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# SUBMANIFOLDS OF CODIMENSION 3 ADMITTING ALMOST CONTACT METRIC STRUCTURE IN A COMPLEX PROJECTIVE SPACE

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ABSTRACT. In this paper we prove the following : Let $M$ be a semi-invariant submanifold with almost contact metric structure $(\phi, \xi, g)$ of codimension 3 in a complex projective space $P_{n+1}\mathbb{C}$. Suppose that the third fundamental form $n$ satisfies $dn = 2\theta\omega$ for a certain scalar $\theta(< \frac{c}{2})$, where $\omega(X,Y) = g(X, \phi Y)$ for any vectors $X$ and $Y$ on $M$. Then $M$ has constant eigenvalues correponding the shape operator $A$ in the direction of the distinguished normal and the structure vector $\xi$ is an eigenvector of $A$ if and only if $M$ is locally congruent to a homogeneous real hypersurface of $P_n\mathbb{C}$.

## 0. Introduction

A submanifold $M$ is called a *CR submanifold* of a Kaehlerian manifold $\tilde{M}$ with complex structure $J$ if there exists a differentiable distribution $T : p \to T_p \subset M_p$ on $M$ such that $T$ is J-invariant and the complementary orthogonal distribution $T^{\perp}$ is totally real, where $M_p$ denotes the tangent space to $M$ at each point $p$ in $M$([1], [20]). In particular, $M$ is said to be a *semi-invariant submanifold* provided that dim $T^{\perp} = 1$. The unit normal vector field in $JT^{\perp}$ is called the *distinguished normal* to the semi-invariant submanifold ([18]). A semi-invariant submanifold admits an induced almost contact metric structure, and many results are known by using this structure ([4], [10], [15], etc.).

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A typical example of a semi-invariant submanifold is real hypersurface. When the ambient manifold $\tilde{M}$ is a complex projective space $P_n\mathbb{C}$, real hypersurfaces were investigated by many geometers in connection with the shape operator and the induced almost contact metric structure ([3], [7], [9], [16], [17], etc.). One of them, the third named author asserts that the following :

**Theorem T([17]).** *Let $M$ be a homogeneous real hyperspace of $P_n\mathbb{C}$. Then $M$ is locally congruent to one of the followings:*

($A_1$) *a geodesic hypersphere (that is, a tube over a hyperplane $P_{n-1}\mathbb{C}$),*
($A_2$) *a tube over a totally geodesic $P_k\mathbb{C}(1 \leq k \leq n-2)$,*
(B) *a tube over a complex quadric $Q_{n-1}$,*
(C) *a tube over $P_1\mathbb{C} \times P_{(n-1)/2}\mathbb{C}$ and $n(\geq 5)$ is odd,*
(D) *a tube over a complex Grassman $G_{2,5}\mathbb{C}$ and $n = 9$,*
(E) *a tube over a Hermitian symmetric space $SO(10)/U(5)$ and $n = 15$.*

Cecil-Ryan ([3]) and Kimura ([9]) extensively investigated a real hypersurface which is realized as a tube of constant radius $r$ over a complex submanifold of $P_n\mathbb{C}$ on which $\xi$ is a principal curvature vector.

On the other hand, submanifolds of codimension 3 addmitting an almost contact metric structure in a complex space form have been studied in ([8], [19]) when the normal connection is L-flat or the distinguished normal is parallel in the normal bundle.

The main purpose of the present paper is to extend Theorem T under certain conditions on a semi-invariant submanifold of codimension 3 in a complex projective space $P_{n+1}\mathbb{C}$, and to give new examples of nontrivial semi-invariant submanifolds in $P_{n+1}\mathbb{C}$.

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## 1. Preliminaries

Let $\tilde{M}$ be a real 2(n+1)-dimensional Kaehlerian manifold equipped with parallel almost complex structure $J$ and a Riemannian metric tensor $G$, which J-Hermitian and covered by a system of coordinate neighborhoods $\{W; y^A\}$.

Let $M$ be a real (2n-1)-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; x^h\}$ and immersed isometrically in $\tilde{M}$ by the immersion $i : M \to \tilde{M}$.

Throughout the present paper the following convention on the range of indices are used, unless otherwise stated :

$$A, B, \cdots = 1, 2, \cdots, 2n+2 \ ; \ i, j, \cdots = 1, 2, \cdots, 2n-1.$$

The summation convention will be used with respect to those system of indices. When the argument is local, $M$ need not to be distinguished from $i(M)$.Thus, for simplicity, a point $p$ in $M$ may be identified with $i(p)$ and a tangent vector $X$ at $p$ may also be identified with the tangent vector $i_*(X)$ at $i(p)$ via the differential $i*$ of $i$. We represent the immersion $i$ locally by $y^A = y^A(x^h)$ and $B_j = (B_j{}^A)$ are also (2n-1)-linearly independent local tangent vectors of $M$, where $B_j{}^A = \partial_j y^A$ and $\partial_j = \partial/\partial x^j$. Three mutually orthogonal unit normals $C$, $D$ and $E$ may then be chosen. The induced Riemannian metric tensor $g$ with components $g_{ji}$ on $M$ is given by $g_{ji} = G(B_j, B_i)$ because the immersion $i$ is isometric.

Denoting by $\nabla_j$ the operator of van der Waerden-Bortolotti covariant differentiation with respect to the induced Riemannian metric, equations of the Gauss for $M$ of $\tilde{M}$ is obtained :

$$\nabla_j B_i = A_{ji} C + K_{ji} D + L_{ji} E, \tag{1.1}$$

where $A_{ji}, K_{ji}$ and $L_{ji}$ are components of the second fundamental forms in the direction of normals $C$, $D$ and $E$ respectively.

Equations of the Weingarten are also given by

$$\begin{aligned} \nabla_j C &= -A_j{}^h B_h + l_j D + m_j E, \\ \nabla_j D &= -K_j{}^h B_h - l_j C + n_j E, \\ \nabla_j E &= -L_j{}^h B_h - m_j C - n_j D, \end{aligned} \tag{1.2}$$

where $A = (A_j{}^h), A_{(2)} = (K_j{}^h)$ and $A_{(3)} = (L_j{}^h)$, which are related by $A_{ji} = A_j{}^r g_{ir}, K_{ji} = K_j{}^r g_{ir}$ and $L_{ji} = L_j{}^r g_{ir}$ respectively, and $l_j, m_j$ and $n_j$ being components of the third fundamental forms.

In the sequel, we denote the normal components of $\nabla_j C$ by $\nabla_j{}^{\perp} C$. The normal vector field $C$ is said to be *parallel* in the normal bundle if we have $\nabla_j{}^{\perp} C = 0$, that is, $l_j$ and $m_j$ vanish identically.

On the other hand, a submanifold $M$ is called a *CR submanifold* of a Kaehlerian manifold $\tilde{M}$ if there exists a differentiable distribution $T : p \rightarrow$

$T_p \subset M_p$ on $M$ satisfying the following conditions, where $M_p$ denotes the tangent space to $M$ at each point $p$ in $M$ :

(1) $T$ is invariant, that is, $JT_p = T_p$ for each $p$ in $M$, (2) the complementary orthogonal distribution $T^\perp : p \to {T_p}^\perp \subset M_p$ is totally real, that is, $J{T_p}^\perp \subset {M_p}^\perp$ for each $p$ in $M$, where ${M_p}^\perp$ denotes the normal space to $M$ at $p \in M$([1], [20], [21]). In particular $M$ is said to be a *semi-invariant submanifold* provided that $dim\ T^\perp = 1$. In this case the unit normal vector field in $JT^\perp$ is called a *distinguished normal* to the semi-invariant submanifold and denoted this by $C$([2], [18]). More precisely, we choose an orthonormal basis $e_1, \cdots, e_{n-1}, e_n$ of $M_p$ in such a way that $e_1, \cdots, e_{n-1} \in T$. Then we see that

$$G(Je_n, e_i) = -G(e_n, Je_i) = -G(e_n, \sum_{k=1}^{n-1} F_{ik}e_k) = 0 \text{ for } i = 1, \cdots, n-1.$$

Also we have $G(Je_n, e_n) = 0$ because $J$ is skew-symmetric. Therefore $Je_n$ is orthogonal to $M_p$. We put $C = -Je_n$. Then we can write

$$(1.3) \qquad JB_i = {\phi_i}^h B_h + \xi_i C, \quad JC = -\xi^h B_h, \quad JD = -E, \quad JE = D$$

in each coordinate neighborhood, where we have put $\phi_{ji} = G(JB_j, B_i), \xi_i = G(JB_i, C), \xi^h$ being associated component of $\xi_h$. By the property of the almost Hermitian structure $J$, it is clear that $\phi_{ji}$ is skew-symmetric. A tensor fied of type (1,1) with components ${\phi_i}^h$ will be denoted by $\phi$. By properties of the almost complex structure $J$ it follows that

$$\begin{aligned} &{\phi_i}^r {\phi_r}^h = -{\delta_i}^h + \xi_i\xi^h, \quad \xi^r {\phi_r}^h = 0, \quad \xi_r {\phi_i}^r = 0, \\ &\xi_r \xi^r = 1, \quad g_{rs} {\phi_j}^r {\phi_i}^s = g_{ji} - \xi_j \xi_i. \end{aligned}$$

Since $J$ is parallel, by differentiating the first equation of (1.3) covariantly along $M$ and using (1.1), (1.2) and (1.3), and by comparing the tangential and normal parts, we find (see [19])

$$(1.4) \qquad \nabla_j {\phi_i}^h = -A_{ji}\xi^h + {A_j}^h \xi_i,$$

$$(1.5) \qquad \nabla_j \xi_i = -A_{jr} {\phi_i}^r,$$

(1.6) $$K_{ji} = -L_{jr}\phi_i{}^r - m_j\xi_i,$$

(1.7) $$L_{ji} = K_{jr}\phi_i{}^r + l_j\xi_i.$$

The last two relations give

(1.8) $$K_{jt}\xi^t = -m_j, \quad L_{jt}\xi^t = l_j,$$

(1.9) $$m_t\xi^t = -k, \quad l_t\xi^t = l$$

where $k = T_rA_{(2)}, l = T_rA_{(3)}$.

Here we may assume that $l = 0$. In fact, for a normal vector $v$ of $M$ we denote by $A_v$ the second fundamental tensor of $M$ in the direction of $v$. Then we have $A_{-v} = -A_v$. Hence there is a unit normal vector $D'$ of $M$ in the plane spanned by two vectors $D$ and $E$ such that $T_rA_{D'} = 0$, which proves our assertion. Therefore we have by (1.9)

(1.10) $$l_t\xi^t = 0.$$

Transforming (1.7) by $\phi_k{}^j$ and using (1.6), we obtain

$$-K_{ik} - m_i\xi_k = K_{st}\phi_i{}^s\phi_k{}^t + \xi_i\phi_{kt}l^t,$$

which implies

$$m_k\xi_i - m_i\xi_k = \xi_i\phi_{kt}l^t - \xi_k\phi_{it}l^t,$$

or, using (1.9)

(1.11) $$\phi_{it}l^t = m_i + k\xi_i.$$

Similarly we have

$$(1.12)\qquad \phi_{ir}m^r = -l_i$$

because of (1.10).

Transforming (1.6) and (1.7) by $L_k{}^i$ and using (1.6), (1.7) and (1.8), we have respectively

$$(1.13)\qquad K_{jr}L_i{}^r + K_{ir}L_j{}^r = -(l_j m_i + l_i m_j),$$

$$(1.14)\qquad L_{ji}^2 - K_{ji}^2 = l_j l_i - m_j m_i.$$

The ambient Kaehlerian manifold $\tilde{M}$ is assumed to be of constant holomorphic sectional curvature c, which is called a *complex space form* and denoted by $M_{n+1}(c)$. Then equations of the Gauss and Codazzi are given by

$$(1.15)\qquad \begin{aligned} R_{kjih} =& \frac{c}{4}(g_{kh}g_{ji} - g_{jh}g_{ki} + \phi_{kh}\phi_{ji} - \phi_{jh}\phi_{ki} - 2\phi_{kj}\phi_{ih}) \\ &+ A_{kh}A_{ji} - A_{jh}A_{ki} + K_{kh}K_{ji} - K_{jh}K_{ki} \\ &+ L_{kh}L_{ji} - L_{jh}L_{ki}, \end{aligned}$$

$$(1.16)\qquad \begin{aligned} &\nabla_k A_{ji} - \nabla_j A_{ki} - l_k K_{ji} + l_j K_{ki} - m_k L_{ji} + m_j L_{ki} \\ &= \frac{c}{4}(\xi_k\phi_{ji} - \xi_j\phi_{ki} - 2\xi_i\phi_{kj}), \end{aligned}$$

$$(1.17)\qquad \nabla_k K_{ji} - \nabla_j K_{ki} + l_k A_{ji} - l_j A_{ki} - n_k L_{ji} + n_j L_{ki} = 0,$$

$$(1.18)\qquad \nabla_k L_{ji} - \nabla_j L_{ki} + m_k A_{ji} - m_j A_{ki} + n_k K_{ji} - n_j K_{ki} = 0,$$

where $R_{kjih}$ is covariant components of the Riemann-Christoffel curvature tensor of $M$, and those of the Ricci by

$$\nabla_k l_j - \nabla_j l_k + A_{kr}K_j{}^r - A_{jr}K_k{}^r + m_k n_j - m_j n_k = 0, \tag{1.19}$$

$$\nabla_k m_j - \nabla_j m_k + A_{kr}L_j{}^r - A_{jr}L_k{}^r + n_k l_j - n_j l_k = 0, \tag{1.20}$$

$$\nabla_k n_j - \nabla_j n_k + K_{kr}L_j{}^r - K_{jr}L_k{}^r + l_k m_j - l_j m_k = \frac{c}{2}\phi_{kj}. \tag{1.21}$$

In the following we need the following definition. The normal connection of a semi-invariant submanifold $M$ of codimension 3 in a complex space form is said to be *L-flat* if it satisfies $dn = \frac{c}{2}\omega$, that is, $\nabla_j n_i - \nabla_i n_j = \frac{c}{2}\phi_{ji}$, where $\omega(X,Y) = g(X,\phi Y)$ for any vectors $X$ and $Y$ on $M$ (p514, [13]).

Differentiating $A\xi = \alpha\xi$ covariantly along $M$, and using (1.5), we find

$$\xi^r \nabla_k A_{jr} = A_{jr}A_{ks}\phi^{rs} - \alpha A_{kr}\phi_j{}^r + (\nabla_k \alpha)\xi_j, \tag{1.22}$$

which together with (1.8) and (1.16) yields

$$\begin{aligned} 2A_{jr}A_{ks}\phi^{rs} - \alpha(A_{kr}\phi_j{}^r - A_{jr}\phi_k{}^r) + \frac{c}{2}\phi_{kj} \\ = \xi_k \nabla_j \alpha - \xi_j \nabla_k \alpha + 2(m_k l_j - m_j l_k). \end{aligned} \tag{1.23}$$

Transvecting $\xi^k$ to this and using $A\xi = \alpha\xi$ , (1.8) and (1.10), we obtain

$$\nabla_j \alpha - (\xi^t \nabla_t \alpha)\xi_j = 2k l_j. \tag{1.24}$$

## 2. The third fundamental forms of semi-invariant submanifolds

In the rest of this paper we shall suppose that $M$ is a real $(2n-1)$-dimensional semi-invariant submanifold of codimension 3 in a complex projective space $P_{n+1}\mathbb{C}$ and that the third fundamental form $n$ satisfies $dn = 2\theta\omega$ for a certain scalar $\theta$ on $M$, that is,

$$(2.1) \qquad \nabla_j n_i - \nabla_i n_j = 2\theta\phi_{ji}.$$

Then we have by (1.21)

$$K_{jr}L_i{}^r - K_{ir}L_j{}^r + l_j m_i - l_i m_j = -2(\theta - \frac{c}{4})\phi_{ji},$$

or, using (1.13)

$$(2.2) \qquad K_{jr}L_i{}^r + l_j m_i = -(\theta - \frac{c}{4})\phi_{ji},$$

which together with (1.8), (1.9) and (1.10) yields

$$(2.3) \qquad K_{jr}l^r = kl_j, \quad L_{jr}m^r = 0.$$

**Remark 2.1.** To write our formulas in a convention form, in the sequel we denote by $h_{(2)} = A_{ji}A^{ji}$, $h = g^{ji}A_{ji}$, $\alpha = A_{ji}\xi^j\xi^i$, $K_{(2)} = K_{ji}K^{ji}$ and $L_{(2)} = L_{ji}L^{ji}$.

Multiplying (2.2) with $\phi^{ji}$ and summing for $j$ and $i$, and using (1.6), (1.8) and (1.11), we find

$$K_{(2)} - k^2 = 2(n-1)(\theta - \frac{c}{4}),$$

which together with (1.8) implies that

$$(2.4) \qquad \| K_{ji} - k\xi_j\xi_i \|^2 = 2(n-1)(\theta - \frac{c}{4}),$$

where $\| F \|^2 = g(F, F)$ for any tensor field $F$ on $M$.

In the same way, we have from (1.7), (1.10), (1.12) and (2.2)

$$(2.5) \qquad L_{(2)} = 2(n-1)(\theta - \frac{c}{4}).$$

Differentiating (2.1) covariantly along $M$ and using (1.4), we obtain

$$\nabla_k(\nabla_j n_i - \nabla_i n_j) = 2(\nabla_k\theta)\phi_{ji} + 2\theta(A_{ki}\xi_j - A_{kj}\xi_i),$$

or, using the first Bianchi identity,

$$(\nabla_k\theta)\phi_{ji} + (\nabla_j\theta)\phi_{ik} + (\nabla_i\theta)\phi_{kj} = 0,$$

which implies $(n-2)\nabla_k\theta = 0$. Thus $\theta(\geq \frac{c}{4})$ is constant if $n > 2$.

**Lemma 2.1.** *Let $M$ be a semi-invariant submanifold of codimension 3 with L-flat normal connection in a complex projective space $P_{n+1}\mathbb{C}$. If the structure vector $\xi$ is an eigenvector of the shape operator $A$ in the direction of the distinguished normal, then we have $A_{(2)} = A_{(3)} = 0$ and $\nabla_j^{\perp} C = 0$.*

**Remark 2.2.** This lemma was proved in [8]. But we give a simpler proof of it here.

*Proof.* By the hypotheses we have $\theta = \frac{c}{4}$. Thus (2.4) and (2.5) are reduce respectively to

$$K_{ji} = k\xi_j\xi_i, \; L_{ji} = 0$$

and hence $m_j = -k\xi_j$ and $l_j = 0$ because of (1.8). It sufficies to show that $k = 0$. In this case (1.19) turns out to be

$$k(\xi_j A_{kr}\xi^r - \xi_k A_{jr}\xi^r) = k(\xi_k n_j - \xi_j n_k),$$

which together with $A\xi = \alpha\xi$ gives

$$k(n_j - x\xi_j) = 0,$$

where $x = n_t\xi^t$.

We also have by (1.18)

$$k\{\xi_k(A_{ji} + n_j\xi_i) - \xi_j(A_{ki} + n_k\xi_i)\} = 0,$$

which implies

$$k(h - \alpha) = 0.$$

Now, let $\Omega_0$ be a set of points such that $k \neq 0$ on $M$ and suppose that $\Omega_0$ be non void. Then we have

$$h - \alpha = 0, \quad n_j = x\xi_j$$

on $\Omega_0$. Differentiating the last equation covariantly along $\Omega_0$ and using (1.5), we find

$$\nabla_k n_j = (\nabla_k x)\,\xi_j - xA_{kr}\phi_j{}^r.$$

Since it is assumed to be $A\xi = \alpha\xi$ and (2.1) with $\theta = \frac{c}{4}$, we verified that

$$\frac{c}{2}\phi_{kj} + x(A_{kr}\phi_j{}^r - A_{jr}\phi_k{}^r) = 0,$$

a contradiction because of $h - \alpha = 0$. This completes the proof.

Transforming (2.2) by ${\phi_k}^i$ and taking account of (1.6) and (1.12), we have

$$K_{jk}^{2} + \xi_j(K_{kr}m^r) + l_j l_k = (\theta - \frac{c}{4})(g_{jk} - \xi_j\xi_k), \tag{2.6}$$

which enable us to obtain

$$\xi_j(K_{kr}m^r) - \xi_k(K_{jr}m^r) = 0.$$

Therefore we have

$$K_{kr}m^r = -(m_r m^r)\xi_k, \tag{2.7}$$

because of (1.8). Thus it follows that

$$K_{ji}{}^2 + l_j l_i - (m_r m^r)\xi_j\xi_i = (\theta - \frac{c}{4})(g_{ji} - \xi_j\xi_i). \tag{2.8}$$

In the same way, we have from (2.2)

$$L_{jr}l^r = km_j + (l_t l^t + k^2)\xi_j. \tag{2.9}$$

Transvecting (2.2) with $m^i$ and making use of (1.11) and (2.3), we obtain

$$(\theta - \frac{c}{4} - m_r m^r)l_j = 0.$$

Similary, we verify, using (2.2) and (2.9), that

$$(\theta - \frac{c}{4} - l_r l^r - k^2)(m_t m^t - k^2) = 0.$$

Now, let $\Omega$ be a set of points such that $l_t l^t \neq 0$ on $M$ and suppose that $\Omega$ be non-empty. Then we have

$$m_r m^r = \theta - \frac{c}{4}, \quad l_r l^r + k^2 = \theta - \frac{c}{4} \tag{2.10}$$

on $\Omega$. From now on, we discuss our arguments on the open subset $\Omega$ of $M$. Then (2.8) turns out to be

$$K_{ji}^{\ 2} = (\theta - \frac{c}{4})g_{ji} - l_j l_i. \tag{2.11}$$

Differentiating this covariantly along $\Omega$, we find

$$K_j^{\ r}\nabla_k K_{ir} + K_i^{\ r}\nabla_k K_{jr} + l_j\nabla_k l_i + l_i\nabla_k l_j = 0, \tag{2.12}$$

from which, taking the skew-symmetric part with respect to indices $k$ and $j$ and making use of (1.17) and (1.19),

$$K_j^{\ r}\nabla_k K_{ir} - K_k^{\ r}\nabla_j K_{ir} + l_j\nabla_k l_i - l_k\nabla_j l_i + K_i^{\ r}(l_j A_{kr} - l_k A_{jr} + n_k L_{jr} - n_j L_{kr}) + l_i(A_j^{\ r}K_{kr} - A_k^{\ r}K_{jr} + n_k m_j - n_j m_k) = 0$$

for any indices $k$, $j$ and $i$. Thus, interchanging indices $k$ and $i$, we have

$$K_j^{\ r}\nabla_i K_{kr} - K_i^{\ r}\nabla_j K_{kr} + l_j\nabla_i l_k - l_i\nabla_j l_k + l_j A_{ir}K_k^{\ r} - l_i A_{jr}K_k^{\ r} + n_i K_k^{\ r}L_{jr} - n_j K_k^{\ r}L_{ir} + l_k(K_i^{\ r}A_{jr} - K_j^{\ r}A_{ir} + n_i m_j - n_j m_i) = 0.$$

Hence, if we use (1.13), (1.17), (1.19) and (2.2), then we get

$$K_j^{\ r}\nabla_k K_{ir} - K_i^{\ r}\nabla_k K_{jr} + l_j\nabla_k l_i - l_i\nabla_k l_j + 2l_j A_{kr}K_i^{\ r} - 2l_i A_{kr}K_j^{\ r} + 2(\theta - \frac{c}{4})n_k\phi_{ji} = 0.$$

Adding this to (2.12), we obtain

$$K_j^{\ r}\nabla_k K_{ir} + l_j(\nabla_k l_i + A_{kr}K_i^{\ r}) - l_i A_{kr}K_j^{\ r} + (\theta - \frac{c}{4})n_k\phi_{ji} = 0. \tag{2.13}$$

Since we have (1.7), (2.3) and (2.11), by transforming $K_h^{\ j}$, we have

(2.14)
$$(\theta - \frac{c}{4})(\nabla_k K_{hi} - n_k L_{hi} + n_k l_h \xi_i - l_i A_{hk}) - l_h(l^r \nabla_k K_{ir})$$
$$+ k l_h(\nabla_k l_i + A_{kr} K_i{}^r) + (A_{kr} l^r) l_h l_i = 0.$$

On the other hand, differentiating the first equation of (2.3) covariantly along $\Omega$, we find

$$l^r \nabla_k K_{jr} + K_j{}^r \nabla_k l_r = k \nabla_k l_j + (\nabla_k k) l_j,$$

which, transvecting $l^j$ and using (2.10),

$$(\nabla_k K_{ji}) l^j l^i = (\theta - \frac{c}{4} - k^2) \nabla_k k.$$

Thus, if we transvect $l^i$ to (2.14) and use (2.9) and (2.10), then we obtain

$$(2.15) \qquad (\nabla_k K_{jr}) l^r = l_j \nabla_k k - l_j A_{kr} l^r + (\theta - \frac{c}{4} - k^2) A_{jk} + n_k\{k m_j + (\theta - \frac{c}{4})\xi_j\}$$

because $\theta - \frac{c}{4} \neq 0$ on $\Omega$, from which, taking the skew-symmetric part and making use of (2.9),

$$(2.16) \qquad l_j(2A_{kr} l^r - \nabla_k k) = l_k(2A_{jr} l^r - \nabla_j k).$$

Therefore it follows that

$$(2.17) \qquad 2A_{jr} l^r - \nabla_j k = \sigma l_j$$

for some function $\sigma$ on $\Omega$. By means of (2.15) and (2.17), the equation (2.14) turns out to be

$$(2.18) \qquad (\theta - \frac{c}{4})(\nabla_k K_{ji} - n_k L_{ji} - l_i A_{jk} - l_j A_{ik}) + \sigma l_k l_j l_i - k l_j n_k m_i + k^2 l_j A_{ik} + k l_j(\nabla_k l_i + A_{kr} K_i{}^r) = 0,$$

from which, taking the skew-symmetric part with respect to $j$ and $i$,

$$kl_j(kA_{ik} - n_k m_i + \nabla_k l_i + A_{kr}K_i{}^r) = kl_i(kA_{jk} - n_k m_j + \nabla_k l_j + A_{kr}K_j{}^r).$$

If we transvect $l^j$ to this and make use of (2.17), we get

$$k(l_t l^t)(kA_{ik} - n_k m_i + \nabla_k l_i + A_{kr}K_i{}^r) = k^2 \sigma l_i l_k.$$

From this and (2.18), we have

$$\nabla_k K_{ji} = n_k L_{ji} + l_i A_{jk} + l_j A_{ik} + \tau l_j l_k l_i \tag{2.19}$$

for some function $\tau$ on $\Omega$. Multiplying $g^{ji}$ to (2.19) and summing for $j$ and $i$, and using (2.17) we have

$$(l_t l^t)\tau = -\sigma. \tag{2.20}$$

Differentiating the first equation of (1.8) covariantly and taking account of (1.5), (1.6) and (2.19), we obtain

$$\nabla_k m_j = -n_k l_j - A_{kr} L_j{}^r. \tag{2.21}$$

Differentiating the first equation of (1.9) covariantly and using (1.11) and (2.21), we find

$$\nabla_j k = 2A_{jr} l^r, \tag{2.22}$$

which together with (2.17) implies that $\tau l^2 = -\sigma$. This means that $\sigma = \tau = 0$ on $\Omega$ by virtue of (2.20). Therefore (2.19) reduces to

$$\nabla_k K_{ji} = n_k L_{ji} + l_i A_{jk} + l_j A_{ik}. \tag{2.23}$$

Substituting (2.23) into (2.13), we obtain

$$n_k K_{jr} L_i{}^r + kl_j A_{ki} + l_j(\nabla_k l_i + A_{kr} K_i{}^r) + (\theta - \frac{c}{4}) n_k \phi_{ji} = 0,$$

which transvect $l^j$ and using (1.11), (2.9) and (2.10),

(2.24) $$\nabla_k l_j = n_k m_j - A_{kr}K_j{}^r - kA_{jk}.$$

Differentiating (1.7) covariantly and using (1.4), (1.5), (1.11) and (2.24), we also find

(2.27) $$\nabla_k L_{ji} = -n_k K_{ji} + m_j A_{ik} + m_i A_{jk}.$$

Differentiating (2.22) covariantly along $\Omega$ and taking account of (2.24), we get

$$\begin{aligned}\nabla_k\nabla_j k = &\ 2(\nabla_k A_{jr})l^r + 2A_j{}^r(n_k m_r - A_{ks}K_r{}^s - kA_{kr}) \\ &+ n_j(2A_{kr}m^r - kn_k),\end{aligned}$$

from which, taking the skew-symmetric part and making use of (1.11), (1.16), (2.3) and (2.9),

$$(\theta - \frac{c}{2})(m_k\xi_j - m_j\xi_k) = 0.$$

Therefore it follows that $(\theta - \frac{c}{2})(m_j + k\xi_j) = 0$ and hence $\theta = \frac{c}{2}$ on $\Omega$ because of (2.10). Thus we have by the first equation of (1.2)

**Lemma 2.2.** *Let $M$ be a semi-invariant submanifold of codimension 3 in $P_{n+1}\mathbb{C}$ satisfying (2.1). If $\theta \neq \frac{c}{2}$, then we have $\nabla_j{}^{\perp}C = -k\xi_j E$ on $M$.*

## 3. Further properties of the third fundamental forms

We continue now, our arguments under the same hypotheses (2.1) as in section 2. Furthermore suppose, throughout this section, that $\theta \neq \frac{c}{2}$ holds and that the structure vector $\xi$ satisfies $A_{jr}\xi^r = \alpha\xi_j$. Then we have by Lemma 2.2

(3.1) $$l_j = 0$$

and hence

(3.2) $$m_j = -k\xi_j$$

because of (1.2). Thus (1.6), (1.7), (1.8), (1.13) and (1.14) are recuded respectively to

(3.3) $$L_{jr}\phi_i{}^r = -K_{ji} + k\xi_j\xi_i,$$

(3.4) $$K_{jr}\phi_i{}^r = L_{ji},$$

(3.5) $$K_{jr}\xi^r = k\xi_j, \quad L_{jr}\xi^r = 0,$$

(3.6) $$L_{jr}K_i{}^r + L_{ir}K_j{}^r = 0,$$

(3.7) $$L_{ji}{}^2 = K_{ji}{}^2 - k^2\xi_j\xi_i.$$

From (3.2) we have

$$\nabla_k m_j = -\xi_j\nabla_k k + kA_{kr}\phi_j{}^r,$$

from which, taking the skew-symmetric part and using (1.20), (3.1) and (3.2),

$$A_{kr}L_j{}^r - A_{jr}L_k{}^r + k(A_{kr}\phi_j{}^r - A_{jr}\phi_k{}^r) = \xi_j\nabla_k k - \xi_k\nabla_j k.$$

Since we have $A\xi = \alpha\xi$, we then have

(3.8) $$\nabla_k k = \lambda\xi_k$$

because of (3.5), where $\lambda = \xi^t\nabla_t k$.

From the last two equations, it is clear that

(3.9) $$A_{kr}L_j{}^r - A_{jr}L_k{}^r = k(A_{jr}\phi_k{}^r - A_{kr}\phi_j{}^r).$$

Similarly, we also have from (1.19), (3.1) and (3.2)

(3.10) $$k(n_j - \mu\xi_j) = 0,$$

(3.11) $$A_{kr}K_j{}^r - A_{jr}K_k{}^r = 0,$$

where $\mu = kn_t\xi^t$.

**Lemma 3.1.** *Let $M$ be a semi-invariant submanifold of codimension 3 in $P_{n+1}\mathbb{C}$ satisfying $dn = 2\theta\omega$, $(\theta \neq \frac{c}{2})$. If it satisfies $A\xi = \alpha\xi$, then $T_r A_{(2)} =$ const.*

*Proof.* Differentiating (3.8) covariantly and making use of (1.5), we find

$$\nabla_k \nabla_j \lambda = \xi_j \nabla_k \lambda - \lambda A_{kr} \phi_j{}^r,$$

which together with $A\xi = \alpha\xi$ yields

$$\lambda(A_{jr}\phi_i{}^r - A_{ir}\phi_j{}^r) = 0. \tag{3.12}$$

On the other hand, by means of (3.1), the equation (1.24) becomes $\nabla_j \alpha = (\xi^t \nabla_t \alpha)\xi_j$. Hence (1.23) implies $\lambda(A_{jr}{}^2 \phi_k{}^r + \frac{c}{4}\phi_{kj}) = 0$ because of (3.1) and (3.12). By the properties of the almost contact metric structure, it follows that

$$\lambda\{h_{(2)} - \alpha^2 + \frac{c}{2}(n-1)\} = 0,$$

which means

$$\lambda\{\|A_{ji} - \alpha\xi_j\xi_i\|^2 + \frac{c}{2}(n-1)\} = 0.$$

Hence $\lambda = 0$ by virtue of $c > 0$ and thus $k =$ *const.* because of (3.8). This complete the proof of Lemma 3.1.

In the following we discuss our arguments the case where $k \neq 0$. Then by (3.10) we have

$$n_j = \mu\xi_j.$$

From this we have

$$\nabla_k n_j = \xi_j \nabla_k \mu - \mu A_{kr} \phi_j{}^r,$$

which implies

$$2\theta\phi_{kj} = \xi_j \nabla_k \mu - \xi_k \nabla_j \mu - \mu(A_{kr}\phi_j{}^r - A_{jr}\phi_k{}^r).$$

$\xi$ being an eigenvector with respect to $A$, it is seen that

$$A_{kr}\phi_j{}^r - A_{jr}\phi_k{}^r = 2\rho\phi_{kj}, \tag{3.13}$$

where we have put $\rho\theta = -\mu$. Thus (3.9) turns out to be

$$A_{jr}L_i{}^r - A_{ir}L_j{}^r = 2\rho k \phi_{ij}. \tag{3.14}$$

Using (1.24), (3.1) and (3.13), the relationship (1.23) becomes

$$A_{jr}A_{ks}\phi^{rs} = (\rho\alpha - \frac{c}{4})\phi_{kj}. \tag{3.15}$$

Applying (3.13) by $A_i{}^j$ and using (3.15), we obtain

$$(\rho\alpha - \frac{c}{4})\phi_{ki} = A_{ir}{}^2\phi_k{}^r + 2\rho A_{ir}\phi_k{}^r.$$

Thus, it follows that

$$A_{ji}^2 + 2\rho A_{ji} = (\rho\alpha - \frac{c}{4})g_{ji} + (\alpha^2 + \rho\alpha + \frac{c}{4})\xi_j\xi_i. \tag{3.16}$$

**Lemma 3.2.** *$\rho$ is nonzero constant if $n > 2$.*

*Proof.* Since we have $\theta\rho = -\mu$, $\rho$ does not vanish because we have $\theta \geq \frac{c}{4}$ and $n_j = \mu\xi_j$.

Differentiating (3.13) covariantly and taking account of (1.4) and (3.16), we find

$$\begin{aligned}&(\nabla_k A_{jr})\phi_i{}^r - (\nabla_k A_{ir})\phi_j{}^r - 2(\nabla_k\rho)\phi_{ji}\\ &= \{\alpha A_{ik} + (\rho\alpha - \frac{c}{4})g_{ik}\}\xi_j - \{\alpha A_{jk} + (\rho\alpha - \frac{c}{4})g_{jk}\}\xi_i.\end{aligned}$$

If we take the cyclic sum with respect to $k, j$ and $i$, and make use of (1.16), then we have

$$(\nabla_k\rho)\phi_{ji} + (\nabla_j\rho)\phi_{ik} + (\nabla_i\rho)\phi_{kj} = 0.$$

Thus, $\rho$ is constant for $n > 2$. This completes the proof of the lemma.

**Lemma 3.3.** *$\alpha$ and $h$ are constant if $k \neq 0$.*

*Proof.* Since we have $\nabla_j\alpha = (\xi^t\nabla_t\alpha)\xi_j$ as is already seen, we can verify, using the same method as in the proof of Lemma 3.1, that $\xi^t\nabla_t\alpha = 0$ and hence $\alpha$ is constant. From (3.13) we obtain

$$\alpha - h = 2(n-1)\rho. \tag{3.17}$$

Thus $h$ is constant because of Lemma 3.2. Therefore Lemma 3.3 is proved.

Since (2.6) is valid by the assumption (2.1), it is , using (3.1), (3.2) and (3.5), verify that

$$K_{ji}^{\ 2} = (\theta - \frac{c}{4})g_{ji} + (k^2 - \theta + \frac{c}{4})\xi_j\xi_i. \tag{3.18}$$

Differentiating (3.18) covariantly and using (1.5), we have

$$\begin{aligned} &K_i^{\ r}(\nabla_k K_{jr}) + K_j^{\ r}(\nabla_k K_{ir}) \\ &= -(k^2 - \theta + \frac{c}{4})(\xi_j A_{kr}\phi_i^{\ r} + \xi_i A_{kr}\phi_j^{\ r}) \end{aligned} \tag{3.19}$$

because $\theta$ and $k$ are both constant.

Using the same method as that used to (2.13) from (2.12), we can derive from (3.19) the following :

(3.20)

$$\begin{aligned} K_j^{\ r}\nabla_k K_{ir} = &- (\theta - \frac{c}{4})n_k\phi_{ji} + \rho(k^2 - \theta + \frac{c}{4})(\xi_k\phi_{ji} + \xi_i\phi_{jk} + \xi_j\phi_{ki}) \\ &- (k^2 - \theta + \frac{c}{4})\xi_j A_{kr}\phi_i^{\ r}, \end{aligned}$$

where we have used (1.17), (3.13) and (3.14). Transvecting $\xi^j$ to this, we get

$$k\xi^r\nabla_k K_{ir} = -(k^2 - \theta + \frac{c}{4})(A_{kr}\phi_i^{\ r} - \rho\phi_{ki}).$$

On the other hand, differentiating the first equation of (3.5) covariantly and taking account of (1.5) and (3.4), we obtain

$$\xi^r\nabla_k K_{ir} = -A_{kr}L_i^{\ r} - kA_{kr}\phi_i^{\ r} \tag{3.21}$$

From the last two equations, it follows that

$$-kA_{kr}L_i^{\ r} = (\theta - \frac{c}{4})A_{kr}\phi_i^{\ r} + \rho(k^2 - \theta + \frac{c}{4})\phi_{ki}. \tag{3.22}$$

Transforming this by $K_j^{\ i}$ and making of (2.2), (3.1) and (3.4), we find

$$(\theta - \frac{c}{4})(A_{kr}L_j{}^r + kA_{kr}\phi_j{}^r) = \rho(k^2 - \theta + \frac{c}{4}),$$

which together with (3.22) yields

$$(3.23) \qquad (k^2 - \theta + \frac{c}{4})\{\rho k L_{ji} - (\theta - \frac{c}{4})(A_{jr}\phi_i{}^r - \rho\phi_{ji})\} = 0.$$

Transforming (3.20) by $K_l{}^j$ and making use of (3.4), (3.5), (3.18) and (3.21), we find

$$(\theta - \frac{c}{4})(\nabla_k K_{li} - n_k L_{li}) = (k^2 - \theta + \frac{c}{4})\{\xi_l(A_{kr}L_i{}^r + \rho k\phi_{ki}) - \rho(\xi_k L_{li} + \xi_i L_{lk}\},$$

from which, taking the skew-symmetric part with respect to indices $l$ and $i$,

$$(k^2 - \theta + \frac{c}{4})\{\xi_l(A_{kr}L_i{}^r + \rho k\phi_{ki} + \rho L_{ki}) - \xi_i(A_{kr}L_l{}^r + \rho k\phi_{kl} + \rho L_{kl})\} = 0.$$

From the last two equations, it follows that

$$(3.24) \qquad \nabla_k K_{ji} = n_k L_{ji} - a(\xi_k L_{ji} + \xi_i L_{jk} + \xi_j L_{ki}),$$

where we have put

$$(3.25) \qquad a(\theta - \frac{c}{4}) = \rho(k^2 - \theta + \frac{c}{4}).$$

Differentiating (3.4) covariantly and using (1.4) and (3.24), we can verify that

(3.26)
$$\begin{aligned}\nabla_k L_{ji} = &- n_k K_{ji} + a(\xi_k K_{ji} + \xi_j K_{ki} + \xi_i K_{kj}) - k(\xi_j A_{ki} + \xi_i A_{kj}) \\ &+ k\{n_k + (2\alpha - a)\xi_k\}\xi_j\xi_i.\end{aligned}$$

If we differentiate (3.24) covariantly and substitute (1.5), we find

$$\begin{aligned}\nabla_l\nabla_k K_{ji} =&(\nabla_l n_k)L_{ji} + n_k\nabla_l L_{ji} + a\{(A_{lr}\phi_k{}^r)L_{ji} + (A_{lr}\phi_i{}^r)L_{jk} + (A_{lr}\phi_j{}^r)L_{ki}\}\\ &- a(\xi_k\nabla_l L_{ji} + \xi_i\nabla_l L_{jk} + \xi_j\nabla_l L_{ki}).\end{aligned}$$

Multiplying this with $\phi^{lk}$ and summing for $l$ and $k$, and taking account of (3.3), (3.4), (3.10), (3.11), (3.17) and (3.26), we obtain

$$\phi^{lk}\nabla_l\nabla_k K_{ji} = (\phi^{lk}\nabla_l n_k)L_{ji} + a\{2(n-1)\rho L_{ji} - A_{jr}L_i{}^r - A_{ir}L_j{}^r\},$$

or, using (2.1) and the Ricci identity for $K_{ji}$,

$$-\frac{1}{2}\phi^{lk}(R_{lkjr}K_i{}^r + R_{lkir}K_j{}^r) = 2(n-1)(\theta - a\rho)L_{ji} - a(A_{jr}L_i{}^r + A_{ir}L_j{}^r).$$

On the other hand we have from (1.15)

$$\phi^{lk}R_{lkji} = \{c(n+1) - 4\theta - 2(\rho\alpha - \frac{c}{4})\}\phi_{ij}.$$

where we have used (2.2) with $l_j = 0$, (3.3), (3.4) and (3.15). Combining with last two equations, it is seen that

$$\{(n+1)(c-2\theta) - 2(\rho\alpha - \frac{c}{4})\}L_{ji} = 2(n-1)a\rho L_{ji} - a(A_{jr}L_i{}^r + A_{ir}L_j{}^r).$$

Multiplying $L^{ji}$ to this and summing for $j$ and $i$, and making use of (2.5), (3.7) and (3.18), we have

$$(n+1)(c-2\theta) - 2(\rho\alpha - \frac{c}{4}) = 2n\rho a. \tag{3.27}$$

**Lemma 3.4.** *$\rho\alpha + \theta - \frac{3}{4}c = 0$ if $k \neq 0$.*

*Proof.* Suppose that $k^2 = \theta - \frac{c}{4}$. Then we have by (3.22)

$$A_{kr}L_i{}^r + kA_{kr}\phi_i{}^r = 0,$$

which together with (3.16) implies that

$$(\rho\alpha - \frac{c}{4})(L_{ji} - k\phi_{ji}) = 0.$$

Thus, it is seen that $\rho\alpha = \frac{c}{4}$. Therefore (3.25) and (3.27) will produce a contradiction because $\theta = \frac{c}{2}$ was assumed. Accordingly we have $k^2 - \theta + \frac{c}{4} = 0$ and hence

$$\rho k L_{ji} - (\theta - \frac{c}{4})(A_{jr}\phi_i{}^r - \rho\phi_{ji}) = 0 \tag{3.28}$$

by virtue of (3.23). If we take the usual norm of this and make use of (3.3), (3.16) and (3.17), then we obtain

$$\rho^2 k^2 = (\theta - \frac{c}{4})(\rho^2 + \rho\alpha - \frac{c}{4}), \tag{3.29}$$

which together with (3.27) gives the required relationship. This completes the proof of Lemma 3.4.

Multiplying (3.14) with $\phi^{ji}$ and summing for $j$ and $i$, and taking account of (3.3), we get

$$A_{ji}K^{ji} = \{\alpha + (n-1)\rho\}k. \tag{3.30}$$

Now, we are going to prove that the distinguished normal $C$ is parallel in the normal bundle. From (1.15) we verify that the Ricci tensor $S$ of $M$ with components $S_{ji}$ is given by

$$S_{ji} = \frac{c}{4}\{(2n+1)g_{ji} - 3\xi_j\xi_i\} + hA_{ji} - A_{ji}^2 + kK_{ji} - K_{ji}^2 - L_{ji}^2, \tag{3.31}$$

which together with (3.5), (3.17) and Lemma 3.4 implies that

$$S_{ji}\xi^j\xi^i = 2(n-1)(\theta - \frac{c}{2}). \tag{3.32}$$

If we multiply (3.31) with $K^{ji}$ and sum for $j$ and $i$, then we obtain

$$S_{ji}K^{ji} = 2(n-1)\{\theta - 2(n-2)\rho^2\}k,$$

where we have used (3.6), (3.16), (3.17), (3.18), (3.30) and Lemma 3.4.

Transforming (3.31) by ${\phi_k}^i$ and using (3.4), (3.7), (3.16), (3.17), (3.18) and Lemma 3.4, we find

$$S_{jr}{\phi_k}^r = \{\frac{c}{4}(2n+1) - \theta\}\phi_{kj} + \{\alpha - 2(n-2)\rho\}A_{jr}{\phi_k}^r + kL_{jk}.$$

Multiplying $L^{jk}$ to this and making use of (2.5), (3.3), (3.30), (3.32), (3.33) and Lemma 3.4, we see that $k(\theta - \frac{c}{4}) = 0$. Therefore we have $\theta = \frac{c}{4}$. Because of Lemma 2.1, it follows that $k = 0$, a contradiction. Thus we have

**Proposition 3.5.** *Let $M$ be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codimension 3 in $P_{n+1}\mathbb{C}$. If it satisfies $dn = 2\theta\omega$ for $\theta \neq \frac{c}{2}$ and $A\xi = \alpha\xi$. Then ${\nabla_j}^{\perp}C = 0$, namely, the distinguished normal is parallel in the normal bundle.*

## 4. Parallel distinguished normal vectors

In this section, we consider a semi-invariant submanifold of codimension 3 satisfying $dn = 2\theta\omega$ in a complex projective space.

Suppose that the distinguished normal $C$ is parallel in the normal bundle. Then we have $l_j = m_j = 0$. Thus, (1.16), (1.17), (1.19) and (1.20) turn out respectively to

$$\nabla_k A_{ji} - \nabla_j A_{ki} = \frac{c}{4}(\xi_k\phi_{ji} - \xi_j\phi_{ki} - 2\xi_i\phi_{kj}), \tag{4.1}$$

$$\nabla_k K_{ji} - \nabla_j K_{ki} = n_k L_{ji} - n_j L_{ki}, \tag{4.2}$$

$$A_{jr}{K_i}^r - A_{ir}{K_j}^r = 0, \quad A_{jr}{L_i}^r - A_{ir}{L_j}^r = 0. \tag{4.3}$$

Since we have $dn = 2\theta\omega$, relationships (2.2) and (2.8) are reduced respectively to

$$K_{jr}L_i{}^r = -(\theta - \frac{c}{4})\phi_{ji}, \tag{4.4}$$

$$K_{ji}{}^2 = (\theta - \frac{c}{4})(g_{ji} - \xi_j\xi_i). \tag{4.5}$$

Since we have $K_{ir}\xi^r = 0$, by differentiating covariantly along $M$ and using (1.7) with $l_j = 0$, we find

$$(\nabla_k K_{ir})\xi^r = -L_{ir}A_k{}^r. \tag{4.6}$$

Differentiating (4.5) covariantly along $M$ and using (1.5), we have

$$K_j{}^r(\nabla_k K_{ir}) + K_i{}^r(\nabla_k K_{jr}) = (\theta - \frac{c}{4})(\xi_j A_{kr}\phi_i{}^r + \xi_i A_{kr}\phi_j{}^r). \tag{4.7}$$

Using the quaite same method as that used to (2.13) from (2.12), we can derive from (4.7) the following :

$$\begin{aligned} 2K_j{}^r\nabla_k K_{ir} = (\theta - \frac{c}{4})\{&2n_k\phi_{ij} + (A_{ir}\phi_j{}^r - A_{jr}\phi_i{}^r)\xi_k \\ &+ (A_{kr}\phi_j{}^r - A_{jr}\phi_k{}^r)\xi_i + (A_{kr}\phi_i{}^r + A_{ir}\phi_k{}^r)\xi_j\}, \end{aligned} \tag{4.8}$$

where we have used (4.2) and (4.4).

In the following, we are going to prove $A_{(2)} = 0$. By means of (4.5), we may only consider the case where $\theta - \frac{c}{4} \neq 0$ because it is already seen that $\theta$ is constant. By (4.2) we can, using $k = l = 0$, verify that $\nabla_r K_j{}^r = L_{jr}n^r$. Thus, multiplying (4.8) with $g^{ki}$ and summing for $k$ and $i$, we find

$$K_j{}^r L_{rs}n^s = (\theta - \frac{c}{4})(\phi_{rj}n^r + \xi^s A_{sr}\phi_j{}^r),$$

which together with (4.4) implies that $\xi^s A_{sr}\phi_j{}^r = 0$ and hence

$$A\xi = \alpha\xi. \tag{4.9}$$

Therefore, if we transvect (4.8) with $\xi^j$ and take account of (1.8) and (4.9), then we obtain

$$(4.10) \qquad A\phi = \phi A.$$

From this and (4.1) we can prove the followings (cf. [7], [11]) :

$$(4.11) \qquad A_{ji}{}^2 = \alpha A_{ji} + \frac{c}{4}(g_{ji} - \xi_j \xi_i),$$

$$(4.12) \qquad \nabla_k A_{ji} = -\frac{c}{4}(\xi_j \phi_{ki} + \xi_i \phi_{kj}).$$

By means of (4.10), the equation (4.8) can be written as

$$K_j{}^r \nabla_k K_{ir} = (\theta - \frac{c}{4})(n_k \phi_{ij} + \xi_k A_{ir} \phi_j{}^r + \xi_i A_{kr} \phi_j{}^r).$$

Transforming by $K_h{}^j$ and using (1.7), (4.3), (4.5) and (4.6), we obtain

$$(4.13) \qquad \nabla_k K_{ji} = n_k L_{ji} - \xi_k A_{jr} L_i{}^r - \xi_i A_{kr} L_j{}^r - \xi_j A_{ir} L_k{}^r,$$

Differentiating (1.7) with $l_j = 0$ covariantly and using (1.4) and (4.13), we have

$$(4.14) \qquad \nabla_k L_{ji} = -n_k K_{ji} + \xi_k A_{jr} K_i{}^r + \xi_i A_{kr} K_j{}^r + \xi_j A_{ir} K_k{}^r,$$

which together (1.8) with $l_j = 0$ and (4.9) implies that

$$(4.15) \qquad T_r(AA_{(2)}) = 0, \quad T_r(A^2 A_{(2)}) = 0$$

because of (4.11).

On the other hand, we have $A_{(2)}\xi = 0$ and $T_r A_{(2)} = 0$ and (4.5), the shape operator $A_{(2)}$ has at most three distinct constant eigenvalues 0, $\sqrt{\theta - \frac{c}{4}}, -\sqrt{\theta - \frac{c}{4}}$ with multiplicities 1, $n-1$, $n-1$ respectively.

By (4.9), (4.10) and (4.11), we also see that $A$ has at most three distinct constant eigenvalues $\alpha, (\alpha+\sqrt{D})/2$, $(\alpha-\sqrt{D})/2$ with multiplicities $1, r, s$ respectively, where $D=\alpha^2+c, r+s=2n-2$.

Since we have $AA_{(2)}=A_{(2)}A$, it follows that $A$ and $A_{(2)}$ are diagonalizable at the same time. Because of (4.15), we have $(\theta-\frac{c}{4})r(\alpha^2+c)=0$. Thus $s=2(n-1)$ and consequently $A$ has two constant eigenvalues $\alpha$ and $(\alpha-\sqrt{D})/2$ with multiplicities 1, $2(n-1)$ repectively. Accordingly the trace $h$ of $A$ is given by

$$h=n\alpha-(n-1)\sqrt{D}. \tag{4.16}$$

Differentiating (4.13) covariantly along $M$ and using (1.5), (1.8), (4.11), (4.12) and (4.13), we find

$$\begin{aligned}\nabla_h\nabla_k K_{ji} =&(\nabla_h n_k)L_{ji}-\frac{c}{4}(K_{ki}\xi_j\xi_h+K_{jh}\xi_k\xi_i+2K_{ih}\xi_j\xi_k)+B_{hkji}\\ &-\alpha(\xi_j\xi_h A_{kr}K_i{}^r+\xi_k\xi_i A_{jr}K_k{}^r+2\xi_j\xi_k A_{ir}K_h{}^r)\\ &+(A_{hs}\phi_j{}^s)(A_{kr}L_i{}^r)+(A_{hs}\phi_k{}^s)(A_{ir}L_j{}^r)+(A_{hs}\phi_i{}^s)(A_{jr}L_k{}^r),\end{aligned}$$

where $B_{hkji}$ is a certain tensor with $B_{hkji}=B_{khji}$, from which, taking the skew-symmetric part with respect to $h$ and $k$, and making use of (2.1), (4.10) and the Ricci identity for $K_{ji}$,

(4.17)

$$\begin{aligned}&R_{khjr}K_i{}^r+R_{khir}K_j{}^r\\ &=2\theta\phi_{hk}L_{ji}-\frac{c}{4}\{\xi_j(\xi_k K_{ih}-\xi_h K_{ik})+\xi_i(\xi_k K_{jh}-\xi_h K_{jk})\}\\ &-\alpha\{\xi_j(\xi_k A_{ir}K_h{}^r-\xi_h A_{ir}K_k{}^r)+\xi_i(\xi_k A_{jr}K_h{}^r-\xi_h A_{jr}K_k{}^r)\}\\ &+(A_{hs}\phi_j{}^s)(A_{kr}L_i{}^r)-(A_{ks}\phi_j{}^s)(A_{hr}L_i{}^r)+(A_{hs}\phi_i{}^s)(A_{kr}L_j{}^r)\\ &-(A_{ks}\phi_i{}^s)(A_{hr}L_j{}^r)+2(A_{hs}\phi_k{}^s)(A_{jr}L_i{}^r).\end{aligned}$$

Multiplying (4.17) with $\phi^{kh}$ and summing for $k$ and $h$, and using (1.6), (1.7), (2.1), (4.10) and (4.11), we find

$$\phi^{kh}(R_{khjr}K_i{}^r+R_{khir}K_j{}^r)=\{c-4(n-1)\theta\}L_{ji}+2(h+\alpha)A_{jr}L_i{}^r. \tag{4.18}$$

On the other hand, we have from (1.15)

$$\phi^{kl}R_{klih} = (cn + \frac{c}{2})\phi_{hi} - 2\alpha A_{hr}\phi_i{}^r + 4K_{hr}L_i{}^r,$$

where we have used (1.7), (1.8), (4.10) and (4.11), which together with (1.7) and (4.5) gives

$$\phi^{kl}(R_{klir}K_j{}^r + R_{kljr}K_i{}^r) = \{8\theta - (2n+3)c\}L_{ji} - 4\alpha A_{jr}L_i{}^r.$$

From this and (4.18), it is seen that

$$(h + 3\alpha)A_{jr}L_i{}^r = \{2(n+1)\theta - (n+2)c\}L_{ji}, \tag{4.19}$$

which implies

$$(h + 3\alpha)(A_{ji} - \alpha\xi_j\xi_i) = \{2(n+1)\theta - (n+2)c\}(g_{ji} - \xi_j\xi_i).$$

If we take the trace of this, then we obtain

$$(h + 3\alpha)(h - \alpha) = 2(n-1)\{2(n+1)\theta - (n+2)c\}. \tag{4.20}$$

In the same way, multiplying $A^{jk}$ to (4.17) and summing for $j$ and $k$, and taking account of (1.6), (1.8), (4.3), (4.9) $\sim$ (4.11), we also have

$$(R_{kjir}K_h{}^r + R_{kjhr}K_i{}^r)A^{ik} = (3\alpha^2 - 2\theta + c)A_{hr}K_j{}^r + \frac{3}{4}c\alpha K_{jh}.$$

On the other hand, we have from (1.15)

$$\begin{aligned}&(R_{kjir}K_h{}^r + R_{kjhr}K_i{}^r)A^{ik}\\ &= (2\theta - 2c - h_{(2)})A_{hr}K_j{}^r + \{(\theta - \frac{c}{2})(h - \alpha) - \frac{c}{4}\alpha\}K_{jh},\end{aligned}$$

where we have used (1.6), (1.7), (4.3), (4.4), (4.5) and (4.11).
From the last two equations, it follows that

$$(4\theta - 3c - h_{(2)} - 3\alpha^2)A_{jr}K_i{}^r = \{c\alpha - (\theta - \frac{c}{2})(h - \alpha)\}K_{ji}, \tag{4.21}$$

which implies

$$(4.22) \qquad (4\theta - 3c - h_{(2)} - 3\alpha^2)(h-\alpha) = 2(n-1)\{c\alpha - (\theta - \frac{c}{2})(h-\alpha)\}.$$

If we take account of (4.11), then (4.22) can be written as

$$2(n+1)(\theta - \frac{3}{4}c)(h-\alpha) - \alpha(h+3\alpha)(h-\alpha) = 2(n-1)c\alpha,$$

or use (4.20),

$$(4.23) \qquad (\theta - \frac{3}{4}c)(h-\alpha) = 2(n-1)\alpha(\theta - \frac{c}{2}).$$

By the way, we have from (4.16) and (4.20)

$$\alpha(\alpha - \sqrt{D}) = 2(\theta - \frac{3}{4}c).$$

Combining (4.16), (4.23) and the last equation, we see that

$$(\theta - \frac{3}{4}c)^2 = \alpha^2(\theta - \frac{c}{2}).$$

From this, (2.5) and (4.5) we have

**Lemma 4.1.** *Let $M$ be a real (2n-1)-dimensional ($n > 2$) semi-invariant submanifold of codimension 3 satisfying $dn = 2\theta\omega$ for a certain scalar $\theta < \frac{c}{2}$ in a complex projective space $P_{n+1}\mathbb{C}$. If the distinguished normal is parallel in the normal bundle, then we have $A_{(2)} = A_{(3)} = 0$.*

Let $N_0(p) = \{\eta \in T_p{}^{\perp}(M) \mid A_\eta = 0\}$ and $H_0(p)$ the maximal J-invariant subspace of $N_0(p)$. As a consequence of Lemma 4.1, we have $A_{(2)} = A_{(3)} = 0$, the orthogonal complement of $H_0(p)$ is invariant under parallel translation with respect to the normal connection because of $\nabla_j{}^{\perp}C = 0$. Thus, by the reduction theorem in [5], [14] and by Lemma 2.2 and Proposition 3.5 we have

**Theorem 4.2.** *Let $M$ be a real (2n-1)-dimensional ($n > 2$) semi-invariant submanfold of codimension 3 in a complex projective space $P_{n+1}\mathbb{C}$. If the structure vector $\xi$ is an eigenvector for the shape operator in the direction of the distinguished normal and the third fundamental tensor $n$ satisfies $dn = 2\theta\omega$ for a certain scalar $\theta (< \frac{c}{2})$, then $M$ is a real hypersurface in a complex projective space $P_n\mathbb{C}$.*

Owing to Theorem T and Theorem 4.2, we have

**Theorem 4.3.** *Let $M$ be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codimension 3 in a complex projective space $P_{n+1}\mathbb{C}$ such that the third fundamental tensor satisfies $dn = 2\theta\omega$ for a certain scalar $\theta(< \frac{c}{2})$, where $\omega(X,Y) = g(X, \phi Y)$ for any vectors $X$ and $Y$ on $M$. Then $M$ has constant eigenvalues corresponding the shape operator $A$ in the direction of distinguished normal and the structure vector $\xi$ is an eigenvector of $A$ if and only if $M$ is locally congruent to a homogeneous real hypersurfaces of $P_n\mathbb{C}$.*

## 5. Examples of a nontrivial semi-invariant submanifold

In this section, we shall give an example of a nontrivial semi-invariant submanifold in $P_n\mathbb{C}$.

Let $p, q (3 \leq p \leq q)$ be integers. We denote by $M_{p,q}\mathbb{C}$ the space of $p \times q$ matrices over $\mathbb{C}$, which can be considered as a complex Euclidean space $\mathbb{C}^{pq}$ with the standard Hermitian inner product. Let denote the unitary group of degree $p$ by $U(p)$. Then the Lie group $G := S(U(p) \times U(q))$ acts on $\mathbb{C}^{pq} \equiv M_{p,q}\mathbb{C}$ as follows :

$$(\sigma, \tau)X = \sigma X \tau^{-1}, \ (\sigma, \tau) \in G, X \in \mathbb{C}^{pq}.$$

Thus we can consider $G$ as a unitary subgroup of $U(pq)$. Remark that this action is nothing but the linear isotropic representation of the compact Hermitian symmetric space $SU(p+q)/S(U(p) \times U(q))$ of type $AIII$(cf. [6]).

Let $\pi$ be the canonical projection of $\mathbb{C}^{pq} - \{0\}$ onto $P_{pq-1}\mathbb{C}$ , and $S^{2pq-1}(r)$ the hypersphere in $\mathbb{C}^{pq}$ of radius $r$ centered at the origin.

Then, for any element $A$ of $\mathbb{C}^{pq} - \{0\}$, the orbit $G(A)$ of $A$ under $G$ is a compact homogeneous submanifold in $S^{2pq-1}(|A|)$, and the space $\pi(G(A))$ is a compact homogeneous submanifolds in $P_{pq-1}\mathbb{C}$. Moreover, for any normal vector $N$ of $G(A)$ in $S^{2pq-1}(|A|)$, the mean curvature of $G(A)$ in the direction $N$ is equal to the one of $\pi(G(A))$ in the direction $\pi_* N$ in $P_{pq-1}\mathbb{C}$.(see e.g. [12] ). In particular, $G(A)$ is minimal in $S^{2pq-1}(|A|)$ if and only if $\pi(G(A))$ is minimal in $P_{pq-1}\mathbb{C}$.

Here, for $i = 1, \cdots, p$ and $\alpha = 1, \cdots, q$, we denote by $E_{i\alpha}$ the element of $M_{p,q}\mathbb{C}$ whose $(i, \alpha)$-entry is 1 and other entries are all 0. In the sequel we shall show

(5.1) If $A = a_1 E_{11} + a_2 E_{22}$ satisfies $a_1 a_2 \neq 0, {a_1}^2 \neq {a_2}^2$, and ${a_1}^2 + {a_2}^2 = r^2$, then $\pi(G(A))$ is a $(4p+4q-11)$-dimensional semi-invariant submanifold in $P_{pq-1}\mathbb{C}$.

By the definition, the tangent space $T_A(G(A))$ of the orbit of $A$ under $G$ is generated by the vectors

$$XA \text{ and } AY,$$

where $X(resp. Y)$ ranges over all skew-Hermitian matrices of degree $p(degree\ q)$. Hence the space $T_A(G(A))$ are spanned over $\mathbb{R}$ by the following vectors :

$$a_1\sqrt{-1}E_{11} + a_2\sqrt{-1}E_{22},\ a_1\sqrt{-1}E_{11} - a_2\sqrt{-1}E_{22},$$
$$E_{12}, \sqrt{-1}E_{12}, E_{21}, \sqrt{-1}E_{21}, E_{i\alpha}, \sqrt{-1}E_{i\alpha}, E_{j\beta}, \sqrt{-1}E_{j\beta},$$

where $1 \leq i \leq 2, 3 \leq \alpha \leq q, 3 \leq j \leq p$ and $1 \leq \beta \leq 2$.
Thus the intersection of the vector space $\sqrt{-1}T_A(G(A))$ and the normal space of $G(A)$ at $A$ in $S^{2pq-1}(r)$ is spanned by the vector

$$a_2\sqrt{-1}E_{11} - a_1\sqrt{-1}E_{22},$$

which shows that $\pi(G(A))$ is semi-invariant in $P_{pq-1}\mathbb{C}$. Since the space $SU(p+q)/S(U(p) \times U(q))$ is irreducible as a symmetric space, our space $\pi(G(A))$ is not trivially semi-invariant, *i.e.*, it satisfies $A_{(2)} \neq 0$ and $A_{(3)} \neq 0$ in the previous notation.

**Remark 5.1.** In the case $p = q = 3$, the space $\pi(G(A))$ is a submanifold of codimension 3 in $P_8\mathbb{C}$.

**Remark 5.2.** We can see that, among the spaces $\pi(G(A))$ satisfying the conditions $0 < a_1 < a_2$ and ${a_1}^2 + {a_2}^2 = r^2$, there is uniqully a minimal one. About this we shall work out in a forthcoming paper.

## References


1. A. Bejancu, *CR-submanifolds of a Kähler manifold* I, Proc. Amer. Math. Soc. **69** (1978), 135–142.
2. D. E. Blair, G.D. Ludden and K. Yano, *Semi-invariant immersion*, Kodai Math. Sem. Rep. **27** (1976), 313–319.
3. T.E. Cecil and P.J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), 481–499.
4. Y.-W. Choe and M. Okumura, *Scalar curvature of a certain CR-submanifold of complex projective space*, Arch. Math. **68** (1997), 340–346.
5. J. Erbacher, *Reduction of the codimension of an isometric immersion*, J. Differential Geom. **3** (1971), 333–340.
6. S. Helgason, *Differential Geometry, Lie group and Symmetric Space*, Academic Press (1978).

7. U-H. Ki, *Cyclic-parallel real hypersurfaces of a complex space form*, Tsukuba J. Math. **12** (1988), 259–268.
8. U-H. Ki and H.-J. Kim, *Semi-invariant submanifolds with lift-flat normal connection in a complex projective space*, to appear in Kyungpook Math. J.
9. M. Kimura, *Real hypersurfaces and complex submanifolds in complex projective space*, Trans. Amer. Math. Soc. **296** (1986), 137–149.
10. J. H. Kwon and J. S. Pak, *CR-submanifolds of $(n-1)$ CR-dimension in a complex projective space*, Saitama Math. J. **15** (1997), 55–65.
11. R. Niebergall and P. J. Ryan, *Real hypersurfaces in complex space forms, in Tight and Taut submanifolds*, Cambridge University Press (1998( T. E. Cecil and S. S. Chern, eds.)), 233–305.
12. M. Okumura, *Totally umbilical submanifolds of a Kaehlerian manifold*, J. Japan Math. Soc. **19** (1967), 317–327.
13. M. Okumura, *Normal curvature and real submanifold of the complex projective space*, Geometriae Dedicata **7** (1978), 509–517.
14. M. Okumura, *Codimension reduction problem for real submanifolds of complex projective space*, Colloq. Math. Soc., Jándos Bolyai **56** (1989), 574–585.
15. M. Okumura and L. Vanhecke, *n-dimensional real submanifolds with $(n-1)$-dimensional maximal holomorphic tangent subspace in complex projective spaces*, Rendiconti del Circolo Mat. di Palermo **43** (1994), 233–249.
16. R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math. **10** (1973), 495–506.
17. R. Takagi, *Real hypersurfaces in a complex projective space with constant principal curvatures* I, II, J. Math. Soc. Japan **27** (1975), 43–53, 507–516.
18. Y. Tashiro, *Relations between the theory of almost complex spaces and that of almost contact spaces( in Japanese)*, Sugaku **16** (1964), 34–61.
19. K. Yano and U-H. Ki, *On $(f,g,u,v,w,\lambda,\mu,\nu)$-structure satisfying $\lambda^2+\mu^2+\nu^2=1$*, Kodai Math. Sem. Rep. **29** (1978), 285–307.
20. K. Yano and M. Kon, *CR submanifolds of Kaehlerian and Sasakian manifolds*, Birkhäuser (1983).
21. K. Yano and M. Kon, *Structures on manifolds*, World Scientific Publ. Co. Singapore (1984).

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