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AN APPLICATION OF FURUTA INEQUALITY

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In this note, by the Furuta inequality, we give an elementary proof of a result of Aluthge and Wang as follows: If T is an invertible w-hyponormal operator, then T^2 is also w-hyponormal.

1. Introduction. Through this note, let T be a bounded linear operator on a Hilbert space $\mathcal{H}$. For positive operators A and B, we write $A \geq B$ if $A - B \geq 0$. We denote $A \succ B$ if A and B are invertible positive operators satisfying $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$. Let $T = U|T|$ be the polar decomposition of T. We define $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. The operator $\widetilde{T}$ is called the Aluthge transformation of T([1]). We denote $\widehat{T} = |\widetilde{T}|^{\frac{1}{2}}\widetilde{U}|\widetilde{T}|^{\frac{1}{2}}$, where $\widetilde{T} = \widetilde{U}|\widetilde{T}|$ is the polar decomposition of $\widetilde{T}$. An operator T is called w-hyponormal if $|\widetilde{T}| \geq |T| \geq |\widetilde{T}^*|$. The notion of w-hyponormal operators was introduced by Aluthge and Wang ([3],[5]). It is known that if T is p-hyponormal, then T^2 is not in general([2]). Indeed, Halmos gave an example of a hyponormal operator T for which T^2 is not hyponormal. But the situation is different for log-hyponormal operators : if T is log-hyponormal, then T^2 is log-hyponormal([4]). Since log-hyponormal operators are invertible w-hyponormal operators, it is natural to ask whether similar result holds for inverible w-hyponormal operators. Aluthge and Wang gave an affirmative answer to this question and proved the following result:

PROPOSITION A ([5], Theorem 5.2). *If T is an invertible w-hyponormal operator, T^2 is also w-hyponormal.*

The aim of this note is to give an elementary proof of Proposition A by the Furuta inequality ([6]).

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2. Proof. We begin with the following theorems:

THEOREM 2.1([5], Corollary 1.2). *An operator T is w-hyponormal if and only if* (1) $|T| \geq (|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}})^{\frac{1}{2}}$ *and* (2) $(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T^*|$.

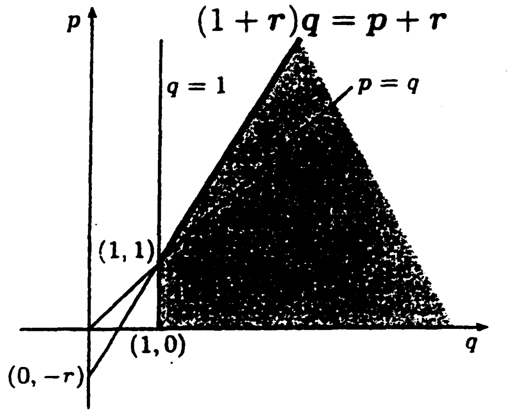
THEOREM 2.2 ([7], Lemma). *Let A and B be invertible positive operators. Then* $(AB^2A)^{\lambda} = AB(BA^2B)^{\lambda-1}BA$ *for any* $\lambda \in \mathbf{R}$.

THEOREM 2.3. *Let T be invertible w-hyponormal. Then we have* $|T| \geq (|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}})^{\frac{1}{2}}$ *if and only if* $(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T^*|$.

Proof. This follows at once from applying Theorem 2.2 with $A = |T|^{\frac{1}{2}}$, $B = |T^*|^{\frac{1}{2}}$ and $\lambda = \frac{1}{2}$.

THEOREM 2.4 ([6], Furuta inequality). *Let A and B be bounded linear operators in a Hilbert space satisfy $A \geq B \geq 0$ and $p, r \geq 0$, $q \geq 1$. If $(1+r)q \geq p+r$, then*

$$A^{\frac{p+r}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}} \quad and \quad (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq B^{\frac{p+r}{q}}.$$



We prepare the following result.

THEOREM 2.5. *Let $A > 0$ and $B > 0$ satisfy $A^{\beta_0} \geq (A^{\frac{\beta_0}{2}}B^{\alpha_0}A^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}}$ for $\alpha_0 > 0$ and $\beta_0 > 0$. Then, for $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$,*

$$A^{\beta} \geq (A^{\frac{\beta}{2}}B^{\alpha}A^{\frac{\beta}{2}})^{\frac{\beta}{\alpha+\beta}}.$$

Proof. Letting $q = \dfrac{p+r}{1+r}$ of the Furuta inequality, we have

$$A^{(1+r)\beta_0} \geq \{A^{\frac{\beta_0 r}{2}}(A^{\frac{\beta_0}{2}}B^{\alpha_0}A^{\frac{\beta_0}{2}})^{\frac{\beta_0 p}{\alpha_0+\beta_0}}A^{\frac{\beta_0 r}{2}}\}^{\frac{1+r}{p+r}} \quad for \quad p \geq 1, r \geq 0.$$

Put $p = \frac{\alpha_0+\beta_0}{\beta_0} \geq 1$. Then we have

$$A^{(1+r)\beta_0} \geq (A^{\frac{1+r}{2}\beta_0}B^{\alpha_0}A^{\frac{1+r}{2}\beta_0})^{\frac{(1+r)\beta_0}{\alpha_0+(1+r)\beta_0}}. \tag{3}$$

Since $r \geq 0$, let $\beta = (1+r)\beta_0$ ($\geq \beta$). Then we have

$$A^{\beta} \geq (A^{\frac{\beta}{2}} B^{\alpha_0} A^{\frac{\beta}{2}})^{\frac{\beta}{\alpha_0+\beta}}.$$

By Theorem 2.2, it is equivalent to $(B^{\frac{\alpha_0}{2}} A^{\beta} B^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta}} \geq B^{\alpha_0}$. And, by the Furuta inequality, we have

$$\{B^{\frac{\alpha_0 r}{2}} (B^{\frac{\alpha_0}{2}} A^{\beta} B^{\frac{\alpha_0}{2}})^{\frac{\alpha_0 p}{\alpha_0+\beta}} B^{\frac{\alpha_0 r}{2}}\}^{\frac{1+r}{p+r}} \geq B^{\alpha_0(1+r)}.$$

Put $p = \frac{\alpha_0+\beta}{\alpha_0} \geq 1$. Then we have

$$(B^{\frac{1+r}{2}\alpha_0} A^{\beta} B^{\frac{1+r}{2}\alpha_0})^{\frac{(1+r)\alpha_0}{\alpha_0(1+r)+\beta}} \geq B^{\alpha_0(1+r)}. \tag{4}$$

Letting $\alpha = (1+r)\alpha_0$ ($\geq \alpha_0$) of (4), we have $(B^{\frac{\alpha}{2}} A^{\beta} B^{\frac{\alpha}{2}})^{\frac{\alpha}{\alpha+\beta}} \geq B^{\alpha}$. Therefore, by Theorem 2.2 we have $A^{\beta} \geq (A^{\frac{\beta}{2}} B^{\alpha} A^{\frac{\beta}{2}})^{\frac{\beta}{\alpha+\beta}}$.
So the proof is complete.

Proof of Proposition A. By Theorems 2.1 and 2.3, T is invertible w-hyponormal if and only if $(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T^*|$. Since $(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T^*|$ implies $(|T^*||T|^2|T^*|)^{\frac{1}{2}} \geq |T^*|^2$ by Theorem 2.5, we have that by Theorem 2.2,

$$(|T^*||T|^2|T^*|)^{\frac{1}{2}} \geq |T^*|^2 \iff |T^2| \geq |T|^2$$

and

$$|T|^2 \geq (|T||T^*|^2|T|)^{\frac{1}{2}} \iff |T^*|^2 \geq |T^{2^*}|.$$

Hence, by Theorem 2.2, we have

$$|T^*|^2 \leq (|T^*||T|^2|T^*|)^{\frac{1}{2}}$$

$$\leq (|T^*||T^2||T^*|)^{\frac{1}{2}}$$

$$= |T^*||T^2|^{\frac{1}{2}}(|T^2|^{\frac{1}{2}}|T^*|^2|T^2|^{\frac{1}{2}})^{\frac{-1}{2}}|T^2|^{\frac{1}{2}}|T^*|.$$

Therefore,

$$|T^2| \geq (|T^2|^{\frac{1}{2}}|T^*|^2|T^2|^{\frac{1}{2}})^{\frac{1}{2}} \geq (|T^2|^{\frac{1}{2}}|T^{2^*}||T^2|^{\frac{1}{2}})^{\frac{1}{2}},$$

which shows that T^2 is w-hyponormal. The proof is complete.

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