2})} + C'\|(k\times x)\cdot\nabla\Phi[a](s)\|,\end{aligned} \tag{7.42}$$

for $0 < s < t < T$. We use the momentum inequality for fractional powers ([15]) to obtain

$$\|\Phi[a](t)\|_{D(A^{(1+\delta)/2})} \le C(t\wedge 1)^{-1+\delta/2}\|a\|_{D(A^{\delta})}, \tag{7.43}$$

from (7.27) and (7.28). In view of (7.42), estimates (7.43) and (7.29) imply that

$$e^{-(t-\cdot)\mathcal{L}}\Phi[a] \in L^1(0,t;D(\mathcal{L})),$$

and, therefore, $w(t) \in D(\mathcal{L})$ for $t > 0$ with estimate

$$\|\mathcal{L}w(t)\| \le C_T\, t^{-1/2+\delta}\|a\|_{D(A^\delta)}, \tag{7.44}$$

for $0 < t < T$. By (6.1), $Aw(t)$ also satisfies the same estimate as (7.44). This combined with (7.35) gives (7.39) for $\delta > 0$ by virtue of (7.14). We also collect (7.35), (7.36) and (7.44) to obtain (7.40) for $\delta > 0$. We have completed the proof. □

To obtain both (7.39) and (7.40) for $\delta = 0$, it is possible to repeat a procedure similar to that in the proof of Lemma 7.6. This will be done below in the proof of Theorem 4.

Proof of (i) of Theorem 4. By Lemma 7.6 we have (7.39) with $\delta = 1/4$. We employ it with a replaced by $\Phi[a](t) \in D(\mathcal{L})$. Estimates (7.27) and (7.28) for $\delta = 0$ together with the momentum inequality then yield

$$\begin{aligned}\left\|Ae^{-(t-s)\mathcal{L}}\Phi[a](s)\right\| &\le C_T\,(t-s)^{-3/4}\|\Phi[a](s)\|_{D(A^{1/4})}\\ &\le C_T\,(t-s)^{-3/4}(s\wedge 1)^{-3/4}\|a\|,\end{aligned} \tag{7.45}$$

for $0 < s < t < T$. Since A is closed, it holds that $w(t) \in D(A)$ for $t > 0$ with estimate

$$\|Aw(t)\| \le C_T\, t^{-1/2}\|a\|, \tag{7.46}$$

for $0 < t < T$. As the first step, we assume (7.10); then we have (7.14). Therefore, it follows from (7.35) with $\delta = 0$ and (7.46) that $e^{-t\mathcal{L}}a \in D(A)$ for $t > 0$ with estimate

$$\left\|Ae^{-t\mathcal{L}}a\right\| \le C_T\, t^{-1}\|a\|, \tag{7.47}$$

for $0 < t < T$. As the next step, we assume only $a \in L^2(\Omega)$ and take $a_j \in C_0^\infty(\Omega)$ such that $a_j \to a$ in $L^2(\Omega)$ as $j \to \infty$. Then estimate (7.47) for a_j implies that $e^{-t\mathcal{L}}a \in D(A)$ for $t > 0$ because A is closed. We obtain also (7.47) for every $a \in L^2(\Omega)$. Hence, the momentum inequality gives (1.9) for $0 < \alpha \le 1$. We have completed the proof. □

Proof of (ii) of Theorem 4. An integration by parts yields

$$|((k\times x)\cdot\nabla\Phi[a](t),\psi)| = |-(\Phi[a](t),(k\times x)\cdot\nabla\psi)| \le \|\,|x|\Phi[a](t)\|\;\|\nabla\psi\|,$$

for $\psi \in C_0^\infty(\Omega)$. This implies that

$$\begin{aligned}&\|(k\times x)\cdot\nabla\Phi[a](t)\|_{H^{-1}(\Omega)}\\ &\le \|\,|x|\Phi[a](t)\|\\ &\le C\left(\|\nabla v_e(t)\|_{\mathbb{R}^3} + \|\nabla v_b(t)\|_D + \|v_e(t)\|_{\mathbb{R}^3} + \|v_b(t)\|_D\right)\\ &\le C(t\wedge 1)^{-1/2}\|a\|.\end{aligned} \tag{7.48}$$

By Lemma 7.6 we have (7.40) with $\delta = 1/4$ and $m = 1$. Using it together with (7.48), (7.27) and (7.28), we obtain

$$\begin{aligned}
&\left\|\mathcal{L}e^{-(t-s)\mathcal{L}}\Phi[a](s)\right\| \\
&\leq C_T\,(t-s)^{-3/4}\|\Phi[a](s)\|_{D(A^{1/4})} \\
&\quad + C'\left\{(t-s)\wedge 1\right\}^{-1/2}\left(\|(k\times x)\cdot\nabla\Phi[a](s)\|_{H^{-1}(\Omega)} + \|\Phi[a](s)\|\right) \\
&\leq C_T\,(t-s)^{-3/4}(s\wedge 1)^{-3/4}\|a\| + C\left\{(t-s)\wedge 1\right\}^{-1/2}(s\wedge 1)^{-1/2}\|a\|,
\end{aligned}$$

for $0 < s < t < T$. Therefore, $w(t) \in D(\mathcal{L})$ for $t > 0$ with estimate

$$\|\mathcal{L}w(t)\| \leq C_T\,t^{-1/2}\|a\|, \tag{7.49}$$

for $0 < t < T$. By (7.14) we gather (7.35), (7.36) and (7.49) to obtain that $e^{-t\mathcal{L}}a \in D(\mathcal{L})$ for all $t > 0$ with estimate (7.40) for $\delta = 0$. Therefore,

$$\mathcal{L}e^{-t\mathcal{L}}a \in C\left([\tau,\infty); L^2(\Omega)\right), \qquad e^{-t\mathcal{L}}a \in C^1\left([\tau,\infty); L^2(\Omega)\right),$$

with

$$\frac{d}{dt}e^{-t\mathcal{L}}a + \mathcal{L}e^{-t\mathcal{L}}a = 0, \qquad t \geq \tau,$$

in $L^2(\Omega)$ for all $\tau > 0$, which implies (1.10) together with (1.11). Applying the momentum inequality to (7.40) with $\delta = 0$, we obtain (1.12) for $0 < \alpha \leq 1$ and $0 < t < 1$. This completes the proof. □

Proof of (iii) of Theorem 4. For $a \in D(\mathcal{L}^\delta) \subset D(A^\delta)$ for $0 < \delta < 1$, we have already obtained (7.40) by Lemmas 7.5 and 7.6. On the other hand, we have also

$$\mathcal{L}^\delta e^{-t\mathcal{L}}a = e^{-t\mathcal{L}}\mathcal{L}^\delta a, \tag{7.50}$$

so that

$$\|\mathcal{L}^\delta e^{-t\mathcal{L}}a\| \leq \|\mathcal{L}^\delta a\|. \tag{7.51}$$

We here observe (7.50) for completeness although the proof is the same as that for generators of analytic semigroups given in literature (for instance [12]). For each $\varepsilon > 0$ and $f \in L^2(\Omega)$, it holds that

$$(\varepsilon + \mathcal{L})^{-\delta} f = \frac{1}{\Gamma(\delta)}\int_0^\infty s^{\delta-1}e^{-s(\varepsilon+\mathcal{L})} f\,ds,$$

where $\Gamma(\cdot)$ is the gamma function. Given $a \in D(\mathcal{L}^\delta)$, we set $f = e^{-t\mathcal{L}}(\varepsilon + \mathcal{L})^\delta a$ in the above formula to obtain $(\varepsilon + \mathcal{L})^{-\delta}e^{-t\mathcal{L}}(\varepsilon + \mathcal{L})^\delta a = e^{-t\mathcal{L}}a$. Therefore, $e^{-t\mathcal{L}}a \in D(\mathcal{L}^\delta)$ with

$(\varepsilon+\mathcal{L})^\delta e^{-t\mathcal{L}}a = e^{-t\mathcal{L}}(\varepsilon+\mathcal{L})^\delta a$. Letting $\varepsilon \to 0$ implies (7.50). Now, by (7.40) and (7.51) we make use of the momentum inequality to get

$$\begin{aligned}\left\|\mathcal{L}^\alpha e^{-t\mathcal{L}}a\right\| &\le C\left\|\mathcal{L}^\delta e^{-t\mathcal{L}}a\right\|^{\frac{1-\alpha}{1-\delta}}\left\|\mathcal{L}e^{-t\mathcal{L}}a\right\|^{\frac{\alpha-\delta}{1-\delta}} \\ &\le Ct^{-\alpha+\delta}\left\|\mathcal{L}^\delta a\right\|^{\frac{1-\alpha}{1-\delta}}\|a\|_{D(A^\delta)}^{\frac{\alpha-\delta}{1-\delta}} \\ &\quad + Ct^{-\frac{m(\alpha-\delta)}{2(1-\delta)}}\left\|\mathcal{L}^\delta a\right\|^{\frac{1-\alpha}{1-\delta}}\left(\|(k\times x)\cdot\nabla a\|_{H^{-m}(\Omega)}+\|a\|\right)^{\frac{\alpha-\delta}{1-\delta}},\end{aligned}$$

for $\delta < \alpha \le 1$ and $0 < t < 1$, which together with (6.3) completes the proof of (1.13). □

Proof of (iv) of Theorem 4. Given $\alpha \in (0,1]$, we fix $\delta \in (0,\alpha)$. Let $a \in C_0^\infty(\Omega) \subset D(\mathcal{L}^\delta)$. Then it follows from (1.13) with $m = 0$ that

$$\left\|\mathcal{L}^\alpha e^{-t\mathcal{L}}a\right\| \le Ct^{-\alpha+\delta}\|a\|_{D(\mathcal{L}^\delta)} + C\|(k\times x)\cdot\nabla a\|,$$

for $0 < t < 1$, which yields (1.15) for $a \in C_0^\infty(\Omega)$. We are also led to (1.14) for $a \in C_0^\infty(\Omega)$ on account of (6.3). We next suppose that $a \in L^2(\Omega)$ and that $(k\times x)\cdot\nabla a \in H^{-1}(\Omega)$. For every $\varepsilon > 0$, we take $a_\varepsilon \in C_0^\infty(\Omega)$ such that $\|a_\varepsilon - a\| < \varepsilon$. By virtue of (1.12) with $m = 1$ we have

$$\left\|\mathcal{L}^\alpha e^{-t\mathcal{L}}a\right\| \le \left\|\mathcal{L}^\alpha e^{-t\mathcal{L}}a_\varepsilon\right\| + Ct^{-\alpha}\varepsilon + Ct^{-\alpha/2}\|(k\times x)\cdot\nabla(a_\varepsilon - a)\|_{H^{-1}(\Omega)},$$

for $0 < t < 1$. We have already obtained (1.15) for $a_\varepsilon \in C_0^\infty(\Omega)$, so that the estimate above gives $\overline{\lim}_{t\to 0}\, t^\alpha\left\|\mathcal{L}^\alpha e^{-t\mathcal{L}}a\right\| \le C\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we arrive at (1.15). By the same approximation procedure as above with the aid of (1.9), we obtain (1.14) for $a \in L^2(\Omega)$. We have completed the proof. □

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References

1. S. Agmon, *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Comm. Pure Appl. Math. **15** (1962), 119–147.
2. S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math. **12** (1959), 623–727; *II*, Comm. Pure Appl. Math. **17** (1964), 35–92.
3. W. Borchers, *Zur Stabilität und Faktorisierungsmethode für die Navier-Stokes Gleichungen inkompressibler viskoser Flüssigkeiten*, Habilitationsschrift, University of Paderborn, 1992.
4. Z.M. Chen and T. Miyakawa, *Decay properties of weak solutions to a perturbed Navier-Stokes system in* $\mathbb{R}^n$, Adv. Math. Sci. Appl. **7** (1997), 741–770.
5. G. DaPrato and A. Lunardi, *On the Ornstein-Uhlenbeck operator in spaces of continuous functions*, J. Funct. Anal. **131** (1995), 94–114.
6. D. Gilbarg and N.S. Trudinger, *"Elliptic Partial Differential Equations of Second Order"*, Springer-Verlag, Berlin, 1977.
7. T. Hishida, *The Stokes operator with rotation effect in exterior domains*, Analysis **19** (1999), 51–67.
8. T. Hishida, *An existence theorem for the Navier-Stokes flow in the exterior of a rotating obstacle*, Arch. Rational Mech. Anal. (to appear).

9. H. Iwashita, *$L_q - L_r$ estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in L_q spaces*, Math. Ann. **285** (1989), 265–288.
10. T. Kato, *Fractional powers of dissipative operators, II*, J. Math. Soc. Japan **14** (1962), 242–248.
11. A. Lunardi and V. Vespri, *Generation of strongly continuous semigroups by elliptic operators with unbounded coefficients in $L^p(\mathbb{R}^n)$*, preprint.
12. A. Pazy, *"Semigroups of Linear Operators and Applications to Partial Differential Equations"*, Springer-Verlag, New York, 1983.
13. J. Prüss and H. Sohr, *On operators with bounded imaginary powers in Banach spaces*, Math. Z. **203** (1990), 429–452.
14. Y. Shibata, *On the global existence of classical solutions of second order fully nonlinear hyperbolic equations with first order dissipation in the exterior domain*, Tsukuba J. Math. **7** (1983), 1–68.
15. H. Tanabe, *"Equations of Evolution"*, Pitman, London, 1979.
16. H. Triebel, *"Interpolation Theory, Function Spaces, Differential Operators"*, North-Holland, Amsterdam, 1978.

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