Nihonkai Math. J.
Vol.10 (1999), 199–209

LAPLACE OPERATOR ON COMPACT GLUED MANIFOLDS

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1. INTRODUCTION

Various studies are developed on the property of the Laplace operator on Riemannian manifolds. In this paper we introduce the concept of *glued manifold* (§2) as an extension of smooth Riemannian manifolds and develop fundamental properties of the Laplace operator on it. Roughly speaking, a glued manifold is a manifold which consists of several pieces of Riemannian manifolds with boundary glued together by isometric mappings between boundaries. For instance, this kind of manifold naturally appears as surface of an object placed in Euclidean space E^3 (See FIGURE 1). On a glued manifold M the Riemannian metric g is thought to be continuous on M but it is thought not to be differentiable on each point of boundaries on which two pieces of Riemannian manifolds are glued together. A glued manifold is an elementary example of a Lipschitz manifold [CP89] [Tel83].

We find that the Laplace operator on functions can be defined on a compact glued manifold (§3). And we find that L^2-space of functions has a orthogonal decomposition into eigen spaces of the Laplace operator just same as on compact smooth Riemannian manifolds (Theorem 1). These results are shown in [CP89] for general compact Lipschitz manifolds, however,

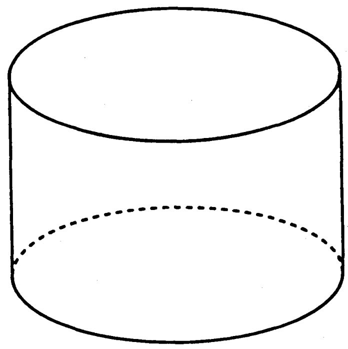

FIGURE 1

The author expresses gratitude to Professor Takushirou Ochiai in Tokyo University, who gave the author foundations of differential geometry and of analysis on manifolds. The author would like to thank Professor Nobuhiro Innami in Niigata University, who motivated the author the idea of the glued manifold and gave him various helpful suggestions on subjects. The author also wishes to thank Professor Kazuo Asano in Niigata University, who showed the author an appropriate literature on elliptic differential operators.

our methods for compact glued manifolds are straightforward extension of the compact smooth Riemannian manifold case. And we find that the eigen function f of the Laplace operator is *smooth* (Theorem 2) in the sense that

1. f is continuous on M and the restriction of f to each piece of smooth Riemannian manifold is infinitely differentiable.
2. grad$f$ is continuous even on each point of boundaries on which two pieces of Riemannian manifolds are glued together.

Finally we prove (Theorem 3) that the inequality known as Lichnerowicz inequality [Lic58] which states the lower bounds for the first non-zero eigenvalue of the Laplace operator on compact smooth Riemannian manifolds still holds on our compact glued manifold in the following form.

We prove that if a compact n-dimensional glued manifold M satisfies,

1. there exists a constant $\delta > 0$ such that Ricci curvature is bounded below by $(n-1)\delta$ on M,
2. pieces of Riemannian manifolds are *non-negatively glued together* (the precise meaning of this condition is explained later),

then the first non-zero eigenvalue $\lambda_1(M)$ of the Laplace operator satisfies

$$\lambda_1(M) \geq n\delta, \tag{1}$$

And we prove that a result of Obata [Oba62] for compact smooth Riemannian manifolds which states that equality holds in inequality (1) if and only if M is isometric to an n-sphere of constant sectional curvature δ is still true for our glued manifold and that, consequently, M is a smooth Riemannian manifold.

2. GLUED MANIFOLDS

2.1. Definition of glued manifold. Let M be a connected topological manifold with dimension n without boundary. We say that (M, g) is a *complete glued manifold* having a decomposition $\Gamma : M = \cup_{\alpha \in \Lambda} M_\alpha$ if the decomposition Γ satisfies the following.

1. Each $(M_\alpha, g_\alpha), g_\alpha = g|M_\alpha$ is a smooth complete Riemannian manifold with smooth boundary B_α and dimension n.
2. $(\mathrm{Int}M_\alpha) \cap M_\beta = \emptyset$ for $\alpha \neq \beta \in \Lambda$, where Int$M_\alpha$ implies the interior of M_α.
3. If M_α and M_β are glued at $p \in B_\alpha \cap B_\beta$, and ther exists a neighborhood U of $p \in M$ such that $U = (U \cap M_\alpha) \cup (U \cap M_\beta)$. $T_pB_\alpha = T_pB_\beta$ and $g_\alpha = g_\beta$ on T_pB_α.

Let B denote $\cup_{\alpha \in \Lambda} B_\alpha$. Note that for each $p \in B$ the number of pieces of Riemannian manifolds which are glued together on the point p is always two.

Throughout this paper, we assume further $M = \cup_{\alpha \in \Lambda} M_\alpha$ is a *compact glued manifold*, which satisfies the following in addition to the conditions above.

1. The index set Λ is a finite set.

2. Each (M_α, g_α) is a compact Riemannian manifold with boundary B_α.

Example 1. The glued manifold of FIGURE 1 is described as follows. Let $M = M_1 \cup M_2 \cup M_3$ be a union of the following three surfaces in the Euclidean space E^3:

$$M_1 := \{(x, y, 0) \in E^3 \mid x^2 + y^2 \leq 1\},$$
$$M_2 := \{(x, y, z) \in E^3 \mid x^2 + y^2 = 1, 0 \leq z \leq 1\},$$
$$M_3 := \{(x, y, 1) \in E^3 \mid x^2 + y^2 \leq 1\},$$

and $g_\alpha, \alpha = 1, 2, 3$ are induced Riemannian metrics from the natural Euclidean metric of E^3. Then, we see that

$$B_1 = \{(x, y, 0) \in E^3 \mid x^2 + y^2 = 1\},$$
$$B_2 = \{(x, y, 1) \in E^3 \mid x^2 + y^2 = 1\},$$
$$B = B_1 \cup B_2.$$

2.2. C^∞ **structure.** We denote Levi-Civita connection on (M_α, g_α) by ∇^α. Let n_α be the inward unit normal vector field to B_α.

For a glued manifold M, we can introduce a canonical C^∞-manifold structure on M, which we describe below. It is sufficient to introduce an appropriate coordinate neighborhood of each point of $p \in B$. Let $p \in M_\alpha \cap M_\beta$. And let U_p be a neighborhood of p in hypersurface B which is diffeomorphic to a open set $V \subset \mathbf{R}^{n-1}$ by a diffeomorphism $\Phi : V \to U_p$. We define a map $\Psi : V \times (-\epsilon, \epsilon) \to M$ as follows.

$$\Psi(x, x_n) := \exp^\alpha_{\Phi(x)}(x_n n_\alpha) \quad (\text{for } x_n \geq 0) \tag{2}$$
$$\Psi(x, x_n) := \exp^\beta_{\Phi(x)}(-x_n n_\beta) \quad (\text{for } x_n < 0)$$

For sufficiently small $\epsilon > 0$, Ψ defines a homeomorphism between $V \times (-\epsilon, \epsilon)$ and some open neighborhood W of p in M. Thus (W, Ψ^{-1}) defines a coordinate neighborhood of $p \in M$ and $\Psi^{-1}|(W \cap M_\alpha)$ and $\Psi^{-1}|(W \cap M_\beta)$ are C^∞-maps. Hence this coordinate neighborhood defines a C^∞-differentiable structure around a point $p \in M_\alpha \cap M_\beta$ so that M should be a C^∞-manifold. As a coordinate neighborhood of a point $p \in M_\alpha \cap M_\beta$, we always use this coordinate neighborhood. We judge the differentiability of functions and several tensors on M relative to this C^∞ structure. The Riemannian metric g is continuous but it may not be C^1-differentiable at each point $p \in B$

2.3. **discrepancy tensor.** We define the second fundamental form h^α and the shape operator L^α on TB_α as usual. For $V, W \in T_pB_\alpha$,

$$h^\alpha(V, W) := g_\alpha(\nabla^\alpha_{\tilde{V}}\tilde{W}, n_\alpha) \tag{3}$$
$$L^\alpha(V) := -\nabla^\alpha_{\tilde{V}} n_\alpha \tag{4}$$

Where above $\tilde{W}$ is an arbitrary vector field on B_α such that $\tilde{W}_p = W$. It is well-known that h^α and L^α are symmetric and that they satisfy the relation

$$h^\alpha(V, W) = g_\alpha(V, L^\alpha(W)). \tag{5}$$

On each point $p \in B_\alpha$ the tangent space T_pM_α has the following orthogonal decomposition.

$$T_pM_\alpha = T_pB_\alpha \oplus \mathbf{R}n_\alpha \tag{6}$$

According to this decomposition, we often use the following decomposition of a vector $X \in T_pM_\alpha$.

$$X = X_\tau + X_n n_\alpha \tag{7}$$

Where above $X_\tau \in T_pB_\alpha$ and $X_n = g_\alpha(X, n_\alpha)$.

On each point $p \in M_\alpha \cap M_\beta$ we investigate the *discrepancy* of two connections ∇^α and ∇^β. For a smooth vector field X on M and for a vector $A \in T_pB_\alpha$,

$$\begin{aligned} \nabla^\alpha_A X &= \nabla^\alpha_A X_\tau + \nabla^\alpha_A (X_n n_\alpha) \\ &= D_A X_\tau + h^\alpha(A, X_\tau)n_\alpha + dX_n(A)n_\alpha - X_n L^\alpha(A) \end{aligned} \tag{8}$$

Where above D denotes Levi-Civita connection on TB. Thus, for $\alpha \neq \beta \in \Lambda$, the difference $\nabla^\alpha_A X - \nabla^\beta_A X$ is computed as follows.

$$\nabla^\alpha_A X - \nabla^\beta_A X = (h^\alpha(A, X_\tau) + h^\beta(A, X_\tau))n_\alpha - X_n(L^\alpha(A) + L^\beta(A)) \tag{9}$$

One can see that the right hand side of the equality above is tensorial about A and X.

Definition 1. For each point $p \in M_\alpha \cap M_\beta$, we define a tensor $S_{\alpha\beta} \in \mathrm{Hom}(T_pB, \mathrm{End}(T_pM))$ as follows.

$$\begin{aligned} S_{\alpha\beta}(A)X &:= \nabla^\alpha_A X - \nabla^\beta_A X \\ &= (h^\alpha(A, X_\tau) + h^\beta(A, X_\tau))n_\alpha - X_n(L^\alpha(A) + L^\beta(A)) \end{aligned} \tag{10}$$

We call $S_{\alpha\beta}$ *discrepancy tensor.*

One can see that $S_{\alpha\beta} = 0$ if and only if $h^\alpha + h^\beta = 0$. And in the case that $h^\alpha + h^\beta = 0$, one can see that Riemannian metric g is C^1-differentiable even on a point $p \in M_\alpha \cap M_\beta$.

Definition 2. For each point $p \in M_\alpha \cap M_\beta$, we define a symmetric tensor T on T_pM as follows. Let $\{e_i\}$ be an orthonormal frame of T_pB.

$$\begin{aligned} T(X,Y) &:= g(S_{\alpha\beta}(X_\tau)Y_\tau, n_\alpha) - X_nY_n \sum_{i=1}^{n-1} g(S_{\alpha\beta}(e_i)n_\alpha, e_i) \\ &= (L^\alpha + L^\beta)(X_\tau, Y_\tau) + \mathrm{tr.}(h^\alpha + h^\beta)X_nY_n \end{aligned} \tag{11}$$

We call T *contracted discrepancy tensor.*

One can see that T is non-negative if and only if $h = h^\alpha + h^\beta$ is non-negative. We regard M as *non-negatively glued together* if T is non-negative on every point on B.

2.4. **geodesic curve.** On a glued manifold a geodesic curve is not necessarily be able to be defined for an arbitrary initial vector. Consider the following example.

Example 2. Let U be the following compact Riemannian manifold with boundary B, the Riemannian metric of which is induced from Euclidean space E^2.

$$\begin{aligned} U &:= \{(x, y) \in E^2 \mid 1 \leq x^2 + y^2 \leq 4\} \\ B = \partial U &= \{(x, y) \in E^2 \mid x^2 + y^2 = 1,\ x^2 + y^2 = 4\} \end{aligned} \tag{12}$$

Let M_1 and M_2 be two copies of U. And let M be a glued manifold which consists of M_1 and M_2 glued together by the identity map on the boundary B. A geodesic curve on M is a straight line on the coordinate system induced from E^2. Then one can see that no geodesic curve exists for an initial vector

$$v_1 = \frac{\partial}{\partial x} \quad \in T_{(0,2)}M \tag{13}$$

And one can see that two geodesic curves could exist for an initial vector

$$v_2 = \frac{\partial}{\partial x} \quad \in T_{(0,1)}M \tag{14}$$

According to this situation, a geodesic curve is not necessarily be extendable infinitely even on a compact glued manifold.

3. Laplace operator

The analytic theory of the Laplace operator and the Green operator on compact glued manifolds proceeds in just the same way as on compact smooth Riemannian manifolds.

Definition 3. We define the Laplace operator $\triangle$ for a function $f \in C^\infty(M)$ as usual.

$$\triangle f := -\mathrm{div}\ \mathrm{grad} f. \tag{15}$$

Remark 1. For $f \in C^\infty(M)$, the function $\triangle f$ is bounded on M but it may be discontinuous at points $p \in M_\alpha \cap M_\beta$.

Let dM_α (resp. dB_α) denote the natural measure on M_α (resp. B_α) induced from the Riemannian metric g. We define L^2-inner product on functions and 1-forms on M as follows.

$$\langle \phi, \psi \rangle_{L^2} := \sum_{\alpha \in \Lambda} \int_{M_\alpha} \phi_\alpha \psi_\alpha \, dM_\alpha \tag{16}$$

$$\langle \eta, \xi \rangle_{L^2} := \sum_{\alpha \in \Lambda} \int_{M_\alpha} g_\alpha(\eta_\alpha, \xi_\alpha) \, dM_\alpha \tag{17}$$

Proposition 1. *For any $\phi, \psi \in C^\infty(M)$, the following identity holds.*

$$\langle d\phi, d\psi \rangle_{L^2} = \langle \phi, \triangle \psi \rangle_{L^2} \tag{18}$$

Proof. By Green's formula, the following identity holds on each M_α.

$$\int_{M_\alpha} g_\alpha(d\phi, d\psi)\, dM_\alpha = -\int_{B_\alpha} \phi \frac{\partial \psi}{\partial n_\alpha}\, dB_\alpha + \int_{M_\alpha} \phi \Delta \psi\, dM_\alpha \tag{19}$$

We sum up the equalities (19) for all $\alpha \in \Lambda$. Since ψ is C^1,

$$\frac{\partial \psi}{\partial n_\alpha} + \frac{\partial \psi}{\partial n_\beta} = 0 \tag{20}$$

holds for $\alpha \neq \beta \in \Lambda$ which satisfy $M_\alpha \cap M_\beta \neq \emptyset$. Thus the first term of the right hand side of the equality (19) cancels for $\alpha \neq \beta \in \Lambda$ which satisfy $M_\alpha \cap M_\beta \neq \emptyset$. Then we get

$$\langle d\phi, d\psi \rangle_{L^2} = \langle \phi, \Delta \psi \rangle_{L^2} \tag{21}$$

□

We define the Sobolev space $W^{1,2}(M)$ as usual. It is well-known that the inclusion $W^{1,2}(M) \hookrightarrow L^2(M)$ is compact on a compact manifold M. And we denote the dual space of $W^{1,2}(M)$ by $(W^{1,2}(M))^*$.

Definition 4. The Laplace operator Δ is defined as an operator with domain $W^{1,2}(M)$ and with range $(W^{1,2}(M))^*$ as usual. For $\phi \in W^{1,2}(M)$,

$$\langle \Delta \phi, \psi \rangle := \langle d\phi, d\psi \rangle_{L^2} \quad \text{for any } \psi \in W^{1,2}(M) \tag{22}$$

(The bracket $\langle \cdot, \cdot \rangle$ in the left hand side of the equality above denotes the natural pairing $(W^{1,2}(M))^* \otimes W^{1,2}(M) \to \mathbf{R}$.)

Remark 2. For C^∞ functions, the definition of the Laplace operator above coincides with the preceding definition in **Definition 3** by the property of **Proposition 1**.

Since the bi-linear form

$$\langle \phi, \psi \rangle_{L^2} + \langle d\phi, d\psi \rangle_{L^2} \tag{23}$$

is equivalent to the $W^{1,2}$-inner product just same as on the smooth Riemannian manifold case, the Laplace operator and Green operator theory works also on our glued manifold without any modification. Then we get the following theorem.

Theorem 1. *There is a direct sum decomposition of $L^2(M)$ into a sum of countably many orthogonal subspaces H_λ. Each H_λ is a finite dimensional vector space, and is an eigen space for the Laplace operator Δ with eigenvalue λ. The eigenvalues λ form a discrete subset of $\mathbf{R}$ and they satisfy $\lambda \geq 0$.*

Theorem 1 is shown in [CP89] for general Lipschitz manifolds.

As for the regularity of the eigen function of the Laplace operator, we find the following theorem.

Theorem 2. *The eigen function of the Laplace operator is C^1-differentiable on M and the restriction of the function to each M_α is C^∞-differentiable.*

Proof. Let f be an eigen function with eigenvalue λ. We shall prove the regularity of the weak solution of the equation

$$L(f) = (\triangle - \lambda)f = 0. \tag{24}$$

By the local property of the differential operator L, one can easily get the regularity of the weak solution in the interior of each M_α by using standard elliptic differential operator theory. (c.f. [GT77])

The point is the regularity in the neighborhood of each point of $M_\alpha \cap M_\beta$. The regularity in problem is shown in **Proposition 9** in Chapter II, §8 from [DL90] for more general *divergence form* differential operators. Our assertion is a direct result of the proposition. □

Corollary 1. *A function $f \in W^{1,2}(M)$ is harmonic* (i.e. $\triangle f = 0$) if and only if f is a constant function on M.

Proof. If f is a constant function on M, it is obvious that f satisfies $\triangle f = 0$. Conversely, if $\triangle f = 0$, one gets $\langle df, df\rangle_{L^2} = 0$. This implies $\langle df_\alpha, df_\alpha\rangle_{L^2(M_\alpha)} = 0$ for each $\alpha \in \Lambda$. Thus f is constant on each M_α. Furthermore, since f is an eigen function of the Laplace operator with eigenvalue zero, f is continuous on M by **Theorem 2**. Hence f is globally constant on M. □

4. Lichnerowicz's inequality and Obata's result

We conclude the paper by proving the analogue of Lichnerowicz's inequality about the first non-zero eigenvalue of the Laplace operator and by proving that Obata's result is still true on our compact glued manifold M. We first prove a lemma which is needed in the proof of the theorem.

Lemma 1. *For $f \in C^\infty(M)$ let $X = \mathrm{grad} f$ and $F = g(X, X)$. And assume $\mathrm{div} X$ is continuous on M. Then the following identity holds on each point $p \in M_\alpha \cap M_\beta$.*

$$\frac{1}{2}\left(\left(\frac{\partial F}{\partial n_\alpha}\right)_p + \left(\frac{\partial F}{\partial n_\beta}\right)_p\right) = T(X_p, X_p) \tag{25}$$

Proof.

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial n_\alpha} g(X, X) &= g(\nabla^\alpha_{n_\alpha} X, X) \\ &= g(\nabla^\alpha_{n_\alpha} X, X_\tau) + X_n g(\nabla^\alpha_{n_\alpha} X, n_\alpha) \end{aligned} \tag{26}$$

First we investigate the first term $g(\nabla^\alpha_{n_\alpha} X, X_\tau)$ of the right hand side of equality (26). We get

$$g(\nabla^\alpha_{n_\alpha} X, X_\tau) = \mathrm{Hess} f(n_\alpha, X_\tau) = \mathrm{Hess} f(X_\tau, n_\alpha) = g(\nabla^\alpha_{X_\tau} X, n_\alpha). \tag{27}$$

Next we investigate the second term $X_n g(\nabla^\alpha_{n_\alpha} X, n_\alpha)$ of the right hand side of the equality (26). Let $\{e_i\}$ be an orthonormal frame of $T_p B_\alpha$. Then $\mathrm{div}^\alpha X$ is expressed as follows.

$$\mathrm{div}^\alpha X = \sum_{i=1}^{n-1} g(\nabla^\alpha_{e_i} X, e_i) + g(\nabla^\alpha_{n_\alpha} X, n_\alpha) \tag{28}$$

The continuity of $\mathrm{div}X$ implies that, for $\alpha \neq \beta \in \Lambda$,

$$\begin{aligned} 0 &= \mathrm{div}^\alpha X - \mathrm{div}^\beta X \\ &= \sum_{i=1}^{n-1} g(\nabla^\alpha_{e_i} X - \nabla^\beta_{e_i} X, e_i) + g(\nabla^\alpha_{n_\alpha} X + \nabla^\beta_{n_\beta} X, n_\alpha). \end{aligned} \tag{29}$$

Thus we get the following identity.

$$g(\nabla^\alpha_{n_\alpha} X + \nabla^\beta_{n_\beta} X, n_\alpha) = -\sum_{i=1}^{n-1} g(S_{\alpha\beta}(e_i)X, e_i) \tag{30}$$

Finally we get

(31)

$$\begin{aligned} \frac{1}{2}\left(\left(\frac{\partial F}{\partial n_\alpha}\right)_p + \left(\frac{\partial F}{\partial n_\beta}\right)_p\right) &= g(S_{\alpha\beta}(X_\tau)X, n_\alpha) - X_n \sum_{i=1}^{n-1} g(S_{\alpha\beta}(e_i)X, e_i) \\ &= T(X, X). \end{aligned}$$

□

Theorem 3. *Let M be a compact n-dimensional glued manifold which satisfies following.*

1. *There exists a constant $\delta > 0$ such that Ricci curvature is bounded below by $(n-1)\delta$ on M.*
2. *The contracted discrepancy tensor T is non-negative on B.*

Then the first non-zero eigenvalue $\lambda_1(M)$ of the Laplace operator satisfies

$$\lambda_1(M) \geq n\delta. \tag{32}$$

And equality holds if and only if M is isometric to an n-sphere of constant sectional curvature δ and consequently the Riemannian metric g is C^∞-differentiable.

Proof. Let $f \in C^1(M)$ and let $X = \mathrm{grad} f$. Then direct computation shows that the following *Lichnerowicz formula* [Lic58] holds on each M_α.

$$-\frac{1}{2}\Delta g(X, X) = |\mathrm{Hess} f|^2 - g(X, \mathrm{grad}(\Delta f)) + \mathrm{Ric}(X, X) \tag{33}$$

One can see that the inequality

$$|\mathrm{Hess} f|^2 \geq \frac{1}{n}(\mathrm{tr.}(\mathrm{Hess} f))^2 = \frac{1}{n}(\Delta f)^2 \tag{34}$$

holds on each point of M_α and that equality holds if and only if Hess$f$ is scalar multiple of the Riemannian metric g on each point. And by the assumption $\mathrm{Ric}(X,X) \geq (n-1)\delta g(X,X)$, we get the following inequality.

$$-\frac{1}{2}\triangle g(X,X) \geq \frac{1}{n}(\triangle f)^2 - g(X, \mathrm{grad}(\triangle f)) + (n-1)\delta g(X,X) \tag{35}$$

If f is an eigen function with eigenvalue λ (i.e. $\triangle f = \lambda f$), we get

$$-\frac{1}{2}\triangle g(X,X) \geq \frac{\lambda^2}{n} f^2 - \lambda g(X,X) + (n-1)\delta g(X,X). \tag{36}$$

Integrating the inequality (36) over M_α, we get

$$\begin{aligned} -\frac{1}{2}\int_{B_\alpha} \frac{\partial}{\partial n_\alpha} g(X,X)\, dB_\alpha \geq & \frac{\lambda^2}{n} ||f||^2_{L^2(M_\alpha)} \\ & - (\lambda - (n-1)\delta) \int_{M_\alpha} g(X,X)\, dM_\alpha. \end{aligned} \tag{37}$$

Since

$$\int_{M_\alpha} g(X,X)\, dM_\alpha = -\int_{B_\alpha} f \frac{\partial f}{\partial n_\alpha}\, dB_\alpha + \int_{M_\alpha} f \triangle f\, dM_\alpha, \tag{38}$$

we get

(39)

$$\begin{aligned} -\frac{1}{2}\int_{B_\alpha} \frac{\partial}{\partial n_\alpha} g(X,X)\, dB_\alpha \geq & \frac{\lambda^2}{n} ||f||^2_{L^2(M_\alpha)} - (\lambda - (n-1)\delta)\, \lambda ||f||^2_{L^2(M_\alpha)} \\ & + (\lambda - (n-1)\delta) \int_{B_\alpha} f \frac{\partial f}{\partial n_\alpha}\, dB_\alpha. \end{aligned}$$

Summing up the inequalities (39) for all $\alpha \in \Lambda$, we get the following inequality by using Lemma 1.

$$-\int_B T(X,X)\, dB \geq -\lambda \frac{n-1}{n} (\lambda - n\delta)\, ||f||^2_{L^2(M)} \tag{40}$$

By the assumption $T(X,X) \geq 0$, we get

$$0 \geq -\lambda \frac{n-1}{n} (\lambda - n\delta)\, ||f||^2_{L^2(M)} \tag{41}$$

Then, if $\lambda > 0$, we get

$$\lambda \geq n\delta. \tag{42}$$

If equality holds in the inequality (42), all the inequalities of the argument above must be equalities. In particular Hess$f$ must be scalar multiple of Riemannian metric g on each point of M. Since $\mathrm{tr.}(\mathrm{Hess}f) = -\triangle f = -\lambda f$, we get

$$\mathrm{Hess}f = -\delta f g. \tag{43}$$

Once the equality (43) is established, one can see that the Obata's result is still true on our glued manifold, as we explain below.

Note that the equality (43) indicates that the Hessian of f is continuous even on each point of $p \in B$.

First we prove that the function f is C^∞-differentiable. Let W be the coordinate neighborhood of a point $p \in M_\alpha \cap M_\beta$ which we mentioned in subsection (2.2) and let $(x_1, x_2, \cdots, x_n)$ denote the coordinate functions on W. And let e_n denote $\frac{\partial}{\partial x_n}$. Since f is continuous and the restriction of f to each M_α is C^∞, it is sufficient to prove

$$\frac{\partial^k f}{\partial x_n^k} \tag{44}$$

is continuous in W. Since $\nabla_{e_n} e_n = 0$ in W, the following identity holds.

$$\nabla^k f(e_n, e_n, \cdots, e_n) = \frac{\partial^k f}{\partial x_n^k} \tag{45}$$

Where above $\nabla^k f$ denotes the k-times covariant derivative tensor of f, which, one can see by using the equality (43) inductively, is continuous. Since the left hand side of the equality (45) is continuous, so is the right hand side. Similarly one can check that $X = \mathrm{grad} f$ is C^∞-differentiable.

Next we explain that on each point $p \in M_\alpha \cap M_\beta$ the continuity of the $\mathrm{Hess} f$ implies that the covariant derivative of the vector field $X = \mathrm{grad} f$ and the covariant derivative by X are independent of the choice of the connections ∇^α or ∇^β. For any vector $V \in T_p B$,

$$\begin{aligned} \nabla^\alpha_V X &= \mathrm{Hess} f(V, \cdot) \\ &= \nabla^\beta_V X. \end{aligned} \tag{46}$$

Thus the covariant derivative of the vector field X is independent of the choice of connections ∇^α or ∇^β. And for any smooth vector field Y,

$$\begin{aligned} \nabla^\alpha_X Y &= \nabla^\alpha_Y X + [X, Y] \\ &= \mathrm{Hess} f(Y, \cdot) + [X, Y] \\ &= \nabla^\beta_Y X + [X, Y] \\ &= \nabla^\beta_X Y. \end{aligned} \tag{47}$$

Thus also the covariant derivative by the vector field X is independent of the choice of connections ∇^α or ∇^β.

Since

$$\nabla_X X = -\delta f X, \tag{48}$$

an integral curve of X is a reparameterazation of a geodesic curve. As we mentioned in subsection (2.4), a geodesic curve on a compact glued manifold is not necessarily be extendable infinitely in general. However, the integral curve of a C^∞-vector field is uniquely determined and extendable infinitely also on our glued manifold.

Based on the facts explained above, one can see that Obata's original proof [Oba62] works also on our glued manifold to prove that M is isometric to an n-sphere of constant sectional curvature δ if one replaces the arguments there using geodesic curves with those using integral curves of X. Consequently the Riemannian metric g is C^∞-differentiable.

□

References

[CP89] Giuseppe DE CECCO and Giuliana PALMIERI, *On the regularity of eigenfunctions of the Laplace operator on a Lipschitz manifold*, J. Math. Pures Appl. **68** (1989), 121–134.

[DL90] Robert Dautray and Jacques-Louis Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, vol. 1, Springer-Verlag, 1990.

[GT77] David Gilbarg and Neil S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 1977.

[Lic58] A. Lichnerowicz, *Geometries des Groupes des Transformationes*, Paris: Dunod, 1958.

[Oba62] M. Obata, *Certain conditions for a Riemannian manifold to be a sphere*, J. Math. Soc. Japan **14** (1962), 333–340.

[Tel83] Nicolae Teleman, *The index of signature operators on Lipschitz manifolds*, Inst. Hautes Etudes Sci. Publ. Math. **58** (1983), 39–78.

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Received September 3, 1999 **Revised October 26, 1999**