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SINGULAR HARMONIC MAPS BETWEEN RANK ONE SYMMETRIC SPACES OF NONCOMPACT TYPE

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ABSTRACT. Regarding the uniqueness for proper harmonic maps, Li and Tam proved the existence of a family of harmonic self-maps of the Poincaré upper-half planes assuming the identity map on the boundary (Ann. of Math. **137** (1993), 167–201). We generalize this example to the other rank one symmetric spaces of noncompact type and investigate their regularity and related properties.

1. INTRODUCTION AND RESULTS

1.1. **Introduction.** There are relatively few examples of *families* of harmonic maps as solutions to the given Dirichlet problem at infinity because of the difficulty of providing general strategies for their construction. In this article, we shall present both a *new family* of harmonic maps between rank one symmetric spaces of noncompact type and a *new technique* for estimating Hölder regularity, in order to better understand the non-uniqueness of solutions to this Dirichlet problem as stated explicitly below.

A Cartan–Hadamard manifold M can be compactified by adding the sphere at infinity ∂M defined by the asymptotic classes of geodesic rays, thereby giving us the compactification of M denoted by $\overline{M} = M \cup \partial M$. This leads us to the following Dirichlet problem between Cartan–Hadamard manifolds:

Dirichlet problem at infinity for harmonic maps

For given Cartan–Hadamard manifolds M, M' and a continuous map $f\colon \partial M \to \partial M'$, find a map $u\colon \overline{M} \to \overline{M}'$ satisfying,

(1) $u|_{\partial M} = f$ *and*

(2) $u|_M\colon M \to M'$ *is a solution to the harmonic map equation.*

In the early nineties, Li and Tam [13], [14] and, simultaneously, Akutagawa [2], carried out ground breaking works by introducing new techniques for existence arguments, when M and M' are both real hyperbolic spaces. In order to simplify these arguments, Bando [3]

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in Economakis's existence argument, we can simplify the nonlinear ordinary differential equation into a translation invariant equation and then utilize the comparison arguments of solutions in conjunction with a diagonal method. We can also use the latter method, with an additional slight modification, in order to construct diverse families of harmonic maps between various tube domains as in Chapters 2 and 3. Secondly, when estimating the Hölder regularity, Li and Tam, and Economakis, utilized the fact that the Jacobian matrix of a geodesic symmetry of the real hyperbolic space is expressed as a product of an orthogonal matrix and a scalar function. However, this fact does not hold true for the other rank one symmetric spaces of noncompact type, and it was therefore necessary to provide an argument which works, even in the cases where the Jacobian matrix can not be expressed as the product described above.

1.2. **Results.** Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$ denote complex or quaternion number fields. Let $n_1 = \dim_{\mathbb{R}}(\mathbb{K}^n)$, $n_2 = \dim_{\mathbb{R}}(\mathrm{Im}(\mathbb{K}))$ for $\mathrm{Im}(\mathbb{K}) = \{a - \bar{a} | a \in \mathbb{K}\}$, where $\bar{a}$ is the conjugation of a. Our following result implies that the assumption of regularity is essential for the uniqueness theorems of Donnelly:

Theorem 1.1. *Suppose that M is either complex or quaternion hyperbolic spaces and $k = 2$. Then exists a family of proper maps $u_\lambda \colon \overline{M} \to \overline{M}$ parameterized by $\lambda \geq 0$ satisfying:*

1. *$u_\lambda|_M \colon M \to M$ is a harmonic diffeomorphism and $u_\lambda|_{\partial M} \colon \partial M \to \partial M$ is the identity map.*
2. *For $\lambda \neq \lambda'$, we have $\mathrm{dist}(u_\lambda(\mathbf{s}), u_{\lambda'}(\mathbf{s})) \sim \exp(\mathcal{N}\mathrm{dist}(\mathbf{s}_0, \mathbf{s}))$ when we let $\mathbf{s} = (\mathbf{x}, \mathbf{t}, \rho) \to \infty$ while binding $|\mathbf{x}|$ and $|\mathbf{t}|$. Here, $\mathbf{s}_0 \in M$ is a fixed point and $\mathcal{N} = \sum_{l=1}^{k} l n_l$ and dist is a geodesic distance function.*
3. *$u_\lambda \in C^{1+\underline{a}}(\overline{M} \setminus \{\infty\}, \overline{M} \setminus \{\infty\})$ for $\underline{a} < (\mathcal{N} + \sqrt{\mathcal{N}^2 + 8\sum_{l=1}^{k} l^2 n_l})/4$.*
4. *$u_\lambda \in C^\alpha(\overline{M}, \overline{M})$ for $\alpha < 1/7$.*
5. *$u_\lambda \notin C^\alpha(\overline{M}, \overline{M})$ for $\alpha > 1/2$ and $\lambda > 0$.*
6. *$u_\lambda \in \Gamma_\alpha^\beta$ for $\alpha < 1/7$, $-\infty < \beta \leq \alpha$.*
7. *$u_\lambda \in C_1^\beta$ for $\beta < -1$.*

Remark 1.1. The first three claims of Theorem 1.1 hold true for Nishikawa and Ueno's k-term Carnot spaces, which include all rank one symmetric spaces of noncompact type [16].

Throughout this article, $C_i = C_i(*, \dots, *)$ $i = 1, \dots, 40$ denote constants depending only on the quantities appearing in parenthesis. In a given context, the same letter C_i

combined Green's function with Hamilton's method [12] as a basis for his own argument. Following this, Donnelly [5], [6] extended their results to prove the existence of harmonic maps between all rank one symmetric spaces of noncompact type [namely, real, complex, quaternion hyperbolic spaces and the Cayley hyperbolic plane]. In order to refine his own results, Donnelly [6] later used Graham's Hölder space [11], where derivatives are assigned weights depending on the direction. Recently, a few attempts were made to generalize their results as can be seen in Nishikawa and Ueno [16] and Ueno [19].

The uniqueness of a solution belonging to $C^1(\overline{M}, \overline{M}')$, in the work of Li and Tam, and $C^2(\overline{M}, \overline{M}')$ or C_3^β $\beta > 2$, in the work of Donnelly, has been established for a given non-degenerate boundary value. In order to confirm that these assumptions of regularity are necessary for the results on uniqueness, we have to solve the problem of constructing more than one harmonic map [for instance, as provided by a family] which induce a given boundary value and are only Hölder continuous when being viewed as maps from $\overline{M}$ to $\overline{M}'$.

With regard to this problem, Li and Tam [14] provided an explicit example of a family of harmonic maps between real hyperbolic planes which induce the identity map on the boundary; these maps are Hölder continuous with the exponent of 1/2 when being viewed as self-maps of $\overline{M}$. Hence, the assumption of regularity cannot be removed from the uniqueness theorem of Li and Tam [14] when the dimension is two. They have constructed this example by, firstly, reducing the harmonic map equation to a nonlinear ordinary differential equation; and, secondly, giving explicitly expressed solutions. In this rare example, we can express solutions for a nonlinear differential equation *explicitly*; but we cannot generally expect this to be the case. Economakis [8] generalized Li and Tam's example to higher dimensional cases by using a contraction mapping theorem, yielding no explicitly expressed solution; his abstractly constructed maps are only Hölder continuous with exponents of less than 1/2 when being viewed as self-maps of $\overline{M}$. Thereby, he also verified that the assumption of regularity cannot be removed from the uniqueness theorem when dimensions are greater than two.

In accordance with these studies, the following problem was suggested by Nishikawa: can we find a family of proper harmonic maps between complex hyperbolic spaces which do not satisfy the assumption of regularity in Donnelly's uniqueness theorems? In the present paper, we shall solve this problem by extending Li and Tam's and Economakis's results to other rank one symmetric spaces. However, our approach is somewhat different from the one they used. Firstly, instead of using a contraction mapping theorem as will, in general, be used to denote different constants depending only on the same set of arguments.

2. Existence theorem

2.1. **Preliminary computations.** Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{C}a$ denote real, complex, quaternion or Cayley number field. Let us set $d = \dim_{\mathbb{R}}(\mathbb{K})$ and $\mathrm{Im}(\mathbb{K}) = \{a - \bar{a} \mid a \in \mathbb{K}\}$. To begin with, we define N as a Lie group whose underlying manifold is $\mathbb{K}^n \times \mathrm{Im}(\mathbb{K})$ with coordinate $(\mathbf{x}, \mathbf{t}) = (\mathbf{x}^1, \dots, \mathbf{x}^n, \mathbf{t})$, where the group law is given by

$$(\mathbf{x}, \mathbf{t}) \cdot (\mathbf{x}', \mathbf{t}') = (\mathbf{x} + \mathbf{x}', \mathbf{t} + \mathbf{t}' + 2\mathrm{Im}(\mathbf{x} \cdot \bar{\mathbf{x}}')).$$

When $\mathbb{K} = \mathbb{C}$, N is called the Heisenberg group. In the following, the left translation of N by $(\mathbf{x}, \mathbf{t})$ shall be denoted as $\tau_{(\mathbf{x},\mathbf{t})}$. Next, we define $S = N \cdot \mathbb{R}_+$ as a semidirect product of N and $\mathbb{R}_+$ given by the dilation $\rho \cdot (\mathbf{x}, \mathbf{t}) = (\rho^{1/2}\mathbf{x}, \rho\mathbf{t})$ and let $\tau_{\mathbf{s}}$ denote the left translation of $N \cdot \mathbb{R}_+$ by $\mathbf{s} = (\mathbf{x}, \mathbf{t}, \rho)$. Now we select a left invariant metric g on S so that $M = (S, g)$ becomes a symmetric space, which is called a real, complex or quaternion hyperbolic space, denoted by $\mathbb{K}H^{n+1}$ $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or the Cayley hyperbolic plane denoted by $\mathbb{C}aH^2$.

In the following, we shall detail a deduction of the explicit formula of the metric g on S in conjunction with the canonical generator of the Lie algebra of S when $(S, g) = \mathbb{C}H^{n+1}$ or $\mathbb{H}H^{n+1}$.

To begin with, let $\{\mathbf{e}^j\}_{j=1}^d$ denote the canonical generator of $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$ given respectively by $\mathbf{e}^1 = 1$ and $\mathbf{e}^2 = \sqrt{-1}$ when $\mathbb{K} = \mathbb{C}$, and $\mathbf{e}^1 = 1$, $\mathbf{e}^2 = \mathbf{i}$, $\mathbf{e}^3 = \mathbf{j}$, $\mathbf{e}^4 = \mathbf{k}$, $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ when $\mathbb{K} = \mathbb{H}$. Utilizing these, we can express $\mathbf{x}^j$ as $\sum_{l=1}^d x^{jl}\mathbf{e}^l$ and $\mathbf{t}$ as $\sum_{l=2}^d t^l\mathbf{e}^l$ and we let $\mathrm{Im}_i(\mathbf{x}^j) = x^{ji}$ denote the $\mathbf{e}^i$ component of $\mathbf{x}^j$. Then, the left invariant extensions in N of tangent vectors $\partial/\partial x^{jl}$ $(1 \le j \le n, 1 \le l \le d)$, $2\partial/\partial t^l$ $(2 \le l \le d)$ at $\mathbf{o} = (0, 0, 1) \in \mathbb{K}^n \times \mathrm{Im}(\mathbb{K}) \times \mathbb{R}_+$ can be computed as follows:

For $\mathbf{x}' \in \mathbb{K}^n$ so that $\mathbf{x}'$ is of the form

$$\mathbf{x}' = (0, \dots, 0, \underset{j-\text{th}}{\varepsilon\mathbf{e}^l}, 0, \dots, 0) \qquad \varepsilon \in \mathbb{R},$$

we can compute

$$\begin{aligned}
\tau_{(\mathbf{x},\mathbf{t})*}(\partial/\partial x^{jl})f &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f((\mathbf{x},\mathbf{t})\cdot(\mathbf{x}',0)) \\
&= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(\mathbf{x}^1,\dots,\mathbf{x}^j+\varepsilon\mathbf{e}^l,\dots,\mathbf{x}^n,\mathbf{t}+2\mathrm{Im}(\mathbf{x}^j\varepsilon\bar{\mathbf{e}}^l)) \\
&= \Big(\frac{\partial}{\partial x^{jl}}+2\sum_{i=2}^{d}\mathrm{Im}_i(\mathbf{x}^j\bar{\mathbf{e}}^l)\frac{\partial}{\partial t^i}\Big)f \\
&= : e_{d(j-1)+l}f,
\end{aligned}$$

$$\tau_{(\mathbf{x},\mathbf{t})*}(2\partial/\partial t^l)f = 2\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f((\mathbf{x},\mathbf{t})\cdot(0,\varepsilon\mathbf{e}^l)) = 2\frac{\partial}{\partial t^l}f =: e_{dn+l-1}f.$$

Similarly, we observe that the left invariant extensions of $\partial/\partial x^{jl}$ $(1\le j\le n, 1\le l\le d)$, $2\partial/\partial t^l$ $(2\le l\le d)$ and $2e_m = 2\partial/\partial\rho$ in $N\cdot\mathbb{R}_+$ are, respectively, given by

$$\begin{aligned}
L_{d(j-1)+l} &= \rho^{1/2}e_{d(j-1)+l} \qquad (1\le j\le n,\ 1\le l\le d), \\
L_{dn+l-1} &= \rho e_{dn+l-1} \qquad (2\le l\le d),
\end{aligned}$$

and

$$L_m = 2\rho e_m.$$

By utilizing these, we define $\mathfrak{n}_1$ and $\mathfrak{n}_2$ by

$$\mathfrak{n}_1 = \mathrm{Span}_{\mathbb{R}}\{L_{d(j-1)+l}\}_{1\le j\le n, 1\le l\le d}, \qquad \mathfrak{n}_2 = \mathrm{Span}_{\mathbb{R}}\{L_{nd+l-1}\}_{2\le l\le d}.$$

Then, for $H = L_m$, we have the following decomposition of the Lie algebra of S:

$$\mathfrak{s} = \mathbb{R}_+\{H\} + \mathfrak{n}_1 + \mathfrak{n}_2,$$

and $\mathfrak{n}_l = \{X\in\mathfrak{s} \mid [H,X] = lX\}$ $(l=1,2)$. Further, for $m = n_1+n_2+1$, we have

$$[e_\alpha, e_\beta] = \sum_{\gamma=1}^{m-1} a_{\alpha\beta}^{\gamma}e_\gamma,$$

where $a_{\alpha\beta}^{\gamma} = 0$, unless $\alpha,\beta\in I_1$, $\gamma\in I_2$ and thereby it holds that $\mathfrak{n}_2 = [\mathfrak{n},\mathfrak{n}]$ is the center of $\mathfrak{n} = \mathfrak{n}_1+\mathfrak{n}_2$.

Having obtained the explicit formula of the canonical generator of the Lie algebra $\mathfrak{s}$, we shall consider the metric g of S. Firstly, since S acts on $N\cdot\mathbb{R}_+$ transitively, an inner product $\langle\cdot,\cdot\rangle$ of the tangent space $T_{\mathbf{o}}(S)$ at $\mathbf{o} = (0,0,1)\in S$, define the left invariant metric g assigning $g_{\mathbf{s}\cdot\mathbf{o}}(V, V') = \langle\tau_{\mathbf{s}*}^{-1}V, \tau_{\mathbf{s}*}^{-1}V'\rangle$ for $V, V'\in T_{\mathbf{s}\cdot\mathbf{o}}(S)$ at each $\mathbf{s}\in S$. At this point, defining the inner product above as

$$\langle\cdot,\cdot\rangle = |d\mathbf{x}|^2 + |d\mathbf{t}|^2/4 + d\rho^2/4,$$

Here $g_{ij} = g(e_i, e_j)$ is a metric of M; $u_*(e_i) = \sum_{\gamma=1}^{m'} u_i^\gamma e'_\gamma$ and $u_{ij}^\gamma = e_i u_j^\gamma$; and, ∇^M and $\nabla^{M'}$ are Levi-Civita connections on TM and TM', respectively; and $\tilde{\nabla}$ is an induced connection on $u^{-1}(TM') = \cup_{p\in M} T_{u(p)}M'$ (see [5, (2.1)]).

Utilizing the computation made above, it is easy to see that the components of the tension field of the map $u\colon M \ni (\mathbf{x}, \mathbf{t}, \rho) \to (\mathbf{x}, \mathbf{t}, \psi(\rho)) \in M$ are given by

$$\begin{aligned}\tau^m(u) &= 4\rho^2\frac{d^2\psi(\rho)}{d\rho^2} - (2\sum_{j=1}^{k} jn_j - 4)\rho\frac{d\psi(\rho)}{d\rho} \\ &\quad +2\sum_{l=1}^{k} l\rho^l\psi(\rho)^{1-l}n_l - 4\rho^2\psi(\rho)^{-1}\left(\frac{d\psi}{d\rho}\right)^2\end{aligned}$$

and $\tau^1(u) = \cdots = \tau^{n_1+n_2}(u) \equiv 0$. Here, we used $u_m^m = d\psi/d\rho$ and $u_i^\gamma = \delta_{i\gamma}$ $(i, \gamma \neq m)$, which are valid because $u_*(e_m) = d\psi/d\rho\, e_m$ and $u_*(e_j) = e_j$ for $j = 1, \cdots, n_1 + n_2$. Given the observation made above, setting $\mathcal{N} = \sum_{j=1}^{k} jn_j$, $\dot{\psi} = d\psi/d\rho$ and $\ddot{\psi} = d^2\psi/d\rho^2$, we have the following:

Lemma 2.1. *Suppose $\psi(\rho)$ is a solution for*

$$\text{(3)}\qquad \begin{cases} \rho\ddot{\psi}(\rho) - (\frac{1}{2}\mathcal{N} - 1)\dot{\psi}(\rho) + \frac{1}{2}\sum_{l=1}^{k} ln_l \left(\frac{\rho}{\psi(\rho)}\right)^{l-1} - (\dot{\psi}(\rho))^2\frac{\rho}{\psi(\rho)} = 0, \\ \psi(0) = 0, \quad \dot{\psi}(0) = 1, \quad \psi(\rho) = -\psi(-\rho) > 0 \ \ \rho > 0. \end{cases}$$

Then $u\colon (\mathbf{x}, \mathbf{t}, \rho) \to (\mathbf{x}, \mathbf{t}, \psi(\rho))$ is a harmonic self-map of M inducing the identity map on the boundary ∂M.

In the next section, we shall establish the existence of a one–parameter family of global solutions for the equation (3) and study their asymptotic behavior. The growth estimates in Proposition 2.2 are used in Subsection 3.3 in order to prove Proposition 3.1.

2.2. An asymptotic analysis of the translation invariant equation.

Theorem 2.1. *There exists a one-parameter family of global solutions $\psi(\rho) = \psi_\lambda(\rho)$ parameterized by $\lambda \geq 0$ for the equation* (3).

The translation invariance of the equation (4) in the following proposition is the key to understanding the non-uniqueness of solutions for (3). By means of the following proposition, in order to prove Theorem 2.1, it suffices to show that there exists a nontrivial global solution $f(t)$ for the equation (4).

Proposition 2.1. $\psi(\rho) = \rho \exp\big(f(\log(|\rho|))\big)$ *is a solution for (3) if and only if f satisfies*

$$(4)\qquad \begin{cases} f''(t) - \frac{1}{2}\mathcal{N} f'(t) - \frac{1}{2}\sum_{l=1}^{k} l n_l \left(1 - e^{-lf(t)}\right) = 0, \\ f(t), f'(t) > 0, \lim_{x \to -\infty} f(t) = 0, \lim_{x \to -\infty} f'(t) = 0, \end{cases}$$

where $f' = df/dt$ and $f'' = d^2f/dt^2$. Given a solution $f(t)$ for (4), we have the solution $f(t + \log(\lambda))$ satisfying (4) for each $\lambda > 0$. Thereby, we see that $\psi_\lambda(\rho) = \rho \exp\big(f(\log(|\rho|\lambda))\big)$ ($\lambda \geq 0$) form a one-parameter family of solutions for (3) parametrized by $\lambda \geq 0$.

Proof. Note that $\log(|\rho|)' = \mathrm{sgn}(\rho)/|\rho| = 1/\rho$. By substituting $\rho \exp\big(f(\log(|\rho|))\big)$ for $\psi(\rho)$ in equation (3), we have

$$\begin{aligned} &\rho\ddot{\psi}(\rho) - (\frac{1}{2}\mathcal{N} - 1)\dot{\psi}(\rho) + \frac{1}{2}\sum_{l=1}^{k} l n_l \left(\frac{\rho}{\psi(\rho)}\right)^{l-1} - (\dot{\psi}(\rho))^2 \frac{\rho}{\psi(\rho)} \\ = \;& \rho(f''(t) + (f'(t)+1)f'(t))\rho^{-1}e^{f(t)} - (\frac{1}{2}\mathcal{N} - 1)(1 + f'(t))e^{f(t)} \\ & + \frac{1}{2}\sum_{l=1}^{k} l n_l \left(\frac{\rho}{\rho e^{f(t)}}\right)^{l-1} - ((1 + f'(t))e^{f(t)})^2 \frac{\rho}{\rho e^{f(t)}} \\ = \;& e^{f(t)}\Big(f''(t) - \frac{1}{2}\mathcal{N}(1 + f'(t)) + \frac{1}{2}\sum_{l=1}^{k} l n_l e^{-lf(t)}\Big) = 0, \end{aligned}$$

where $t = \log(|\rho|)$. Since $f(t) > 0$ and $f(t) \to 0$ as $t \to -\infty$ we have $\psi(0) = 0$ and $\psi(\rho) \to \infty$ as $\rho \to \infty$. Moreover, $f(t) \to 0$ as $t \to -\infty$ and $f'(t) \to 0$ imply that $\dot{\psi}(0) = 1$.

Conversely, if $\psi(\rho)$ satisfies (3), then we can verify that $f(t) = \log(\psi(\exp(t))) - t$ satisfies (4) as can be seen in the following: for $\rho = e^t$, $\dot{\psi}(0) = 1$ and $\psi(0) = 0$ being the case, it holds that $f(t) = \log(\psi(e^t)/e^t) \to 0$ and $f'(t) = \dot{\psi}(e^t)e^t/\psi(e^t) - 1 \to 0$ as $t \to -\infty$. Since $e^{-f(t)} = \rho/\psi(\rho)$, we have

$$\begin{aligned} & f''(t) - \frac{1}{2}\mathcal{N} f'(t) - \frac{1}{2}\sum_{l=1}^{k}(1 - e^{-lf(t)}) l n_l \\ = \;& \frac{\rho}{\psi(\rho)}\Big(\rho\ddot{\psi}(\rho) - (\frac{1}{2}\mathcal{N} - 1)\dot{\psi}(\rho) + \frac{1}{2}\sum_{l=1}^{k}\Big(\frac{\rho}{\psi(\rho)}\Big)^{l-1} l n_l - \frac{\rho}{\psi(\rho)}\dot{\psi}(\rho)^2\Big) = 0. \end{aligned}$$

□

Proof of Theorem 2.1. By setting $X(t) = f(t)$ and $Y(t) = f'(t)$, we can express the equation above as two first–order ordinary differential equations:

$$\frac{dY}{dt}(t) = \frac{1}{2}\mathcal{N}Y(t) + \frac{1}{2}\sum_{l=1}^{k}(1 - e^{-lX(t)})ln_l,$$
$$\frac{dX}{dt}(t) = Y(t).$$

Consequently, we have

$$\begin{aligned}\frac{dY}{dX} &= \frac{\mathcal{N}}{2} + \frac{\sum_{l=1}^{k}(1 - e^{-lX})ln_l}{2Y} \\ &= \frac{\mathcal{N}}{2} + E(X)\frac{X}{Y},\end{aligned}$$

where

$$E(X) = \frac{\sum_{l=1}^{k}(1 - e^{-lX})ln_l}{2X}.$$

At this point, it should be noted that $E(X)$ is a monotone decreasing function of $X > 0$.

In the following, in order to show the global existence of the solution $Y(X)$ which satisfies $Y(0) = 0$, we shall solve the following equations:

$$\text{(5)} \qquad \begin{cases} \dfrac{dY}{dX} = \dfrac{\mathcal{N}}{2} + E(X)\dfrac{X}{Y}, \\ Y(X) \to 0,\ dY/dX \to a\ \ (X \to 0), \quad a = \mathcal{N}/2 + E(0)/a > 0. \end{cases}$$

The condition $dY/dX \to a \ (X \to 0)$ corresponds to the requirement that $\lim_{X\to 0} dY/dX = \lim_{X\to 0}(\mathcal{N}/2 + E(X)X/Y)$. We shall define a constant c as follows:

$$c = E(0) = \frac{1}{2}\sum_{l=1}^{k} l^2 n_l.$$

Then a constant a in (5) is given by

$$a = (\mathcal{N} + \sqrt{\mathcal{N}^2 + 16c})/4.$$

Our strategy to complete the proof of Theorem 2.1 is as follows: in Step 1, it will be shown that $Y(X)$ exists globally; in Step 2, by using $Y(X)$, we shall solve $f'(t) = Y(f(t))$ with boundary values $f, f' \to 0 \ \ t \to -\infty$, and thereby we will establish the global existence of a solution $f(t)$ for the equation (4). Once the global existence of $f(t)$ is established, the proof of Theorem 2.1 will be completed by using Proposition 2.1.

Step 1: The methodology we shall use to prove the global existence of $Y(X)$ for (5) is as follows:

Note that the right-hand side is C^∞ for the variables $Y > 0$ and $X > 0$. This means that for any $\varepsilon_0 > 0$, the solution $Y(X)$ with an initial value $Y(\varepsilon_0) > 0$ exists locally for

$X > \varepsilon_0$ and that dY/dX does not diverge at finite X as long as $X(>0)$ and $Y(>0)$ are finite. Accordingly, in order to prove that $Y(X)$ exists globally on $[\varepsilon_0, \infty)$, it suffices to create positive functions which limit the behavior of $Y(X)$ from above and below for all $X > \varepsilon_0$. We can then verify that neither $Y(X)$ nor dY/dX diverges at any finite time.

Supposing that $\bar{c} > c > \underline{c} \geq 0$ and that

$$\underline{a} = \mathcal{N}/2 + \underline{c}/\underline{a} > 0, \quad \bar{a} = \mathcal{N}/2 + \bar{c}/\bar{a} > 0,$$

we have $\underline{a} < a < \bar{a}$. Next, given any $\varepsilon_0 > 0$, let us solve

$$\begin{aligned} \frac{d\overline{Y}(X)}{dX} &= \frac{\mathcal{N}}{2} + \bar{c}\frac{X}{\overline{Y}}, \\ \frac{dY(X)}{dX} &= \frac{\mathcal{N}}{2} + E(X)\frac{X}{Y}, \qquad (6) \\ \frac{d\underline{Y}(X)}{dX} &= \frac{\mathcal{N}}{2} + \underline{c}\frac{X}{\underline{Y}} \end{aligned}$$

with these initial values:

$$\underline{a}\varepsilon_0 = \underline{Y}(\varepsilon_0) < Y(\varepsilon_0) < \overline{Y}(\varepsilon_0) = \bar{a}\varepsilon_0.$$

Clearly, $\underline{Y}(X) = \underline{a}X$ and $\overline{Y}(X) = \bar{a}X$ are the solutions for the first and third equations.

This being understood, let us observe the following lemma:

Lemma 2.2. *Given any $T_0 > \varepsilon_0$, it holds that*

$$\underline{Y}(X) \quad < Y(X) < \quad \overline{Y}(X) \tag{7}$$

on $[\varepsilon_0, T_0]$ for all $\underline{c}$ and $\bar{c}$ satisfying

$$0 \leq \underline{c} < E(T_0), \quad c < \bar{c}.$$

Proof. Since $E(X)X/Y > 0$ we have $dY/dX > \mathcal{N}/2$, and thereby $Y(X) > 0$ is monotone increasing. Thus we can conclude that Y is bounded by X axis and that dY/dX does not diverge at a finite time from (5). Consequently, let us next assume that $\overline{Y}$ cannot bind Y from above, and accordingly, there exists an initial intersection of $\overline{Y}$ and Y at $X_0 < \infty$.

On one hand, since $\overline{Y}(\varepsilon_0) > Y(\varepsilon_0)$ and $\overline{Y}(X)$ meet $Y(X)$ for the first time at X_0, we have

$$\left[\frac{d(\overline{Y} - Y)}{dX}\right]\Bigg|_{X=X_0} \leq 0.$$

On the other hand, given that $E(X)$ is a monotone decreasing function of X and that $E(0) = c < \bar{c}$, it follows that

$$\left[\frac{d\overline{Y}}{dX} - \frac{dY}{dX}\right]\Bigg|_{X=X_0} = (\bar{c} - E(X_0))\frac{X_0}{\overline{Y}(X_0)} > 0.$$

Here we have obtained a contradiction, which implies the global existence of Y on $[\varepsilon_0, \infty)$. So, in order to verify (7), let us further assume that $\underline{Y}$ cannot bind Y from below, and accordingly, there exists an initial intersection of $\underline{Y}$ and Y at $X_0 < T_0$. Then we obtain $[d(Y - \underline{Y})/dX]|_{X=X_0} \leq 0$. On the contrary, our assumption $E(X_0) \geq E(T_0) > \underline{c}$ provides

$$\left[\frac{dY}{dX} - \frac{d\underline{Y}}{dX}\right]\Bigg|_{X=X_0} = (E(X_0) - \underline{c})\frac{X_0}{\underline{Y}(X_0)} > 0.$$

Thereby, we have obtained another contradiction. □

Utilizing the Lemma above, for each integer $j > 0$, we can obtain the solution $Y = Y_j$ which has an initial value $Y(\varepsilon_0) = a\varepsilon_0$ on $[\varepsilon_0, 1]$, defined as $\varepsilon_0 = 1/j$. From the conclusions made above, it can be noted that each element of the sequence $\{Y_j\}$ satisfies:

$$\underline{Y} < Y_j < \overline{Y} \qquad \text{on} \qquad [1/j, 1].$$

Furthermore, it should be noted that the equation (5) combined with (7) provides the upper and lower bounds for dY/dX and d^2Y/dX^2. Given these, when we examine $\{Y_j\}$ firstly on $[1/2, 1]$, by using the Ascoli-Arzelà theorem, we have a sub-sequence $\{Y_{j_i}\}$ converging in $C^1([1/2, 1])$. Then, secondly, focusing on $[1/3, 1]$, by utilizing the Ascoli-Arzelà theorem once again, we can select a sub-sequence $\{Y_{j_{i_l}}\}$ converging in $C^1([1/3, 1])$. Upon continuing, we obtain a diagonal sub-sequence converging to Y_∞ locally in $C^1((0, 1])$. Thus we have obtained a solution of (5) satisfying

$$\underline{Y}(X) < Y_\infty(X) < \overline{Y}(X) \qquad \text{on } (0, 1].$$

Having already established the existence of solution Y for (5) with an initial value $Y(1) = Y_\infty(1)$ satisfying $\underline{Y}(1) < Y_\infty(1) < \overline{Y}(1)$ as in Lemma 2.2, the continuation of Y_∞ provides a solution Y on $(0, \infty)$.

At this point, it should be noted that, when dividing (7) by X, it holds that

$$\underline{a} < \frac{Y(X)}{X} < \overline{a}.$$

By setting $0 < X < T_0 \to 0$ in order to let $\overline{c}, \underline{c} \to c$ and $\overline{a}, \underline{a} \to a$, we observe that

$$\frac{Y(X)}{X} \to a \qquad X \to 0.$$

Hence, we obtained our desired global solution $Y(X)$ for (5) defined on $(0, \infty)$.

Step 2: It is important to note that $Y(X) \in C^\infty$ by induction: firstly, (5) implies $Y(X) \in C^1$, and secondly, $Y(X) \in C^k$ $(k \geq 1)$ implies that the left-hand side of (5) [that is, dY/dX] is also C^k thereby giving $Y(X) \in C^{k+1}$. Consequently, we see that a solution for $f'(t) = Y(f(t))$ exists locally and $f'(t)$ does not diverge unless $f(t)$ diverges. We shall prove that $f(t)$ exists globally on $\mathbb{R}$ by showing that $f(t)$ does not diverge at a finite time t.

Given $t_0 \in \mathbb{R}$, let us firstly solve

$$\overline{f}'(t) = \overline{a}\overline{f}(t), \quad f'(t) = Y(f(t)), \quad \underline{f}'(t) = \underline{a}\,\underline{f}(t)$$

for $t > t_0$ with these initial values: $\overline{f}(t_0) = \overline{f}_0 > f(t_0) = f_0 > \underline{f}(t_0) = \underline{f}_0 > 0$. According to (7), as long as $0 < f < T_0$, it holds that

$$\overline{f}' - f' = \overline{a}\overline{f} - Y(f) \geq \overline{a}(\overline{f} - f), \qquad f' - \underline{f}' = Y(f) - \underline{a}\underline{f} \geq \underline{a}(f - \underline{f}),$$

and hence

$$\overline{f} - f \geq e^{(t-t_0)\overline{a}}(\overline{f}_0 - f_0), \qquad f - \underline{f} \geq e^{(t-t_0)\underline{a}}(f_0 - \underline{f}_0). \tag{8}$$

Substituting both $\overline{f} = \overline{f}_0 e^{(t-t_0)\overline{a}}$ and $\underline{f} = \underline{f}_0 e^{(t-t_0)\underline{a}}$ for each respective term of (8), and noting that $f_0 e^{(t-t_0)\overline{a}} \leq T_0$ implies

$$t \leq \frac{1}{\overline{a}} \log(T_0/f_0) + t_0,$$

we have,

$$f_0 e^{(t-t_0)\overline{a}} \geq f(t) \geq f_0 e^{(t-t_0)\underline{a}}, \tag{9}$$

for the time interval $[t_0, \overline{a}^{-1}\log(T_0/f_0) + t_0]$. Since T_0 can be infinity, we can observe that $f(t)$ and $f'(t)$, which exist locally, do not diverge at a finite time. Thereby we obtain the global solution $f(t)$ for the time interval $[t_0, \infty)$.

Secondly, in order to see the behavior of $f(t)$ for $t \leq t_0$, by setting $\tilde{t}_0 = -t_0$, we shall solve the following equations:

$$-\overline{g}'(t) = \overline{a}\,\overline{g}(t), \quad -g'(t) = Y(g(t)), \quad -\underline{g}'(t) = \underline{a}\,\underline{g}(t)$$

for $t \geq \tilde{t}_0$ with these initial values: $\overline{g}(\tilde{t}_0) = \overline{g}_0 > g(\tilde{t}_0) = g_0 > \underline{g}(\tilde{t}_0) = \underline{g}_0 > 0$. By adapting the argument given in the above for $t \geq t_0$ to $t \geq \tilde{t}_0$, it holds that

$$-(\overline{g}' - g') = \overline{a}\,\overline{g} - Y(g) \geq \overline{a}(\overline{g} - g), \qquad -(g' - \underline{g}') = Y(g) - \underline{a}\,\underline{g} \geq \underline{a}(g - \underline{g}),$$

and hence, it follows that

$$\overline{g} - g \leq e^{-(t-\tilde{t}_0)\overline{a}}(\overline{g}_0 - g_0), \qquad g - \underline{g} \leq e^{-(t-\tilde{t}_0)\underline{a}}(g_0 - \underline{g}_0).$$

By substituting both $\overline{g} = \overline{g}_0 e^{-(t-\tilde{t}_0)\overline{a}}$ and $\underline{g} = \underline{g}_0 e^{-(t-\tilde{t}_0)\underline{a}}$ for each respective term in the above, we have

$$g_0 e^{-(t-\tilde{t}_0)\underline{a}} \leq g(t) \leq g_0 e^{-(t-\tilde{t}_0)\overline{a}}, \tag{10}$$

for $t > \tilde{t}_0$. Since $f(-t) = g(t)$ when $t > \tilde{t}_0$, we have obtained the solution $f(t)$ for the time interval $(-\infty, t_0]$. Hence, we have a positive solution $f(t)$ for the equation (4) for all $t \in \mathbb{R}$, which is in C^∞ by using an argument similar to that in the case of $Y(X)$. By means of Proposition 2.1, we are led to the family of solutions $\psi_\lambda(\rho)$ for the equation (3) and thereby we have completed the proof of Theorem 2.1. □

Proof of the first claim of Theorem 1.1. Since $\dot{\psi}_\lambda(\rho) > 0$ for all $\rho > 0$, we have $Ju = (\partial s^i \circ u/\partial s^j) > 0$ for all $(\mathbf{x}, \mathbf{t}, \rho) \in N \cdot \mathbb{R}_+$, where $(s^1, \dots, s^m) = (x^{11}, \dots, t^d, \rho)$. Since Theorem 2.1 is at our disposal, we can prove this claim by using Lemma 2.1. □

Proposition 2.2. *For $C_0 > 0$, $\mathcal{N} = \sum_{l=1}^k l n_l$ and $\delta > 1$, there exist positive constants $C_1(C_0, \mathcal{N}), C_2(C_0, \mathcal{N})$ and $C_3(C_0, \mathcal{N}, \delta)$ so that, for $\rho > C_0$, the following inequalities hold:*

$$\psi(\rho) \geq \rho \exp(C_1 \rho^{\mathcal{N}/2}), \tag{11}$$

$$\dot{\psi}(\rho) \leq C_2 \rho^{3\mathcal{N}/2} \psi(\rho), \tag{12}$$

$$\dot{\psi}(\rho) \leq C_3 \psi(\rho)^\delta \qquad (\delta > 1). \tag{13}$$

Proof. Firstly, let us set $f_0 := f(\log(t_0))$ for $t_0 = \log(\lambda C_0)$.

By setting $\underline{c} = 0$ and $\underline{a} = \mathcal{N}/2$, accordingly, thereby we shall set $T_0 = \infty$, so that the inequality (9) holds for the time interval $[t_0, \infty)$. Owing to this inequality, we have

$$\psi(\rho) \geq \rho \exp\big(f_0 e^{\mathcal{N}(\log(\lambda\rho) - \log(\lambda C_0))/2}\big) = \rho \exp(f_0 (C_0^{-1}\rho)^{\mathcal{N}/2}) = \rho \exp(C_1 \rho^{\mathcal{N}/2}),$$

where $C_1 = f_0 C_0^{-\mathcal{N}/2}$. Thus, we have obtained the inequality (11).

Secondly, given that each n_l is a non-negative integer, it holds that

$$\sum_{l=1}^k l^2 n_l \leq \sum_{l=1}^k l^2 n_l^2 \leq (\sum_{l=1}^k l n_l)^2 = \mathcal{N}^2,$$

and, thereby, it follows that

$$a = \Big(\mathcal{N} + \sqrt{\mathcal{N}^2 + 8\textstyle\sum_{l=1}^k l^2 n_l}\Big)/4 \leq \mathcal{N}.$$

Recognizing that $\bar{a}$ in (7) and (9) can be arbitrarily close to a, we shall set $\bar{a}$ so that

$$\mathcal{N} < \bar{a} \leq 3\mathcal{N}/2$$

holds. Since $C_0^{-1}\rho \geq 1$ from our assumption $\rho \geq C_0$, we have

$$C_0^{-(3\mathcal{N}/2) + \bar{a} - 1} \rho^{(3\mathcal{N}/2) - \bar{a} + 1} \geq 1. \tag{14}$$

Moreover, by using (7) and (9), we have $f'(t) = Y(f(t)) \leq \overline{a}f(t) \leq \overline{a}f_0 e^{\overline{a}(t-t_0)}$. Hence,

$$\begin{aligned}\dot{\psi}(\rho) &= (1 + f'(\log(\lambda\rho)))\rho^{-1}\psi(\rho) \\ &\leq \left(1 + \overline{a}f_0 e^{(\log(\lambda\rho)-\log(\lambda C_0))\overline{a}}\right)\rho^{-1}\psi(\rho) \\ &\leq \left(1 + \max(1, \overline{a}f_0)\left(\rho C_0^{-1}\right)^{\overline{a}}\right)\rho^{-1}\psi(\rho) \\ &\leq 2\max(1, \overline{a}f_0)C_0^{-\overline{a}}\rho^{\overline{a}-1}\psi(\rho) \\ &\leq C_4 C_0^{-(3\mathcal{N}/2)-1}\rho^{3\mathcal{N}/2}\psi(\rho) = C_2\rho^{3\mathcal{N}/2}\psi(\rho),\end{aligned}$$

where $C_4 = 2\max(1, \overline{a}f_0)$ and $C_2 = C_4 C_0^{-(3\mathcal{N}/2)-1}$. Thus, we have obtained (12).

Thirdly, since the inequality (11) is at our disposal, combining this with our assumption $\rho > C_0$, we have $\psi(\rho) \geq \rho\exp(C_1\rho^{\mathcal{N}/2}) > C_0\exp(C_1\rho^{\mathcal{N}/2})$. Hence

$$\rho^{\mathcal{N}/2} \leq C_1^{-1}\log(\psi(\rho)C_0^{-1}). \tag{15}$$

Combining (15) with (12) and setting $\eta(\rho) = \psi(\rho)C_0^{-1}$, we have a constant C_3 so that

$$\begin{aligned}\dot{\psi}(\rho) &\leq C_2\rho^{3\mathcal{N}/2}\psi(\rho) \leq C_2 C_0^{-1}\left(C_1^{-1}\log(\eta(\rho))\right)^3(\eta(\rho))^{1-\delta}\eta(\rho)^{\delta} \\ &= C_2 C_0^{-1}C_1^{-3}\left(\frac{3}{\delta-1}\frac{\log(\eta(\rho)^{\frac{\delta-1}{3}})}{\eta(\rho)^{\frac{\delta-1}{3}}}\right)^3\eta(\rho)^{\delta} \leq C_3\psi(\rho)^{\delta} \qquad (\delta > 1).\end{aligned}$$

Here, we have deduced this last inequality by observing that $\eta(\rho) = C_0^{-1}\psi(\rho) \geq \exp(C_1 C_0^{\alpha\mathcal{N}})$ for $\rho > C_0$ according to (11), and that $\log(x)/x$ is a bounded function on $x > C_0$. Hence, we have (13). □

Lemma 2.3. *There exists* $C_5 = C_5(\mathcal{N})$ *so that*

$$\lim_{\rho\to\infty}\frac{f(\log(\lambda\rho))}{(\mathcal{N}/2)^{-1}\lambda^{\mathcal{N}/2}\rho^{\mathcal{N}/2}} = C_5.$$

Proof. Let h be the solution for $h''(t) = \mathcal{N}h'(t)/2 + \mathcal{N}/2$ with these initial values: $h'(t_0) = f'(t_0) > 0$ and $h(t_0) = f(t_0) > 0$. Noting that

$$\frac{d(h'-f')}{dt} = \frac{1}{2}\mathcal{N}(h'-f') + \frac{1}{2}\sum_{l=1}^{k} e^{-lf(t)}ln_l \geq \frac{1}{2}\mathcal{N}(h'-f'),$$

we find that $h'(t) \geq f'(t)$ for $t \geq t_0$, and thereby it follows that

$$h'(t) = e^{\mathcal{N}(t-t_0)/2}(f'(t_0)+1) - 1 \geq f'(t).$$

Dividing the above by $e^{\mathcal{N}t/2}$, we see that $f'(t)e^{-\mathcal{N}t/2}$ is bounded from above. Moreover, since $f(t) > 0$, we have

$$\frac{d(f'(t)e^{-\mathcal{N}t/2})}{dt} = e^{-\mathcal{N}t/2}\left(f''(t) - \frac{1}{2}\mathcal{N}f'(t)\right) = \frac{1}{2}e^{-\mathcal{N}t/2}\sum_{l=1}^{k}(1 - e^{-lf(t)})ln_l > 0,$$

thereby it is apparent that $f'(t)e^{-\mathcal{N}t/2}$ is monotone increasing. Given the observation made above, we have a constant C_5 so that

$$C_5 = \lim_{t\to\infty} \frac{f'(t)}{e^{\mathcal{N}t/2}} = \lim_{t\to\infty} \frac{f(t)}{(\mathcal{N}/2)^{-1} e^{\mathcal{N}t/2}}. \tag{16}$$

□

Proof of the second claim of Theorem 1.1.

Let $(\mathbf{n}, \rho(t))$ be the geodesic of $N \cdot \mathbb{R}_+$ joining $(\mathbf{n}, \rho_0)$ and $\mathbf{s} := (\mathbf{n}, \rho)$, satisfying $\rho(0) = \rho_0$ and $\rho(1) = \rho$. Given that

$$\operatorname{dist}((\mathbf{n}, \rho_0), (\mathbf{n}, \rho)) = \left| \int_0^1 \sqrt{\frac{\rho'(t)^2}{4\rho(t)^2}} dt \right| = \left| \frac{1}{2} \int_{\rho_0}^{\rho} \frac{d\rho}{\rho} \right| = \left| \log(\rho^{1/2}/\rho_0^{1/2}) \right|, \tag{17}$$

by using Lemma 2.3, we have

$$\begin{aligned} \frac{\operatorname{dist}(u_\lambda(\mathbf{s}), u_{\lambda'}(\mathbf{s}))}{\rho^{\mathcal{N}/2}} &= \left| \frac{\log(\psi(\lambda\rho))}{\rho^{\mathcal{N}/2}} - \frac{\log(\psi(\lambda'\rho))}{\rho^{\mathcal{N}/2}} \right| = \left| \frac{f(\log(\lambda\rho))}{\rho^{\mathcal{N}/2}} - \frac{f(\log(\lambda'\rho))}{\rho^{\mathcal{N}/2}} \right| \\ &\to C_5 (\mathcal{N}/2)^{-1} |(\lambda^{\mathcal{N}/2} - \lambda'^{\mathcal{N}/2})| \quad \text{as } \mathbf{s} = (\mathbf{n}, \rho) \to \infty. \end{aligned} \tag{18}$$

Since the line segment joining $(\mathbf{n}, \rho_0)$ and $(\mathbf{n}, \rho)$ is a geodesic of $N \cdot \mathbb{R}_+$, we have

$$\begin{aligned} \operatorname{dist}((\mathbf{n}, \rho_0), (\mathbf{n}, \rho)) &\le \operatorname{dist}((\mathbf{n}_0, \rho_0), (\mathbf{n}, \rho)) \\ &\le \operatorname{dist}((\mathbf{n}_0, \rho_0), (\mathbf{n}, \rho_0)) + \operatorname{dist}((\mathbf{n}, \rho_0), (\mathbf{n}, \rho)). \end{aligned}$$

Applying the exponential function $\exp(*)$ to the above, we have a constant C_6 so that

$$\begin{aligned} \exp(\mathcal{N} \operatorname{dist}((\mathbf{n}, \rho_0), (\mathbf{n}, \rho))) &\le \exp(\mathcal{N} \operatorname{dist}((\mathbf{n}_0, \rho_0), (\mathbf{n}, \rho))) \\ &\le C_6 \exp(\mathcal{N} \operatorname{dist}((\mathbf{n}, \rho), (\mathbf{n}, \rho_0))), \end{aligned}$$

this is because $\operatorname{dist}((\mathbf{n}_0, \rho_0), (\mathbf{n}, \rho_0)) < \infty$ from our assumption $|\mathbf{n}| < \infty$ when $(\mathbf{n}, \rho) \to \infty$. Furthermore, combining this with (17), we obtain

$$\exp(\mathcal{N} \operatorname{dist}((\mathbf{n}_0, \rho_0), (\mathbf{n}, \rho))) \sim \exp(\mathcal{N} \operatorname{dist}((\mathbf{n}, \rho), (\mathbf{n}, \rho_0))) = \rho^{\mathcal{N}/2} / \rho_0^{\mathcal{N}/2}$$

when we let $(\mathbf{n}, \rho) \to \infty$ while keeping $|\mathbf{n}|$ finite. Combining the above with (18), we have proven the claim. □

Proposition 2.3. *Near $\rho = 0$, we have*

$$\psi(\rho) = \rho + o(\rho|\rho|^{\underline{a}}) \quad \textit{for any } \underline{a} < \left(\mathcal{N} + \sqrt{\mathcal{N}^2 + 8 \textstyle\sum_{l=1}^{k} l^2 n_l}\right) \Big/ 4. \tag{19}$$

Proof. Fix solution $f(-t) = g(t)$ and let $t = -\log(|\rho|)$. Dividing (9) by $e^{-t\underline{a}}$, we have

$$g(t) e^{t\underline{a}} \ge g(t_0) e^{t_0 \underline{a}} e^{t(\underline{a} - \bar{a})}.$$

Consequently, we know that $g(t)e^{t\underline{a}}$ is bounded from below. Further to this, we have

$$\frac{d(e^{\underline{a}t}g(t))}{dt} = \underline{a}e^{t\underline{a}}g(t) + e^{t\underline{a}}g'(t) = e^{t\underline{a}}\Big(\underline{a}g(t) + g'(t)\Big) = e^{t\underline{a}}\Big(\underline{Y}(g) - Y(g)\Big) < 0,$$

thereby, we can observe that $g(t)e^{t\underline{a}}$ is monotone decreasing. Given these considerations, we have a constant $C_7(\underline{c})$, so that

$$C_7 = \lim_{t\to\infty} \frac{g(t)}{e^{-t\underline{a}}}.$$

Now, let us suppose $C_7 \neq 0$. By using de l'Hôpetal's theorem, we have

$$a = \lim_{X\to 0}\frac{dY}{dX} = -\lim_{t\to\infty}\frac{g''(t)}{g'(t)} = -\lim_{t\to\infty}\frac{g'(t)}{g(t)} = -\lim_{t\to\infty}\frac{g'(t)}{e^{-t\underline{a}}}\lim_{t\to\infty}\frac{e^{-t\underline{a}}}{g(t)} = \underline{a},$$

which contradicts $\underline{a} < a$. Consequently, we must have $C_7(\underline{c}) = 0$ and $\lim_{\rho\to 0} f(\log(|\rho|))/|\rho|^{\underline{a}} = 0$, thereby we have

$$\begin{aligned} \exp(f(\log(|\rho|))) &= 1 + f(\log(|\rho|))\sum_{n=1}^{\infty}\frac{f(\log(|\rho|))^{n-1}}{n!} \\ &= 1 + |\rho|^{\underline{a}}\Big(f(\log(|\rho|))/|\rho|^{\underline{a}}\Big)\sum_{n=1}^{\infty}\frac{f(\log(|\rho|))^{n-1}}{n!} \\ &= 1 + o(|\rho|^{\underline{a}}). \end{aligned}$$

Furthermore, for a given t_0 and T_0 satisfying $T_0 > f(t_0)$, we have

$$\begin{aligned} f_0e^{(t-t_0)\underline{a}} \leq\ & f(t) \leq f_0e^{(t-t_0)\overline{a}} \quad \text{on} \quad [t_0, \overline{a}^{-1}\log(T_0/f_0) + t_0], \\ f_0e^{(t-t_0)\overline{a}} \leq\ & f(t) \leq f_0e^{(t-t_0)\underline{a}} \quad \text{on} \quad (-\infty_0, t_0], \end{aligned}$$

for all $\underline{c}$ satisfying $0 \leq \underline{c} < E(T_0)$. Since $f(t_0) \to 0$ as $t_0 \to -\infty$, we can let $T_0 \to 0$ so that $E(T_0) \to c$ as $t_0, t \to -\infty$. This leads us to conclude the following:

$$\psi(\rho) = \rho + o(\rho|\rho|^{\underline{a}}), \qquad \underline{a} < \Big(\mathcal{N} + \sqrt{\mathcal{N}^2 + 8\textstyle\sum_{l=1}^{k} l^2 n_l}\,\Big)\Big/4.$$

□

This lemma proves the third claim of Theorem 1.1.

3. The boundary regularity

3.1. **Notation.** Let σ denote the geodesic symmetry of $N \cdot \mathbb{R}_+$ at $(0,0,1)$ which will be given in Lemma 5.2. By the third claim of Theorem 1.1, it suffices to estimate the regularity of $\tilde{u}_\lambda := \sigma \circ u \circ \sigma$ only near a small neighborhood of $(0,0,0)$. We shall use u and $\tilde{u}$ as abbreviations for u_λ and $\tilde{u}_\lambda$, respectively.

3.2. Regularity of $\tilde{u}$.

Definition 3.1. *For* $\mathbf{s} = (\mathbf{x}, \mathbf{t}, \rho) \in \mathbb{K}^n \times \mathrm{Im}(\mathbb{K}) \times \mathbb{R}_+$, *let* $\|\cdot\|_H : M \to \mathbb{R}_{\geq 0}$ *be*

$$\|\mathbf{s}\|_H := \left||\mathbf{x}|^2 + \mathbf{t} + \rho\right|^{1/2} = \left((|\mathbf{x}|^2 + \rho)^2 + |\mathbf{t}|^2\right)^{1/4}.$$

Here $|\cdot|$ *denotes the Euclidean norm.*

Remark 3.1. It holds that $\rho\circ\sigma = \rho/\|\mathbf{s}\|_H^4$.

The following observation will be useful in order to prove Lemma 3.8:

Lemma 3.1. $\|\sigma(\mathbf{s})\|_H = 1/\|\mathbf{s}\|_H$.

Proof.

$$\begin{aligned}\|\sigma(\mathbf{s})\|_H^2 &= \left||\mathbf{x}\circ\sigma|^2 + \mathbf{t}\circ\sigma + \rho\circ\sigma\right| \\ &= \left||\mathbf{x}|^2 - \mathbf{t} + \rho\right|\left||\mathbf{x}|^2 + \mathbf{t} + \rho\right|^{-2} = \|\mathbf{s}\|_H^{-2}.\end{aligned}$$

□

Lemma 3.2. *For* $|\mathbf{s}| \leq 5^{-1/2}$, *we have*

$$C_8|\mathbf{s}| \leq \|\mathbf{s}\|_H \leq C_9|\mathbf{s}|^{1/2}. \tag{20}$$

Proof.

$$\begin{aligned}\|\mathbf{s}\|_H^4 &= (|\mathbf{x}|^2 + \rho)^2 + |\mathbf{t}|^2 \\ &\leq (|\mathbf{s}|^2 + |\mathbf{s}|)^2 + |\mathbf{s}|^2 \leq 5|\mathbf{s}|^2 \ (\leq 1),\end{aligned} \tag{21}$$

$$\begin{aligned}|\mathbf{s}|^2 &= |\mathbf{x}|^2 + |\mathbf{t}|^2 + \rho^2 \\ &\leq \|\mathbf{s}\|_H^2 + \|\mathbf{s}\|_H^4 + \|\mathbf{s}\|_H^4 \leq 3\|\mathbf{s}\|_H^2.\end{aligned} \tag{22}$$

This completes the estimate. □

Lemma 3.3. $\|(u\circ\sigma)(\mathbf{s})\|_H^{-1}\|\mathbf{s}\|_H^{-1} \leq 1$.

Proof. By using Definition 3.1, we have $\|(\mathbf{x}, \mathbf{t}, \rho')\|_H \geq \|(\mathbf{x}, \mathbf{t}, \rho)\|_H$ when $\rho' \geq \rho$. Since $\psi(\rho) \geq \rho$, by using Lemma 3.1, we have

$$\begin{aligned}\|u\circ\sigma(\mathbf{s})\|_H &= \|(\mathbf{x}\circ\sigma, \mathbf{t}\circ\sigma, \psi(\rho\circ\sigma))\|_H \geq \|(\mathbf{x}\circ\sigma, \mathbf{t}\circ\sigma, \rho\circ\sigma)\|_H \\ &= \|\sigma(\mathbf{s})\|_H = \|\mathbf{s}\|_H^{-1}.\end{aligned}$$

□

Lemma 3.4. *For* $|\mathrm{s}| \leq 5^{-1/2}$, *we have*

$$|\tilde{u}(\mathrm{s})| \leq C_{10}|\mathrm{s}|^{1/2}.$$

Proof. By using Lemmas 3.1 and 3.3, we have

$$\|\tilde{u}\|_H = \|\sigma\circ u\circ\sigma\|_H = \|u\circ\sigma\|_H^{-1} = \|u\circ\sigma\|_H^{-1}\|\mathrm{s}\|_H^{-1}\|\mathrm{s}\|_H \leq \|\mathrm{s}\|_H. \tag{23}$$

Note that we have $\|\tilde{u}\|_H \leq 5^{1/4}|\mathrm{s}|^{1/2}$ by using the inequalities (21) and (23). In particular, we also have $\|\tilde{u}\|_H \leq 1$. Hence, by using the inequality (22), we have $3^{-1/2}|\tilde{u}| \leq \|\tilde{u}\|_H$. In summary, we have obtained $C_8|\tilde{u}| \leq C_9|\mathrm{s}|^{1/2}$, which completes the proof of the lemma. □

Proposition 3.1. *For* $|\mathrm{s}| \leq 5^{-1/2}$, *we have*

$$\|J\tilde{u}\| \leq C_{11}|\mathrm{s}|^{-(\delta+2)} \textit{ for any } \delta > 1.$$

Here $\|J\tilde{u}\| = \left(\mathrm{Tr}({}^t J\tilde{u}\cdot J\tilde{u})\right)^{1/2} = \left(\sum_{ll'} |\partial(s^l\circ\tilde{u})/\partial s^{l'}|^2\right)^{1/2}$ *for* $(s^1, \ldots, s^m) = (x^{11}, \ldots, t^d, \rho)$.

The proof of this proposition shall be made in the next section.

Lemma 3.5. *Let* $|\mathrm{s}_1| \leq |\mathrm{s}_2| \leq 1/2$. *Let* s *be any point on the line segment joining* s_1 *and* s_2. *Suppose that we have*

$$\|J\tilde{u}\| \leq C_{12}|\mathrm{s}|^{-\beta}, \quad |\tilde{u}(\mathrm{s})| \leq C_{13}|\mathrm{s}|^{\gamma}$$

with $\beta + \gamma \geq 1 + \varepsilon$ *for* $\varepsilon > 0$. *Then*

$$|\tilde{u}(\mathrm{s}_1) - \tilde{u}(\mathrm{s}_2)| < C_{14}|\mathrm{s}_1 - \mathrm{s}_2|^{\gamma/(\beta+\gamma)}.$$

Proof. In the case of $|\mathrm{s}_2|^{\beta+\gamma} \leq |\mathrm{s}_1 - \mathrm{s}_2|$, it holds that

$$|\tilde{u}(\mathrm{s}_1) - \tilde{u}(\mathrm{s}_2)| \leq |\tilde{u}(\mathrm{s}_1)| + |\tilde{u}(\mathrm{s}_2)| \leq C_{13}|\mathrm{s}_1|^{\gamma} + C_{13}|\mathrm{s}_2|^{\gamma} \leq 2C_{13}|\mathrm{s}_1 - \mathrm{s}_2|^{\gamma/(\beta+\gamma)}.$$

When $|\mathrm{s}_2|^{\beta+\gamma} \geq |\mathrm{s}_1 - \mathrm{s}_2|$, for any point s on the line segment from s_1 to s_2, we have $|\mathrm{s}| \geq |\mathrm{s}_2| - |\mathrm{s} - \mathrm{s}_2| \geq |\mathrm{s}_2| - |\mathrm{s}_1 - \mathrm{s}_2| \geq |\mathrm{s}_2|(1 - |\mathrm{s}_2|^{\beta+\gamma-1}) \geq |\mathrm{s}_2|(1 - 2^{-\varepsilon})$, and thereby $|\mathrm{s}|^{-\beta} \leq C_{15}|\mathrm{s}_1 - \mathrm{s}_2|^{-\beta/(\beta+\gamma)}$. By using the mean value inequality, we have

$$|\tilde{u}(\mathrm{s}_1) - \tilde{u}(\mathrm{s}_2)| \leq \|J\tilde{u}(\mathrm{s})\||\mathrm{s}_1 - \mathrm{s}_2| \leq C_{12}|\mathrm{s}|^{-\beta}|\mathrm{s}_1 - \mathrm{s}_2| \leq C_{15}C_{12}|\mathrm{s}_1 - \mathrm{s}_2|^{\gamma/(\beta+\gamma)}.$$

□

Proof of the fourth claim of Theorem 1.1.

Combining Lemmas 3.4 and 3.5 with Proposition 3.1, and noting that $1/2 \ / \ (1/2+2+\delta) = 1/(5+2\delta)$, we have proven the claim. □

3.3. **The estimate of $\|J\tilde{u}\|$.** The purpose of this subsection is to prove Proposition 3.1.

Lemma 3.6. *For any path $\mathbf{s}_t$ satisfying $|\mathbf{s}_t| \le 5^{-1/2}$, and $\rho\circ\sigma(\mathbf{s}_t) \to 0$ as $t \to \infty$, we have $J\tilde{u} \to \mathrm{Id}$ (the identity matrix) as $t \to \infty$.*

Proof. Let $\mathbf{s}_\infty = (\mathbf{x}_\infty, \mathbf{t}_\infty, \rho_\infty)$ be any point in $\overline{M}$ so that $\rho\circ\sigma(\mathbf{s}_t) \to 0$ as $\mathbf{s}_t \to \mathbf{s}_\infty$. Since $\rho\circ\sigma(\mathbf{s}_\infty) = 0$, it holds that $\mathbf{x}_\infty \ne 0$ or $\mathbf{t}_\infty \ne 0$. Moreover, since $|\mathbf{s}_t| \le 5^{-1/2}$, we also have $\mathbf{x}_\infty\circ\sigma \ne 0$ or $\mathbf{t}_\infty\circ\sigma \ne 0$. In addition, $\sigma \in C^\infty$ near $\mathbf{s}_\infty = (\mathbf{x}_\infty, \mathbf{t}_\infty, \rho_\infty)$ when $\mathbf{x}_\infty \ne 0$ or $\mathbf{t}_\infty \ne 0$. We also have $u = \big(\mathbf{x}, \mathbf{t}, \rho + o(\rho^{\underline{a}+1})\big)$ according to Proposition 2.3. If a function $f \in C^\infty$ near $\mathbf{x}, \mathbf{t} \ne 0$, it holds that $f\big(\mathbf{x}, \mathbf{t}, \rho + o(\rho^{\underline{a}+1})\big) = f(\mathbf{x}, \mathbf{t}, \rho) + o(\rho^{\underline{a}+1})$ and we therefore have $f\big(\mathbf{x}\circ\sigma, \mathbf{t}\circ\sigma, \rho\circ\sigma + o((\rho\circ\sigma)^{\underline{a}+1})\big) = f(\mathbf{x}\circ\sigma, \mathbf{t}\circ\sigma, \rho\circ\sigma) + o\big((\rho\circ\sigma)^{\underline{a}+1}\big)$. By applying this observation to $\partial(s^i\circ\sigma)/\partial s^l$, we have

$$\frac{\partial(s^i\circ\sigma)}{\partial s^l}\big(\mathbf{x}\circ\sigma, \mathbf{t}\circ\sigma, \rho\circ\sigma + o((\rho\circ\sigma)^{\underline{a}+1})\big) = \frac{\partial(s^i\circ\sigma)}{\partial s^l}(\mathbf{x}\circ\sigma, \mathbf{t}\circ\sigma, \rho\circ\sigma) + o\big((\rho\circ\sigma)^{\underline{a}+1}\big).$$

Utilizing the above and the chain rule, we have

$$\begin{aligned}
\frac{\partial(s^i\circ\tilde{u})}{\partial s^j} &= \sum_{l,k=1}^{m} \frac{\partial(s^i\circ\sigma)}{\partial s^l}(u\circ\sigma)\cdot\frac{\partial(s^l\circ u)}{\partial s^k}(\sigma)\cdot\frac{\partial(s^k\circ\sigma)}{\partial s^j} \\
&= \sum_{l=1}^{m-1} \frac{\partial(s^i\circ\sigma)}{\partial s^l}(u\circ\sigma)\cdot\frac{\partial(s^l\circ\sigma)}{\partial s^j} + \frac{\partial(s^i\circ\sigma)}{\partial s^m}(u\circ\sigma)\cdot\frac{\partial(s^m\circ\sigma)}{\partial s^j}\cdot\dot{\psi}(\rho\circ\sigma) \\
&= \sum_{l=1}^{m-1}\Big(\frac{\partial(s^i\circ\sigma)}{\partial s^l}(\sigma) + o\big((\rho\circ\sigma)^{\underline{a}+1}\big)\Big)\cdot\frac{\partial(s^l\circ\sigma)}{\partial s^j} \\
&\quad +\Big(\frac{\partial(s^i\circ\sigma)}{\partial s^m}(\sigma) + o\big((\rho\circ\sigma)^{\underline{a}+1}\big)\Big)\cdot\frac{\partial(s^m\circ\sigma)}{\partial s^j}\cdot\big(1 + o((\rho\circ\sigma)^{\underline{a}})\big) \\
&= \sum_{l=1}^{m} \frac{\partial(s^i\circ\sigma)}{\partial s^l}(\sigma)\cdot\frac{\partial(s^l\circ\sigma)}{\partial s^j} + o\big((\rho\circ\sigma)^{\underline{a}+1}\big) \\
&= \delta_{ij} + o\big((\rho\circ\sigma)^{\underline{a}+1}\big).
\end{aligned}$$

This completes the proof of the lemma. □

Remark 3.2. Given Lemma 3.6, in order to prove Proposition 3.1, it suffices to estimate $\|J\tilde{u}\|$ for sufficiently large $\rho\circ\sigma$. In what follows, we shall assume $\rho\circ\sigma = \rho/\|\mathbf{s}\|_H^4 \ge C_0$ for a constant $C_0 > 0$, accordingly.

In the following, we shall prove Proposition 3.2, which will be the key to making an estimation of $\|J\tilde{u}\|$. To begin with, let us note that the left invariant orthonormal frames obtained in previous section, can be expressed as follows:

(24)
$$
\begin{aligned}
&\text{when } \mathbb{C}H^{n+1},\\
&\begin{cases}
L_{2j-1} = \rho^{1/2}\,(\partial/\partial x^{j1} + 2x^{j2}\partial/\partial t^2)\,,\\
L_{2j} = \rho^{1/2}\,(\partial/\partial x^{j2} - 2x^{j1}\partial/\partial t^2)\,,\\
L_{2n+1} = 2\rho\partial/\partial t^2,\\
L_{2n+2} = 2\rho\partial/\partial\rho,
\end{cases}\\
&\text{when } \mathbb{H}H^{n+1},\\
&\begin{cases}
L_{4j-3} = \rho^{1/2}\,(\partial/\partial x^{j1} + 2x^{j2}\partial/\partial t^2 + 2x^{j3}\partial/\partial t^3 + 2x^{j4}\partial/\partial t^4)\,,\\
L_{4j-2} = \rho^{1/2}\,(\partial/\partial x^{j2} - 2x^{j1}\partial/\partial t^2 - 2x^{j4}\partial/\partial t^3 + 2x^{j3}\partial/\partial t^4)\,,\\
L_{4j-1} = \rho^{1/2}\,(\partial/\partial x^{j3} + 2x^{j4}\partial/\partial t^2 - 2x^{j1}\partial/\partial t^3 - 2x^{j2}\partial/\partial t^4)\,,\\
L_{4j} = \rho^{1/2}\,(\partial/\partial x^{j4} - 2x^{j3}\partial/\partial t^2 + 2x^{j2}\partial/\partial t^3 - 2x^{j1}\partial/\partial t^4)\,,\\
L_{4n+l-1} = 2\rho\partial/\partial t^l \quad (2 \le l \le 4),\\
L_{4n+4} = 2\rho\partial/\partial\rho,
\end{cases}\\
&\qquad\qquad (1 \le j \le n).
\end{aligned}
$$

Regarding these, for a point $\mathbf{s} \in N \cdot \mathbb{R}_+$, let us define the matrices $\mathcal{T} = (\tau_{ij}(\mathbf{s}))$ when $\mathbb{C}H^{n+1}$ by

$$
(\tau_{ij}(\mathbf{s})) = \begin{pmatrix}
1 & & & & & \\
& 1 & & & & \\
& & \cdots & & & \\
& & & 1 & & \\
-2x^{12} & 2x^{11} & \cdots & 2x^{n1} & 2\rho^{1/2} & \\
& & & & & 2\rho^{1/2}
\end{pmatrix},
$$

and when $\mathbb{H}H^{n+1}$ by

$$(\tau_{ij}(\mathrm{s})) = \begin{pmatrix} 1 & & & & & & & & & \\ & 1 & & & & & & & & \\ & & 1 & & & & & & & \\ & & & 1 & & & & & & \\ & & & \cdots & & \cdots & & & & \\ & & & & & 1 & & & & \\ -2x^{12} & 2x^{11} & -2x^{14} & 2x^{13} & \cdots & 2x^{n3} & 2\rho^{1/2} & & & \\ -2x^{13} & 2x^{14} & 2x^{11} & -2x^{12} & \cdots & -2x^{n2} & & 2\rho^{1/2} & & \\ -2x^{14} & -2x^{13} & 2x^{12} & 2x^{11} & \cdots & 2x^{n1} & & & 2\rho^{1/2} & \\ & & & & & & & & & 2\rho^{1/2} \end{pmatrix},$$

and let $\mathcal{T}^{-1} = (\tau^{ij}(\mathrm{s}))$ denote their inverse matrices. Utilizing these, we can express the frame $\{L_i\}$ as

$$L_i = \rho^{1/2} \sum_{l=1}^{m} \tau_{li}(\mathrm{s}) \frac{\partial}{\partial s^l},$$

and the dual frame $\{L_j^*\}$ as

$$L_j^* = \rho^{-1/2} \sum_{l=1}^{m} \tau^{jl}(\mathrm{s}) ds^l,$$

where $(s^1, \ldots, s^m) = (x^{11}, \ldots, t^d, \rho)$. Indeed, we can see the following:

$$L_i(L_j^*) = \sum_{l=1}^{m} \tau_{li}\tau^{jl} = \delta_i^j.$$

Next, for the column vector $ds = {}^t(ds^1, \ldots, ds^m)$, let us denote the column vector $\mathbb{L}^*$ as

$$\mathbb{L}^* = \begin{pmatrix} L_1^* \\ \vdots \\ L_m^* \end{pmatrix} = \rho^{-1/2}\mathcal{T}^{-1}(\mathrm{s}) \begin{pmatrix} ds^1 \\ \vdots \\ ds^m \end{pmatrix} = \rho^{-1/2}\mathcal{T}^{-1}(\mathrm{s}) \cdot ds.$$

Utilizing this notation, we can express $g = {}^t\mathbb{L}^* \cdot \mathbb{L}^*$ because $\{L_j\}$ is an orthonormal frame. Moreover, since σ is an isometry, we have

$$g = {}^t\, \mathbb{L}^* \cdot \mathbb{L}^* = {}^t\Big(\big(\rho\circ\sigma(\mathrm{s})\big)^{-1/2}\mathcal{T}(\sigma(\mathrm{s}))^{-1}d\sigma\Big)\cdot\Big(\big(\rho\circ\sigma(\mathrm{s})\big)^{-1/2}\mathcal{T}(\sigma(\mathrm{s}))^{-1}d\sigma\Big). \tag{25}$$

Furthermore, for the Jacobi matrix $J\sigma = (\partial(s^i\circ\sigma)/\partial s^j)$, we can express the column vector $d\sigma = {}^t(d(s^1\circ\sigma), \dots, d(s^m\circ\sigma))$ as follows:

$$d\sigma = \begin{pmatrix} d(s^1\circ\sigma) \\ \vdots \\ d(s^m\circ\sigma) \end{pmatrix} = J\sigma\cdot d\mathrm{s} = J\sigma\cdot\mathcal{T}(\mathrm{s})\rho^{1/2}\rho^{-1/2}\mathcal{T}(\mathrm{s})^{-1}d\mathrm{s} = J\sigma\cdot\mathcal{T}(\mathrm{s})\rho^{1/2}\mathbb{L}^*.$$

By substituting the above $d\sigma$ for each respective term in (25), we can obtain the following proposition:

Proposition 3.2.

$$\rho^{1/2}\left(\rho\circ\sigma(\mathrm{s})\right)^{-1/2}\mathcal{T}(\sigma(\mathrm{s}))^{-1}\cdot J\sigma(\mathrm{s})\cdot\mathcal{T}(\mathrm{s})$$

is an orthogonal matrix.

Remark 3.3. Observing that

$$\begin{aligned} \tau_{\mathrm{s}*}(\sum_{i=1}^m a_i\partial/\partial s^i) &= \sum_{i=1}^m a_i\tau_{\mathrm{s}*}(\partial/\partial s^i) \\ &= \sum_{j\in I_1} a^j\rho^{1/2}\sum_{l=1}^m \tau_{lj}\partial/\partial s^l + 2^{-1}\sum_{j\in I_2}\rho^{1/2}a^j\sum_{l=1}^m\tau_{lj}\partial/\partial s^l \\ &\quad +2^{-1}a^m\rho^{1/2}\sum_{l=1}^m\tau_{lm}\partial/\partial s^l, \end{aligned}$$

and that $(\tau_{lj}(\mathrm{s}))$ is non-singular, we can confirm the following well-known fact: the linear map

$$\tau_{\mathrm{s}*}: T_o(M)\ni(a^1,\dots,a^m)\to(a^1,\dots,a^m)\begin{pmatrix}\rho^{1/2} & & & & \\ & \cdots & & & \\ & & 2^{-1}\rho^{1/2} & & \\ & & & \cdots & \\ & & & & 2^{-1}\rho^{1/2}\end{pmatrix}(\tau_{lj}(\mathrm{s}))\in T_{\mathrm{s}\cdot o}(M)$$

is non-singular; this fact is in consistent with the fact that multiplication τ_{s} (the left translation) is a diffeomorphism of the Lie group.

By means of Proposition 3.2, we have the following lemma:

Lemma 3.7. *For* $|\mathrm{s}| < 5^{-1/2}$ *and* $\rho/\|\mathrm{s}\|_H^4 > C_0$*, we have*

$$\|J\tilde{u}\| \le C_{17}\|u\circ\sigma\|_H^{-2}\|\mathrm{s}\|_H^{-2}\psi(\rho\circ\sigma)^{-1/2}\dot{\psi}(\rho\circ\sigma)|\mathrm{s}|^{-2}.$$

Proof. To begin with, by using the chain rule, we have

$$
\begin{aligned}
J\tilde{u} &= J\sigma(u\circ\sigma)\cdot Ju(\sigma)\cdot J\sigma \\
&= \left(\rho\circ\sigma(u\circ\sigma)\right)^{1/2}\left(\rho\circ u(\sigma)\right)^{-1/2}\mathcal{T}(\sigma(u\circ\sigma)) \\
&\quad \cdot\left(\rho\circ u(\sigma)\right)^{1/2}\left(\rho\circ\sigma(u\circ\sigma)\right)^{-1/2}\cdot\mathcal{T}(\sigma(u\circ\sigma))^{-1}\cdot J\sigma(u\circ\sigma)\cdot\mathcal{T}(u\circ\sigma) \qquad (26)\\
&\quad \cdot\mathcal{T}(u\circ\sigma)^{-1}\cdot Ju(\sigma)\cdot\left(\rho\circ\sigma(\mathbf{s})\right)^{1/2}\rho^{-1/2}\cdot\mathcal{T}(\sigma(\mathbf{s})) \\
&\quad \cdot\rho^{1/2}\left(\rho\circ\sigma(\mathbf{s})\right)^{-1/2}\mathcal{T}^{-1}(\sigma(\mathbf{s}))\cdot J\sigma(\mathbf{s})\cdot\mathcal{T}(\mathbf{s}) \qquad (27)\\
&\quad \cdot\mathcal{T}(\mathbf{s})^{-1}.
\end{aligned}
$$

Noting that (26) and (27) are orthogonal matrices, we shall now use Proposition 3.2 to obtain

$$
\begin{aligned}
\|J\tilde{u}\| \;\le\; & m^2\left(\rho\circ\sigma(u\circ\sigma)\right)^{1/2}\left(\rho\circ u(\sigma)\right)^{-1/2}(\rho\circ\sigma)^{1/2}\rho^{-1/2} \\
& \cdot\|\mathcal{T}(\sigma(u\circ\sigma))\|\|\mathcal{T}(u\circ\sigma)^{-1}\cdot Ju(\sigma)\cdot\mathcal{T}(\sigma)\|\|\mathcal{T}(\mathbf{s})^{-1}\|,
\end{aligned}
$$

where $m = n_1 + n_2 + 1$.

Next, we shall evaluate each term of the above individually.

Firstly, by using Remark 3.1, we have

$$\left(\rho\circ\sigma(u\circ\sigma)\right)^{1/2}\left(\rho\circ u(\sigma)\right)^{-1/2}(\rho\circ\sigma)^{1/2}\rho^{-1/2} = \|u\circ\sigma\|_H^{-2}\|\mathbf{s}\|_H^{-2}.$$

Secondly, we can observe that

$$\|\mathcal{T}(\mathbf{s})\|^2 = \mathrm{Tr}({}^t\mathcal{T}(\mathbf{s})\cdot\mathcal{T}(\mathbf{s})) = n_1 + 4n_2(|\mathbf{x}|^2+\rho) + 4\rho.$$

Furthermore, according to the proof of Lemma 3.4, it holds that $|\rho\circ\tilde{u}| \le |\mathbf{x}\circ\tilde{u}|^2 + |\rho\circ\tilde{u}| \le \|\tilde{u}\|_H^2 \le 1$. Hence we have

$$\|\mathcal{T}(\sigma(u\circ\sigma))\|^2 \;\le\; n_1 + 4n_2 + 4.$$

Thirdly, we can easily see that $\mathcal{T}(u\circ\sigma)^{-1}\cdot Ju(\sigma)\cdot\mathcal{T}(\sigma)$ is a diagonal matrix with these entries:

$$1,\quad \left(\rho\circ\sigma/\psi(\rho\circ\sigma)\right)^{1/2},\quad \left(\rho\circ\sigma/\psi(\rho\circ\sigma)\right)^{1/2}\dot{\psi}(\rho\circ\sigma).$$

In the first case, by using our construction of ψ, we have $\rho\circ\sigma/\psi(\rho\circ\sigma) = \exp\left(-f(\log(\lambda\rho\circ\sigma))\right) \le 1$, thereby, we have

$$\left(\rho\circ\sigma/\psi(\rho\circ\sigma)\right)^{1/2} \le 1.$$

In the second case, since $\psi(\rho) = \rho\exp(f(\log(\lambda\rho)))$ and $f' > 0$, it holds that

$$\left(\rho\circ\sigma/\psi(\rho\circ\sigma)\right)^{1/2}\dot{\psi}(\rho\circ\sigma) = \left(1 + f'(\log(\lambda\,\rho\circ\sigma))\right)\exp(2^{-1}f(\log(\lambda\,\rho\circ\sigma))) > 1.$$

Comparing them, we have

$$\|\mathcal{T}(u\circ\sigma)^{-1}\cdot Ju(\sigma)\cdot\mathcal{T}(\sigma(\mathbf{s}))\| \le m\dot{\psi}(\rho\circ\sigma)\,\left(\rho\circ\sigma/\psi(\rho\circ\sigma)\right)^{1/2}.$$

Fourthly, because $|\mathbf{s}| < 5^{-1/2}$ implies $\rho < 1$ and $|\mathbf{x}| < 1$, we may observe

$$\begin{aligned}\|\mathcal{T}(\mathbf{s})^{-1}\|^2 &= n_1 + 4^{-1}n_2(|\mathbf{x}|^2+1)\rho^{-1} + \rho^{-1}4^{-1} \\ &\le n_1 + (3n_2+1)4^{-1}\rho^{-1} \\ &\le \max(n_1, (3n_2+1)4^{-1})\rho^{-1}.\end{aligned}$$

Finally, by using the inequality (20), we obtain

$$(\rho\circ\sigma)^{1/2}\rho^{-1/2} = 1/\|\mathbf{s}\|_H^2 \le C_8^2|\mathbf{s}|^{-2}.$$

To summarize, we have completed the proof of the lemma. □

Lemma 3.8. *Given $\delta > 1$, for $|\mathbf{s}| < 5^{-1/2}$ and $\rho/\|\mathbf{s}\|_H^4 > C_0$, we have*

$$\|(u\circ\sigma)(\mathbf{s})\|_H^{-2}\|\mathbf{s}\|_H^{-2}\dot{\psi}(\rho\circ\sigma)\psi(\rho\circ\sigma)^{-1/2} \le C_{19}/|\mathbf{s}|^{2\delta-1}. \tag{28}$$

Proof. To begin with, it should be noted that the estimation for $3/2 > \delta > 1$, is sufficient to complete the proof, from our assumption $1 \le 1/|\mathbf{s}|$ (as in $|\mathbf{s}| < 5^{-1/2}$).

Next, by using Definition 3.1, we have the following:

$$\begin{aligned}&\|u\circ\sigma\|_H^{-2}\|\mathbf{s}\|_H^{-2}\psi(\rho\circ\sigma)^{-1/2}\dot{\psi}(\rho\circ\sigma) \\ &= \frac{\dot{\psi}(\rho\circ\sigma)\psi(\rho\circ\sigma)^{-1/2}}{\left(\left(\|\mathbf{s}\|_H^2|\mathbf{x}\circ\sigma|^2 + \|\mathbf{s}\|_H^2\psi(\rho\circ\sigma)\right)^2 + |\mathbf{t}\circ\sigma|^2\|\mathbf{s}\|_H^4\right)^{1/2}} =: R_3.\end{aligned}$$

Here, for the sake of simplicity, we shall set $\eta = \|\mathbf{s}\|_H^2\psi(\rho\circ\sigma)$ in the following.

In order to complete our estimation, we shall divide our discussion into two separate parts, with careful consideration of the following identity as in Lemma 3.1:

$$1 = \|\sigma(\mathbf{s})\|_H^4\|\mathbf{s}\|_H^4 = \|\mathbf{s}\|_H^4|\mathbf{t}\circ\sigma|^2 + (\|\mathbf{s}\|_H^2|\mathbf{x}\circ\sigma|^2 + \rho/\|\mathbf{s}\|_H^2)^2. \tag{29}$$

In the first case, when $\|\mathbf{s}\|_H^4|\mathbf{t}\circ\sigma|^2 \ge 1/2$, by using (13) and (20) we obtain,

$$\begin{aligned}R_3 &\le \frac{C_3\psi(\rho\circ\sigma)^{\delta-1/2}}{(1/2+\eta^2)^{1/2}} = \frac{C_3\eta^{\delta-1/2}}{(1/2+\eta^2)^{1/2}}\frac{1}{\|\mathbf{s}\|_H^{2\delta-1}} \\ &\le \frac{C_3C_{21}}{\|\mathbf{s}\|_H^{2\delta-1}} \le \frac{C_4C_{21}C_{22}}{|\mathbf{s}|^{2\delta-1}},\end{aligned}$$

where $C_{22} = C_8^{1-2\delta}$.

In the second case, when $\|\mathbf{s}\|_H^4 |\mathbf{t}\circ\sigma|^2 \le 1/2$, it follows that

$$R_3 \le \frac{\dot\psi(\rho/\|\mathbf{s}\|_H^4)\psi(\rho/\|\mathbf{s}\|_H^4)^{-1/2}}{\|\mathbf{s}\|_H^2|\mathbf{x}\circ\sigma|^2 + \|\mathbf{s}\|_H^2\psi(\rho\circ\sigma)} =: R_4.$$

Firstly, when $\|\mathbf{s}\|_H^2|\mathbf{x}\circ\sigma|^2 \ge 1/2$, by using (13) and (20) once again, we have

$$R_4 \le \frac{C_3\psi(\rho\circ\sigma)^{\delta-1/2}}{1/2+\eta} = \frac{C_3\eta^{\delta-1/2}}{1/2+\eta}\frac{1}{\|\mathbf{s}\|_H^{2\delta-1}} \le \frac{C_3C_{23}}{\|\mathbf{s}\|_H^{2\delta-1}} \le \frac{C_3C_{23}C_{22}}{|\mathbf{s}|^{2\delta-1}}.$$

Secondly, when $\|\mathbf{s}\|_H^2|\mathbf{x}\circ\sigma|^2 \le 1/2$, by utilizing the identity (29) in coordination with $\|\mathbf{s}\|_H^4|\mathbf{t}\circ\sigma|^2 \le 1/2$, we have $(\|\mathbf{s}\|_H^2|\mathbf{x}\circ\sigma|^2+\rho/\|\mathbf{s}\|_H^2)^2 \ge 1/2$, which implies that $\|\mathbf{s}\|_H^2|\mathbf{x}\circ\sigma|^2+\rho/\|\mathbf{s}\|_H^2 \ge 1/\sqrt{2}$, and thereby it follows that

$$\rho/\|\mathbf{s}\|_H^2 \ge 1/\sqrt{2} - \|\mathbf{s}\|_H^2|\mathbf{x}\circ\sigma|^2.$$

Combining this with, $\|\mathbf{s}\|_H^2|\mathbf{x}\circ\sigma|^2 \le 1/2$ we have $\rho/\|\mathbf{s}\|_H^2 \ge 1/\sqrt{2} - 1/2$ and thus,

$$\|\mathbf{s}\|_H^2/\rho \le (1/\sqrt{2}-1/2)^{-1}.$$

Making use of this inequality with (11) and (12), we have

$$\begin{aligned} R_4 &\le \frac{C_2\left(\rho/\|\mathbf{s}\|_H^4\right)^{3\mathcal{N}/2}}{\psi(\rho/\|\mathbf{s}\|_H^4)^{1/2}\|\mathbf{s}\|_H^2} \le \frac{C_2\left(\rho/\|\mathbf{s}\|_H^4\right)^{3\mathcal{N}/2+1}\left(\rho/\|\mathbf{s}\|_H^4\right)^{-1}}{\left(\rho/\|\mathbf{s}\|_H^4\right)^{1/2}\exp\left(2^{-1}C_1\left(\rho/\|\mathbf{s}\|_H^4\right)^{\mathcal{N}/2}\right)\|\mathbf{s}\|_H^2} \\ &= C_2\left(\|\mathbf{s}\|_H^2/\rho\right)\left(\rho/\|\mathbf{s}\|_H^4\right)^{3\mathcal{N}/2+1/2}\exp\left(-2^{-1}C_1\left(\rho/\|\mathbf{s}\|_H^4\right)^{\mathcal{N}/2}\right) < \infty. \end{aligned}$$

To summarize, we have verified (28). □

Proof of Proposition 3.1. By summarizing the estimation in Lemmas 3.7 and 3.8, we can prove Proposition 3.1. □

3.4. The fifth claim of Theorem 1.1. *Proof of the fifth claim of Theorem 1.1.*

Define two paths $\mathbf{s}_i(\tau)$ $(i = 1, 2)$ by

$$\begin{aligned} &\rho\circ\mathbf{s}_1(\tau) = \tau^4\psi^{-1}(1/\tau^2),\ \rho\circ\mathbf{s}_2(\tau) = 0, \\ &\|\mathbf{s}_1(\tau)\|_H = \tau,\ \|\mathbf{s}_2(\tau)\|_H = (t^2\circ\mathbf{s}_1(\tau))^{1/2} = (t^2\circ\mathbf{s}_2(\tau))^{1/2}, \\ &\mathbf{x}\circ\mathbf{s}_i(\tau) = t^3\circ\mathbf{s}_i(\tau) = t^4\circ\mathbf{s}_i(\tau) = 0 \quad i = 1, 2. \end{aligned}$$

To begin with, we shall note that $\|\mathbf{s}_1(\tau)\|_H^4 = \tau^8\psi^{-1}(1/\tau^2)^2 + (t^2\circ\mathbf{s}_1(\tau))^2$ implies

$$(t^2\circ\mathbf{s}_1(\tau)/\|\mathbf{s}_1(\tau)\|_H^2)^2 = 1 - \tau^4\psi^{-1}(1/\tau^2)^2.$$

Since $\rho\circ\tilde u(\mathbf{s}_1) = \psi(\rho\circ\sigma(\mathbf{s}_1))/(\psi(\rho\circ\sigma(\mathbf{s}_1))^2 + (t^2\circ\mathbf{s}_1(\tau)/\|\mathbf{s}_1(\tau)\|_H^4)^2)$, we have

$$\begin{aligned} |\tilde u(\mathbf{s}_1(\tau)) - \tilde u(\mathbf{s}_2(\tau))| &\ge |\rho\circ\tilde u(\mathbf{s}_1) - \rho\circ\tilde u(\mathbf{s}_2)| \\ &= \rho\circ\tilde u(\mathbf{s}_1) = \tau^2/(2 - \tau^4\psi^{-1}(1/\tau^2)^2). \end{aligned}$$

If $\tilde{u}$ is ε–Hölder continuous, we have

$$|\tilde{u}(\mathbf{s}_1(\tau)) - \tilde{u}(\mathbf{s}_2(\tau))| \le C_{28}|\mathbf{s}_1(\tau) - \mathbf{s}_2(\tau)|^\varepsilon = C_{28}\tau^{4\varepsilon}\psi^{-1}(1/\tau^2)^\varepsilon.$$

This implies that

$$\tau^2/(2 - \tau^4\psi^{-1}(1/\tau^2)^2) \le C_{28}\tau^{4\varepsilon}\left(\psi^{-1}(1/\tau^2)\right)^\varepsilon,$$

thereby we obtain

$$(30)\qquad \begin{aligned} 1 &\le (2 - \tau^4\psi^{-1}(1/\tau^2)^2)C_{28}\tau^{2(2\varepsilon-1)}\left(\psi^{-1}(1/\tau^2)\right)^\varepsilon \\ &\le 2C_{28}\tau^{2(2\varepsilon-1)}\left(\psi^{-1}(1/\tau^2)\right)^\varepsilon. \end{aligned}$$

For $\rho' = \rho_\circ\sigma(\mathbf{s}_1) = \rho_\circ\mathbf{s}_1/\tau^4$, we have $1/\tau^2 = \psi(\rho') = \rho'\exp(f(\log(\lambda\rho')))$, we therefore obtain $\log(1/\tau^2) = \log(\rho') + f(\log(\lambda\rho'))$. Thus, by using (16) we have

$$\begin{aligned} C_5 &= \lim_{\rho'\to\infty}\frac{f(\log(\rho'))}{(\mathcal{N}/2)^{-1}e^{\mathcal{N}\log(\rho')/2}} \\ &= \lim_{\rho'\to\infty}\frac{\log(1/\tau^2) - \log(\rho')}{(\mathcal{N}/2)^{-1}\rho'^{\mathcal{N}/2}} \\ &= \lim_{\rho'\to\infty}\frac{\log(1/\tau^2)}{(\mathcal{N}/2)^{-1}\rho'^{\mathcal{N}/2}}. \end{aligned}$$

Hence, it follows that

$$\psi^{-1}(1/\tau^2) = \rho' \sim \left(\log(1/\tau^2)\right)^{2/\mathcal{N}}.$$

Combining the above with (30) and further supposing that $\varepsilon > 1/2$, we observe that on one hand, the right side of (30) tends to be 0 when $\tau \to 0$, but on the other hand, the left side is one. This leads to a contradiction. □

4. Graham's non-isotropic Hölder spaces

In this section, we shall estimate the regularity of coordinate functions of our constructed maps in terms of Graham's non-isotropic Hölder spaces.

To begin with, let $\mathbf{n} = (\mathbf{x}, \mathbf{t})$ denote a point of $N = \mathbb{K}^n \times \mathrm{Im}(\mathbb{K})$ and let us further define the Heisenberg distance function d_N of N by

$$d_N(\mathbf{n}_0, \mathbf{n}_1) = \left(|\mathbf{x}_0 - \mathbf{x}_1|^4 + |\mathbf{t}_0 - \mathbf{t}_1 - 2\mathrm{Im}(\mathbf{x}_0 \cdot \bar{\mathbf{x}}_1)|^2\right)^{1/4}.$$

This distance function has a good property for scaling, that is,

$$(31)\qquad \rho d_N(\mathbf{n}, \mathbf{n}') = d_N(\rho\cdot\mathbf{n}, \rho\cdot\mathbf{n}'),$$

where $\rho \cdot \mathbf{n} = \rho \cdot (\mathbf{x}, \mathbf{t}) = (\rho^{1/2}\mathbf{x}, \rho\mathbf{t})$ is the dilation. Then, by utilizing this distance function, Folland and Stein's Hölder space Γ_β is defined by a set of functions f on N satisfying:

$$|f(\mathbf{n}_1) - f(\mathbf{n}_2)| \leq C_{29} d_N(\mathbf{n}_1, \mathbf{n}_2)^\beta \qquad \text{for all } \mathbf{n}_1, \mathbf{n}_2 \in N.$$

Following these, extensive research has been made into the properties of this Γ_β space. Well-known inclusion relationships are given below:

Lemma 4.1.

$$C^\beta \subset \Gamma_\beta \subset C^{\beta/2}.$$

This can be proven by noting that

$$C_{30} d_N(\mathbf{n}_1, \mathbf{n}_2) \leq |\mathbf{n}_1 - \mathbf{n}_2| \leq C_{31} d_N(\mathbf{n}_1, \mathbf{n}_2)^{1/2} \tag{32}$$

for small $\mathbf{n}_1, \mathbf{n}_2$.

Given these defined spaces, let us consider the boundary value h of the coordinate functions of our harmonic maps. We can observe the following:

$$\begin{aligned} |t^l \circ h(\mathbf{n}_1) - t^l \circ h(\mathbf{n}_2)| &\leq C_{32} d_N(\mathbf{n}_1, \mathbf{n}_2)^\beta \qquad \text{for all } \beta \leq 2 \qquad (2 \leq l \leq d), \\ |x^{jl} \circ h(\mathbf{n}_1) - x^{jl} \circ h(\mathbf{n}_2)| &\leq C_{33} d_N(\mathbf{n}_1, \mathbf{n}_2)^\beta \qquad \text{for all } \beta \leq 1 \qquad (1 \leq j \leq n, 1 \leq l \leq d), \end{aligned}$$

thereby, we can conclude the following:

Corollary 4.1. *$u|_{\partial M} \in \Gamma_\beta$ for $\beta \leq 1$.*

Taking Γ_β spaces into account, Graham [11] defined Hölder spaces on M whose members have boundary values belonging to Γ_β, by utilizing a discretization of the invariant metric g, namely, the distance function:

$$d(\mathbf{s}_1, \mathbf{s}_2)^2 = \frac{|\Delta \mathbf{x}|^2}{\rho} + \frac{|\Delta \rho|^2}{\rho^2} + \frac{|\Delta t - 2\mathrm{Im}(\mathbf{x}_1 \cdot \Delta \bar{\mathbf{x}})|^2}{\rho^2}, \quad \text{for } \mathbf{s}_i = (\mathbf{x}_i, \mathbf{t}_i, \rho_i) \in M, (i = 1, 2),$$

where $\rho = \min(\rho_1, \rho_2)$, $\Delta \mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$, $\Delta \mathbf{t} = \mathbf{t}_1 - \mathbf{t}_2$ and $\Delta \rho = \rho_1 - \rho_2$. In order to identify the above with Graham's expression [11, (6.2)], we may compute the following:

$$\begin{aligned} d(\mathbf{s}_1, \mathbf{s}_2)^2 &= |(\rho^{-1/2}\Delta \mathbf{x}, \rho^{-1}\Delta \rho, \rho^{-1}(\Delta \mathbf{t} - 2\mathrm{Im}(\mathbf{x}_1 \cdot \Delta \bar{\mathbf{x}})))|^2 \\ &= |\rho^{-1} \cdot (\mathbf{n}_1 \cdot \mathbf{n}_2^{-1}, \rho_1 - \rho_2)|^2. \end{aligned}$$

By using this distance function, Graham defined the two-parameter family of Hölder spaces Γ_α^β as follows: for $-\infty < \beta \leq \alpha$, $0 < \alpha < 1$, a function f on M is in Γ_α^β if

$$|f(\mathbf{s}_1) - f(\mathbf{s}_2)| \leq C_{34} \rho^{\beta/2} d(\mathbf{s}_1, \mathbf{s}_2)^\alpha \qquad \text{for all } \mathbf{s}_1, \mathbf{s}_2 \in M.$$

In this section, we shall assume that all functions on $M \cup \{\rho = 0\}$ are compactly supported, by multiplying smooth cut-off functions if required.

Now let us mention the following theorem by Graham [11, Theorem 6.17, Proposition 6.7]:

Theorem 4.1. *Suppose that $\alpha \geq \beta > 0$. Then we have $C^\alpha \subset \Gamma_\alpha^\beta$. Moreover, $f \in \Gamma_\alpha^\beta$ implies $f(\cdot, \rho) \in \Gamma_\beta$ (uniformly in ρ). Consequently, f has a boundary value $f(\cdot, 0)$ belonging to Γ_β.*

As an implication of this theorem, we may make a cursory characterization of Γ_α^β as a space of functions which belong to C^α in the interior and whose boundary values belong to Γ_β. An analogous space to Γ_α^β was examined by Graham, where the interior regularity was measured in terms of C^k norms, rather than C^α norms. This space is defined in the following way:

For multi indices $\gamma = (\gamma^1, \dots, \gamma^m)$, let us set

$$D^\gamma = e_1^{\gamma^1} \cdots e_m^{\gamma^m}, \quad |\gamma| = \sum_{i=1}^{m} \gamma^i, \quad \mathrm{wt}(\gamma) = \sum_{i \in I_1} \gamma^i + \sum_{i \in I_2} 2\gamma^i + 2\gamma^m.$$

Then, he defined C_k^β as a space of functions f satisfying

$$|D^\gamma f| \leq C_{35} \rho^{(\beta - \mathrm{wt}(\gamma))/2},$$

for all multi indices γ satisfying $|\gamma| \leq k$, taking into account the appropriate weight for each derivative.

Proposition 4.1 (Proposition 6.15 [11]). $C_1^\beta \subset \Gamma_\alpha^\beta$.

It should be further remarked that these spaces Γ_α^β and C_k^β are invariant when group actions are being applied:

Proposition 4.2 (Proposition 6.7 [11]). *$f \in \Gamma_\alpha^\beta$ if and only if $f \circ \tau_{\mathbf{s}} \in \Gamma_\alpha^\beta$, and $f \in C_k^\beta$ if and only if $f \circ \tau_{\mathbf{s}} \in C_k^\beta$.*

Motivated by his work, Donnelly [6] adopted Graham's space C_k^β in order to study the Dirichlet problem at infinity for harmonic maps, thereby proving the uniqueness of the solution within C_3^β for $\beta > 2$. Following their dialectic, let us estimate the regularity of the coordinate functions of our harmonic maps $\tilde{u}$ near the origin on one chart given in (36).

To begin with, we shall remark the following:

$$\|J\tilde{u} \cdot \mathcal{T}(\mathbf{s})\rho^{1/2}\|^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} (L_j s^i \circ \tilde{u})^2 \geq \sum_{i=1}^{m} \sum_{|\gamma|=1} \left(|D^\gamma s^i \circ \tilde{u}| \rho^{\mathrm{wt}(\gamma)/2}\right)^2.$$

Similar to Lemma 3.7, supposing that $\rho/\|\mathbf{s}\|_H^4 > C_0$ [as given in Remark 3.2], we have

Lemma 4.2.

$$\|J\tilde{u}\cdot\mathcal{T}(\mathbf{s})\rho^{1/2}\| \le C_7\|u\circ\sigma\|_H^{-2}\|\mathbf{s}\|_H^{-2}\dot{\psi}(\rho\circ\sigma)\big(\rho\circ\sigma/\psi(\rho\circ\sigma)\big)^{1/2}\rho^{1/2}.$$

Here, it should be remarked that because $\mathcal{T}(\mathbf{s})$ is multiplied from the right side of $J\tilde{u}$, the unwelcome term $\|\mathcal{T}^{-1}(\mathbf{s})\|$ is not present at this time as it is in Lemma 3.7.

Proceeding with, let us evaluate the right-hand side of the above. Firstly, $\|\mathbf{s}\|_H^4 > \rho^2$ implies that

$$\big(\rho\circ\sigma\big)^{1/2}\rho^{1/2} = (\rho/\|\mathbf{s}\|_H^4)^{1/2}\rho^{1/2} \le 1.$$

Secondly, by utilizing the proof of Lemma 3.8, we have

$$\|u\circ\sigma\|_H^{-2}\|\sigma\|_H^{-2}\dot{\psi}(\rho\circ\sigma)\psi(\rho\circ\sigma)^{-1/2} \le C_{38}\|\mathbf{s}\|_H^{1-2\delta}.$$

Finally, combining $\|\mathbf{s}\|_H^{1-2\delta} \le \rho^{(1-2\delta)/2}$ with the above, we have concluded the following:

Proposition 4.3. *$\tilde{u} \in C_1^\beta$ for $\beta < -1$.*

Hence, we have verified that the assumption of regularity cannot be removed from Donnelly's theorem [6].

Next, let us consider the space Γ_β^α. To start with, we remark the following:

Lemma 4.3. ([11, Lemma 6.4]) *Let $\mathbf{n}_1, \mathbf{n}_2 \in N$, and suppose that $\mathbf{n}_1$ varies over a bounded set. Then $C_{36}|\mathbf{n}_1 - \mathbf{n}_2| \le |\mathbf{n}_1\cdot\mathbf{n}_2^{-1}| \le C_{37}|\mathbf{n}_1 - \mathbf{n}_2|$.*

Utilizing the above as in Graham's proof [11, Proposition 6.8], we have the following rough estimate:

$$\begin{aligned}
|s^i\circ\tilde{u}(\mathbf{s}_1) - s^i\circ\tilde{u}(\mathbf{s}_2)| &\le C_{38}(|\mathbf{x}_1-\mathbf{x}_2|^2 + |\mathbf{t}_1-\mathbf{t}_2|^2 + |\rho_1-\rho_2|^2)^{\alpha/2}\\
&\le C_{39}(|\mathbf{x}_1-\mathbf{x}_2|^2 + |\mathbf{t}_1-\mathbf{t}_2-2\mathrm{Im}(\mathbf{x}_1\cdot(\bar{\mathbf{x}}_1-\bar{\mathbf{x}}_2))|^2 + |\rho_1-\rho_2|^2)^{\alpha/2}\\
&= C_{39}\rho^{\alpha/2}\Big(\frac{|\mathbf{x}_1-\mathbf{x}_2|^2}{\rho} + \frac{\rho|\mathbf{t}_1-\mathbf{t}_2-2\mathrm{Im}(\mathbf{x}_1\cdot(\bar{\mathbf{x}}_1-\bar{\mathbf{x}}_2))|^2}{\rho^2} + \frac{\rho|\rho_1-\rho_2|^2}{\rho^2}\Big)^{\alpha/2}\\
&\le C_{40}\rho^{\alpha/2}\Big(\frac{|\mathbf{x}_1-\mathbf{x}_2|^2}{\rho} + \frac{|\mathbf{t}_1-\mathbf{t}_2-2\mathrm{Im}(\mathbf{x}_1\cdot(\bar{\mathbf{x}}_1-\bar{\mathbf{x}}_2))|^2}{\rho^2} + \frac{|\rho_1-\rho_2|^2}{\rho^2}\Big)^{\alpha/2}
\end{aligned}$$

for $\alpha < 1/7$ near the origin. In the last inequality, we allowed for the possibility that the right-hand side diverges. Hence, we find that near $\rho = 0$, each coordinate function of $\tilde{u}$ is in Γ_α^β for $-\infty < \beta \le \alpha$ for $\alpha < 1/7$. Thereby, once again, we can verify that the assumption of regularity cannot be removed from Donnelly's theorem [6].

5. The Cayley transformation

In this section, we review some facts from hyperbolic geometry. In the following, $\mathbb{K}$ denotes $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

5.1. **The homogeneous model of $\mathbb{K}H^n$.** Let $V^{1,n}(\mathbb{K})$ be the vector space $\mathbb{K}^{n+1}$ together with the unitary structure given by

$$\Phi(z,w) = -\bar{z}_0 w_0 + \bar{z}_1 w_1 + \cdots + \bar{z}_n w_n.$$

An automorphism g of $V^{1,n}(\mathbb{K})$ is said to be a unitary transformation. The group of unitary transformations $G = SO(1,n)$, $SU(1,n)$ or $Sp(1,n)$ is a subgroup of $SL(n+1,\mathbb{K})$, which is $\mathbb{K}$–linear, that is

$$\Phi(g(z), g(w)) = \Phi(z,w).$$

Define V_{-1} by

$$V_{-1} = \{\zeta = (\zeta^0, \ldots, \zeta^n) \in \mathbb{K}^{n+1} | \Phi(\zeta,\zeta) = -1\}.$$

A projection map $P\colon V_{-1} \to P(V_{-1})$ is given by using the following equivalence relations:

$$\zeta \sim \zeta' \text{ if and only if there exists } \lambda \in \mathbb{K} \setminus \{0\} \text{ so that } \zeta = \zeta'\lambda.$$

Since $\Phi(\zeta,\zeta) = -1 < 0$ implies

$$|\zeta^1|^2 + \cdots + |\zeta^n|^2 < |\zeta^0|^2,$$

we have $|\zeta^0|^2 \neq 0$. Hence, $P(V_-)$ is identified with

$$B^n_{\mathbb{K}} = \{(w^1, \ldots, w^n) \in \mathbb{K}^n \mid \sum_{j=1}^{n} |w^j|^2 < 1\},$$

by identifying $[\zeta] \in P(V_-)$ and $w \in B^n_{\mathbb{K}}$ for $(w^j = \zeta^j(\zeta^0)^{-1}\ j = 1, \cdots, n)$. Then in the coordinate representation, the map $P\colon V_{-1} \to B^n_{\mathbb{K}}$ is given by

$$P(\zeta) = w, \quad w^j = \zeta^j(\zeta^0)^{-1}.$$

5.2. **The Cayley transformation as determined by $\pm k_n \in \partial\mathbb{K}H^n$.** Let $\{k_1, \ldots, k_n\}$ and $\{e_0, \ldots, e_n\}$ be the standard basis of $P(\mathbb{K}^{n+1})$ and $\mathbb{K}^{n+1}$, respectively. An element g in G is said to be parabolic if g leaves exactly one point on the boundary fixed. Firstly, we shall focus on the particular boundary point $k_n = (0, \ldots, 0, 1) \in \partial\mathbb{K}H^n$. Since $k_n = P(e_0 + e_n)$, an element $g \in G$ leaves k_n fixed if and only if $g(e_0 + e_n) = (e_0 + e_n)\lambda$ for $\lambda \in \mathbb{K}\setminus\{0\}$. Because of this, in order to examine the parabolic subgroup, a different basis

$\hat{e}_j = \sum_{i=1}^n e_i d_{ij}$ which contains a multiple of $e_0 + e_n$ is often used. Following this, we shall change the basis as follows:

$$\begin{aligned}\hat{e}_0 &= (e_0 - e_n)/\sqrt{2},\\ \hat{e}_n &= (e_0 + e_n)/\sqrt{2},\\ \hat{e}_j &= e_j \quad 1 \le j \le n-1.\end{aligned}$$

In the form of a matrix, this change of basis is provided by

$$D = \begin{pmatrix} 1/\sqrt{2} & & -1/\sqrt{2} \\ & E_{n-1} & \\ 1/\sqrt{2} & & 1/\sqrt{2} \end{pmatrix},$$

where E_{n-1} is the identity matrix of the degree $n-1$. The linear transformation $C = D^{-1}$ or the projective transformation which it induces is called the Cayley transformation. In the coordinate representation, C is given by

$$C\colon (\zeta^0, \zeta^1, \dots, \zeta^{n-1}, \zeta^n) \to ((\zeta^0 - \zeta^n)/\sqrt{2}, \zeta^1, \dots, \zeta^{n-1}, (\zeta^0 + \zeta^n)/\sqrt{2}).$$

Viewed as a projective transformation,

$$\begin{aligned}\eta^j &= -\sqrt{2} w^j (1 - w^n)^{-1},\\ \eta^n &= (1 + w^n)(1 - w^n)^{-1},\end{aligned}$$

which maps an open ball

$$B^n_{\mathbb{K}} = \{w \in \mathbb{K}^n \mid \sum_{k=1}^n |w^k|^2 < 1\}$$

to the Siegel domain of type II

$$\Sigma = \{\eta \in \mathbb{K}^n \mid \mathrm{Re}(\eta^n) > \sum_{j=1}^{n-1} |\eta^j|^2/2\}.$$

Our convention in the case of $\mathbb{K} = \mathbb{C}$ differs from that of Graham's [11, p. 444]. His z^n is $\sqrt{-1}\eta^n$, z^j is $\sqrt{2}\eta^j$ and his w is $-w$, respectively. Following Graham's change of variable [11, p. 444], setting a new coordinate $(\mathbf{x}, \mathbf{t}, \rho)$ by

(33) $$\mathbf{x}^j = \eta^j/\sqrt{2}, \quad \rho = \mathrm{Re}(\eta^n) - \textstyle\sum_{j=1}^{n-1} |\eta^j|^2/2, \quad -\mathbf{t} = \mathrm{Im}(\eta^n),$$

thus we have obtained a diffeomorphism from $B^n_{\mathbb{K}}$ to $\mathbb{K}^{n-1} \times \mathrm{Im}(\mathbb{K}) \times \mathbb{R}_+$ given in the following:

$$\begin{aligned} \mathbf{x}^j &= -w^j(1-w^n)^{-1}, \\ -\mathbf{t}+\rho &= (1+w^n)(1-w^n)^{-1} - \sum_{j=1}^{n-1} |w^j|^2 |1-w^n|^{-2}. \end{aligned} \tag{34}$$

At this point it should be noted that the group $\hat{G} = D^{-1}GD$ preserves $D^{-1}(V_{-1})$, thereby the action of G on $B^n_{\mathbb{K}}$ is converted by the Cayley transformation C into the action of $\hat{G}$ in Σ. Furthermore, C maps k_n to the point $\infty = C(k_n) \in \partial\Sigma$. Thus, the isotropy group of k_n in G corresponds to the isotropy group of ∞ in $\hat{G}$.

Secondly, we shall focus on another boundary point $-k_n = (0, \ldots, 0, -1) \in \partial\mathbb{K}H^n$. Noting that $P(-e_0 + e_n) = -k_n$, we can perform the same computation by replacing k_n with $-k_n$. In the coordinate representation, we have

$$C\colon (\zeta^0, \zeta^1, \ldots, \zeta^{n-1}, \zeta^n) \to ((-\zeta^0) - \zeta^n)/\sqrt{2}, \zeta^1, \ldots, \zeta^{n-1}, ((-\zeta^0) + \zeta^n)/\sqrt{2}).$$

Viewed as a projective transformation,

$$\begin{aligned} \eta^j &= -\sqrt{2}(-w^j)\big(1-(-w^n)\big)^{-1}, \\ \eta^n &= \big(1+(-w^n)\big)\big(1-(-w^n)\big)^{-1}. \end{aligned}$$

If we set a new coordinate $(\mathbf{x}, \mathbf{t}, \rho)$ according to (33) again, then the formula above is given in the following form:

$$\begin{aligned} \mathbf{x}^j &= -(-w^j)\big(1-(-w^n)\big)^{-1}, \\ -\mathbf{t}+\rho &= \big(1+(-w^n)\big)\big(1-(-w^n)\big)^{-1} - \sum_{j=1}^{n-1} |w^j|^2 |1-(-w^n)|^{-2}. \end{aligned} \tag{35}$$

Thus, we constructed the Cayley transformations as determined by the two boundary points k_n and $-k_n$. We shall denote these as Ψ_{k_n} and Ψ_{-k_n}, respectively. They satisfy $\Psi_{k_n} = (-\mathrm{Id}) \circ \Psi_{-k_n}$. Hence, we have the following boundary charts of $\overline{B^n_{\mathbb{K}}} = B^n_{\mathbb{K}} \cup S^{n_1+n_2}$ ($S^{n_1+n_2}$ being a n_1+n_2 dimensional sphere) given by the Cayley transformations Ψ_{k_n} and Ψ_{-k_n}:

$$\begin{aligned} \Psi_{k_n} &: \mathbb{R}^{n_1+n_2} \times \mathbb{R}_{\geq 0} \to \overline{B^n_{\mathbb{K}}} \setminus \{k_n\}, \\ \Psi_{-k_n} &: \mathbb{R}^{n_1+n_2} \times \mathbb{R}_{\geq 0} \to \overline{B^n_{\mathbb{K}}} \setminus \{-k_n\}. \end{aligned} \tag{36}$$

5.3. **A linear fractional transformation.** $G = SO(1,n)_o$, $SU(1,n)$, or $Sp(1,n)$ acts on $P(V_-) = B^n_{\mathbb{K}}$, as a linear fractional transformation $B^n_{\mathbb{K}} \ni w \to s(w) \in B^n_{\mathbb{K}}$ given by

$$w^i \circ s(w) = (s_{i0} + \sum_{j=1}^{n} s_{ij} w^j)(s_{00} + \sum_{j=1}^{n} s_{0j} w^j)^{-1} \text{ for } s = (s_{ij}) \in G.$$

There is an Iwasawa decomposition $G = KAN$ where K coincides with a stabilizer subgroup of G that leaves the origin of $P(V_-)$ fixed. Given that $S = NA$ is diffeomorphic to $G/K \cong P(V_-)$, we shall examine the Lie algebra $\mathfrak{s}$ of S. The $\mathfrak{a}$-gradation of $\mathfrak{s}$ is given by $\mathfrak{s} = \mathbb{R}\{\mathbf{H}\} + \mathfrak{n}_1 + \mathfrak{n}_2$,

$$\mathbf{H} = \begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix}, \quad \mathbf{X}_l = \begin{pmatrix} & -\bar{\mathbf{x}}^l & \\ -\mathbf{x}^l & & \mathbf{x}^l \\ & -\bar{\mathbf{x}}^l & \end{pmatrix}, \quad \mathbf{T} = \frac{1}{2}\begin{pmatrix} -\mathbf{t} & & \mathbf{t} \\ & & \\ -\mathbf{t} & & \mathbf{t} \end{pmatrix},$$

$$\mathfrak{n}_1 = \{\sum_{l=1}^{n-1} \mathbf{X}_l \mid (\mathbf{x}^1, \dots, \mathbf{x}^{n-1}) \in \mathbb{K}^{n-1}\}, \quad \mathfrak{n}_2 = \{\mathbf{T} \mid \mathbf{t} \in \mathrm{Im}(\mathbb{K})\},$$
$$\mathfrak{n}_i = \{X \in \mathfrak{s} \mid ad(\mathbf{H})X = iX\} \quad (i = 1, 2),$$

where, each $\mathbf{X}_l$ has four entries depending on $\mathbf{x}^j$, $-\mathbf{x}^j$ and $-\bar{\mathbf{x}}^j$ placed in the $(l+1)$-st column and $(l+1)$-st row and the other entries vanish. Understanding that any element of $\mathfrak{n}$ is provided as a linear combination $\sum_{l=1}^{n-1} \mathbf{X}^l + \mathbf{T}$, we can express any element of S [being defined as $(s_{ij}) = \exp(\sum_{l=1}^{n-1} \mathbf{X}^l + \mathbf{T}) \exp(s\mathbf{H})$] as follows:

$$(s_{ij}) = \begin{pmatrix} \mathrm{Ch}(s) + e^{-s}(|\mathbf{x}|^2 - \mathbf{t})/2 & -\bar{\mathbf{x}}^1 & \cdots & -\bar{\mathbf{x}}^{n-1} & \mathrm{Sh}(s) + s^{-s}(-|\mathbf{x}|^2 + \mathbf{t})/2 \\ -e^{-s}\mathbf{x}^1 & & & & e^{-s}\mathbf{x}^1 \\ \cdots & 1 & & & \cdots \\ \cdots & & & 1 & \cdots \\ -e^{-s}\mathbf{x}^{n-1} & & & & e^{-s}\mathbf{x}^{n-1} \\ \mathrm{Sh}(s) + e^{-s}(|\mathbf{x}|^2 - \mathbf{t})/2 & -\bar{\mathbf{x}}^1 & \cdots & -\bar{\mathbf{x}}^{n-1} & \mathrm{Ch}(s) + s^{-s}(-|\mathbf{x}|^2 + \mathbf{t})/2 \end{pmatrix}.$$

Recognizing that an action of an element of S to $B^n_{\mathbb{K}}$ is a fractional transformation, we can observe that (s_{ij}) maps the origin of $B^n_{\mathbb{K}}$ to

$$\begin{aligned} w^l &= -e^{-s}\mathbf{x}^l(\mathrm{Ch}(s) + e^{-s}(|\mathbf{x}|^2 - \mathbf{t})/2)^{-1} \\ &= -2\mathbf{x}^l(e^{2s} + 1 + |\mathbf{x}|^2 - \mathbf{t})^{-1}, \\ w^n &= (\mathrm{Sh}(s) + e^{-s}(|\mathbf{x}|^2 - \mathbf{t})/2)(\mathrm{Ch}(s) + e^{-s}(|\mathbf{x}|^2 - \mathbf{t})/2)^{-1} \\ &= (e^{2s} - 1 + |\mathbf{x}|^2 - \mathbf{t})(e^{2s} + 1 + |\mathbf{x}|^2 - \mathbf{t})^{-1}. \end{aligned}$$

So, by substituting ρ for e^{2s}, we have obtained a diffeomorphism $M \ni (\mathbf{x}, \mathbf{t}, \rho) \to (w^1, \ldots, w^n) \in P(V_-)$ given by

$$\begin{aligned} w^l &= -2\mathbf{x}^l\big(|\mathbf{x}|^2 - \mathbf{t} + \rho + 1\big)^{-1} \quad (1 \le l \le n-1), \\ w^n &= \big(|\mathbf{x}|^2 - \mathbf{t} + \rho - 1\big)\big(|\mathbf{x}|^2 - \mathbf{t} + \rho + 1\big)^{-1}. \end{aligned}$$

Since the point ∞ determined by $\mathbb{R}_+$ directions is mapped to $k_n \in \partial B^n_{\mathbb{K}}$, this diffeomorphism shall be denoted by Φ_{k_n}. Recognizing that

$$\begin{aligned} 1 - w^n &= 2\big(|\mathbf{x}|^2 - \mathbf{t} + \rho + 1\big)^{-1}, \\ 1 + w^n &= 2\big(|\mathbf{x}|^2 - \mathbf{t} + \rho\big)\big(|\mathbf{x}|^2 - \mathbf{t} + \rho + 1\big)^{-1}, \end{aligned}$$

we can obtain the inverse of this diffeomorphism given by

$$\begin{aligned} \mathbf{x}^l &= -w^l\big(1 - w^n\big)^{-1}, \\ -\mathbf{t} + \rho &= \big(1 + w^n\big)\big(1 - w^n\big)^{-1} - \sum_{l=1}^{n-1} |w^l|^2 |1 - w^n|^{-2}. \end{aligned}$$

At this point, it should be noted that this diffeomorphism is identical to the Cayley transformation (34). By setting $\Phi_{-k_n} = (-\mathrm{Id})\circ\Phi_{k_n}$, we have boundary charts of $\overline{B^n_{\mathbb{K}}} = B^n_{\mathbb{K}} \cup S^{n_1+n_2}$ given by

$$(37) \qquad \begin{aligned} \Phi_{k_n} &: \mathbb{R}^{n_1+n_2} \times \mathbb{R}_{\ge 0} \to \overline{B^n_{\mathbb{K}}} \setminus \{k_n\}, \\ \Phi_{-k_n} &: \mathbb{R}^{n_1+n_2} \times \mathbb{R}_{\ge 0} \to \overline{B^n_{\mathbb{K}}} \setminus \{-k_n\}. \end{aligned}$$

As we can observe from Chen and Greenberg [4, Proposition 2.3.1], the left invariant metric on $P(V_-) = B^n_{\mathbb{K}}$ is $dw{\cdot}d\bar{w}$ at the origin $\mathbf{0} = (0, \ldots, 0) \in B^n_{\mathbb{K}}$. Furthermore, we can verify the following:

Lemma 5.1.

$$d(w\circ\Psi_{k_n}){\cdot}\overline{d(w\circ\Psi_{k_n})}|_{\mathbf{o}} = d(w\circ\Psi_{-k_n}){\cdot}\overline{d(w\circ\Psi_{-k_n})}|_{\mathbf{o}} = d\mathbf{x}{\cdot}d\bar{\mathbf{x}} + d\mathbf{t}d\bar{\mathbf{t}}/4 + d\rho^2/4$$

for $\mathbf{o} = (0,0,1) = \Psi^{-1}_{k_n}(\mathbf{0})$. *This determines the canonical left invariant metric g so that $(N \cdot \mathbb{R}_+, g)$ is a symmetric space.*

Since the geodesic symmetry of the ball $B^n_{\mathbb{K}}$ at the origin $\mathbf{0} \in B^n_{\mathbb{K}}$ is $B^n_{\mathbb{K}} \ni w \to -w \in B^n_{\mathbb{K}}$, the geodesic symmetry of $N \cdot \mathbb{R}_+$ at $\mathbf{o} = (0,0,1) = \Phi^{-1}_{k_n}(\mathbf{0}) = \Phi^{-1}_{-k_n}(\mathbf{0}) \in N \cdot \mathbb{R}_+$ is provided by

$$\sigma = \Phi^{-1}_{k_n}\circ\Phi_{-k_n} = \Phi^{-1}_{k_n}\circ(-\mathrm{Id})\circ\Phi_{k_n} : \mathbb{R}^{n_1+n_2} \times \mathbb{R}_{\ge 0} \setminus \{(0,0)\} \to \mathbb{R}^{n_1+n_2} \times \mathbb{R}_{\ge 0} \setminus \{(0,0)\}.$$

This will be the coordinate transformation of the coordinate system as given in (36). Regarding the estimation in Proposition 3.2, it should be noted that $\sigma^*(g) = g$, this is because $w \to -w$ is an isometry of $(B^n_{\mathbb{K}}, g_B)$ and it holds that $g = \Psi^*_{k_n}(g_B) = \Psi^*_{-k_n}(g_B)$.

Lemma 5.2. *The explicit formula of σ is as follows:*

$$\begin{aligned}
\mathbf{x}^l{\circ}\sigma &= -\mathbf{x}^l\left(|\mathbf{x}|^2 - \mathbf{t} + \rho\right)^{-1}, \\
-\mathbf{t}{\circ}\sigma + (\rho{\circ}\sigma) &= (\mathbf{t}+\rho)\left||\mathbf{x}|^2 + \mathbf{t} + \rho\right|^{-2}.
\end{aligned}$$

Proof. Recognizing that

$$\begin{aligned}
(1+w^n)(1-w^n)^{-1} &= (1+w^n)\left(1-\overline{w}^n\right)\left(1-\overline{w}^n\right)^{-1}(1-w^n)^{-1} \\
&= \left(1-\overline{w}^n + w^n + |w^n|^2\right)|1-w^n|^{-2},
\end{aligned}$$

$$\begin{aligned}
(1-w^n)^{-1}(1+w^n) &= (1-w^n)^{-1}\left(1-\overline{w}^n\right)^{-1}\left(1-\overline{w}^n\right)(1+w^n) \\
&= |1-w^n|^{-2}\left(1-\overline{w}^n + w^n + |w^n|^2\right),
\end{aligned}$$

we obtain

$$|\mathbf{x}|^2 - \mathbf{t} + \rho = (1+w^n)(1-w^n)^{-1} = (1-w^n)^{-1}(1+w^n),$$

and thereby

$$\left(|\mathbf{x}|^2 - \mathbf{t} + \rho\right)^{-1} = (1-w^n)(1+w^n)^{-1} = (1+w^n)^{-1}(1-w^n).$$

Utilizing the above, we find that

$$\begin{aligned}
\mathbf{x}^l{\circ}\sigma &= -(-w^l)\left(1-(-w^n)\right)^{-1} \\
&= -(-w^l)\left(1-w^n\right)^{-1}\left(1-w^n\right)\left(1+w^n\right)^{-1} \\
&= -\mathbf{x}^l\left(|\mathbf{x}|^2 - \mathbf{t} + \rho\right)^{-1},
\end{aligned}$$

$$\begin{aligned}
-\mathbf{t}{\circ}\sigma + \rho{\circ}\sigma &= \left(1-(-w^n)\right)^{-1}\left(1+(-w^n)\right) - |\mathbf{x}{\circ}\sigma|^2 \\
&= \left(1+w^n\right)^{-1}\left(1-w^n\right) - |\mathbf{x}{\circ}\sigma|^2 \\
&= \left(|\mathbf{x}|^2 - \mathbf{t} + \rho\right)^{-1} - |\mathbf{x}{\circ}\sigma|^2 \\
&= (\mathbf{t}+\rho)\left||\mathbf{x}|^2 + \mathbf{t} + \rho\right|^{-2}.
\end{aligned}$$

□

Remark 5.1. Generally speaking, we should note that the regularity of the coordinate functions of maps depends on the coordinate system. Let us consider the diffeomorphisms $\mathbf{f}_{2\alpha}\colon N \times \mathbb{R}_+ \ni (\mathbf{x}, \mathbf{t}, \rho) \to (\mathbf{x}, \mathbf{t}, \rho^{2\alpha}) \in N \times \mathbb{R}_+$ for an integer $2\alpha > 0$, and a coordinate system given by $\mathbf{f}_{2\alpha}\circ\Psi_{k_n}$ and $\mathbf{f}_{2\alpha}\circ\Psi_{-k_n}$, instead of (36). Then our harmonic maps are of the form $\mathbf{f}_{2\alpha}^{-1}\circ u\circ\mathbf{f}_{2\alpha}\colon N\times\mathbb{R}_+\ni(\mathbf{x},\mathbf{t},\rho)\to(\mathbf{x},\mathbf{t},\psi(\rho^{2\alpha})^{1/2\alpha})\in N\times\mathbb{R}_+$, where $\psi(\rho^{2\alpha})^{1/2\alpha}=(\rho^{2\alpha}+o(\rho^{2\alpha}|\rho|^{2\alpha\underline{a}}))^{1/2\alpha}=\rho+o(\rho|\rho|^{2\alpha\underline{a}})$. This affects the third claim of Theorem 1.1. In order to be consistent with the C^2 statement in Donnelly [5], we may have to consider the above with $\alpha=1$. Although we can obtain the fourth claim without specifying α by using the same argument, we had to leave out this further complication for want of readability. But, it is easy to see from our results in this article that the coordinate functions of our map cannot be C^1 for the case of $\alpha=1$. Therefore, we can conclude that the assumption of regularity cannot be removed from Donnelly's theorem [5].

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