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SOME SPECTRAL MAPPING THEOREMS FOR p-HYPONORMAL OPERATORS

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Abstract

The purpose of this paper is to show some spectral mapping theorems for p-hyponormal operators, using the concept of spectral homotopy property.

1 Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator means a bounded linear operator on $\mathcal{H}$. An operator T is said to be a p-hyponormal operator if $(T^*T)^p - (TT^*)^p \geq 0$(see [1]). If $p = 1$, T is called hyponormal and if $p = 1/2$, T is called semi-hyponormal. The set of all p-hyponormal operators in $B(\mathcal{H})$ is denoted by p-H$(\mathcal{H})$(or p-H). Let p-HU$(\mathcal{H})$(or p-HU) denote the set of all operators in p-H$(\mathcal{H})$ with equal defect and nullity. Hence for $T \in p$-HU$(\mathcal{H})$ we may assume that the operator U in a polar decomposition $T = U|T|$ is unitary. The set of all hyponormal operators, all semi-hyponormal operators and all semi-hyponormal operators with unitary U in $B(\mathcal{H})$ is denoted by HN, SH and SHU, respectively. Throughout this paper, let $0 < p < 1/2$. For an operator T, we denote the spectrum, the approximate point spectrum and the residual spectrum by $\sigma(T)$, $\sigma_a(T)$ and $\sigma_r(T)$, respectively. A point $z \in \mathbb{C}$ is in the normal approximate point spectrum $\sigma_{na}(T)$ if there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $(T - z)x_n \to 0$ and $(T - z)^*x_n \to 0$.
D. Xia(cf. [6] or chapter VI of [7]) studied spectral mapping theorem under a class of functional transformation $\varphi(U|T|) = \xi(U)\psi(|T|)$. For a p-hyponormal operator we define as follows: Let $T = U|T| \in B(\mathcal{H})$ (the polar decomposition of T) be p-hyponormal. Let ξ and ψ be Baire functions on $\sigma(U)$ and $\sigma(|T|)$, respectively. Then we define the functional transformation $\varphi_{\{\xi,\psi\}}$ by

$$\varphi_{\{\xi,\psi\}}(T) = \xi(U)(\psi(|T|^{2p}))^{\frac{1}{2p}}.$$

Also we define a mapping $\varphi_{\{\xi,\psi\}}(\cdot)$ in the complex plane by $\varphi_{\{\xi,\psi\}}(\rho e^{i\theta}) = \xi(e^{i\theta})(\psi(\rho^{2p}))^{\frac{1}{2p}}$. Under the functional transformation $\varphi_{\{\xi,\psi\}}$, we study the following formulae:

$$\sigma_{na}(\varphi_{\{\xi,\psi\}}(T)) = \sigma_a(\varphi_{\{\xi,\psi\}}(T)), \tag{1}$$

$$\sigma_*(\varphi_{\{\xi,\psi\}}(T)) = \varphi_{\{\xi,\psi\}}(\sigma_*(T)), \tag{2}$$

where $\sigma_* = \sigma_a,\ \sigma_r$ or σ.
In this paper, we show the following theorem:

Theorem. *Let $T = U|T| \in p-$HU. $\xi_t \in \mathcal{A}_0(\sigma(U))$, $t \in [0,1]$, such that $\xi_t(z)$ is a continuous function of $t \in [0,1]$ for each $z \in \sigma(U)$ and $\xi_0(z) \equiv z$, $\xi_1(z) \equiv \xi(z)$. Let $\psi \in \mathcal{T}_0(\sigma(|T|))$, $\frac{\psi(\rho^{2p})^{\frac{1}{2p}}}{\rho^{2p}}$ be monotone decreasing and denote $\varphi_. \equiv \varphi_{\{\xi,\psi\}}$.*
If ξ_t satisfies the following condition:

$$\delta_{\xi_t} \le m((|T|^{2p}_{(+)})^{-\frac{1}{2}}|T|^{2p}_{(-)}(|T|^{2p}_{(+)})^{-\frac{1}{2}}), \tag{3}$$

then (1) *and* (2) *hold.*

2 Proof.

To show the theorem, we prepare for some notations. Let E be a bounded closed set on the real line $\mathbf{R}$, $M(E)$ be the class of all bounded real Baire functions on E and K_ψ be the singular integral operator defined on $L^2(E)$ by

$$(K_\psi f)(x) = s - \lim_{\epsilon\to 0+} \frac{1}{2\pi}\int_E \frac{\psi(x)-\psi(y)}{x-(y+i\epsilon)} f(y)dy.$$

Put $S(E) = \{\psi | \psi \in M(E), K_\psi \ge 0\}$. Let $\mathcal{T}(E)$ be the class of all strictly monotone increasing continuous function on E. Let

$$M_0(E) = \{\psi \in M(E), \psi(x) \ge 0 \text{and } x \in E \text{ and } \psi(0) = 0\}.$$

Also, we let $S_0(E) = M_0(E) \cap S(E)$ and $\mathcal{T}_0(E) = M_0(E) \cap \mathcal{T}(E)$. If γ is a closed set in the unit circle $\mathbf{T}$, then the class of all complex Baire functions on γ, whose values are in $\mathbf{T}$, is denoted by $M_0(\gamma)$. In case of $\gamma \subset \mathbf{T}$, let K_ξ be the singular integral operator defined on $L^2(\gamma)$ by

$$(K_\xi f)(e^{i\theta}) = s - \lim_{\epsilon\to 0+} \frac{1}{2\pi}\int_\gamma \frac{1-\xi(e^{i\theta})\overline{\xi(e^{i\eta})}}{1-e^{i\theta}e^{-i\eta}(1-\epsilon)} f(e^{i\eta})d\eta.$$

Denote $S_0(\gamma) = \{\xi | \xi \in M_0(\gamma),\ K_\xi \geq 0\}$. Let $\gamma \subset \mathbf{T}$ and $\xi : \ \gamma \to \gamma$ be a homeomorphism preserving the derection. Denote the set of all these functions ξ by $\mathcal{T}_0(\gamma)$. Let $\gamma \subset \mathbf{T}$ be a closed set . Suppose that $\xi \in \mathcal{T}_0(\gamma)$. If there exists a nonnegative number q such that

$$|(\mathcal{P}(\overline{\xi}g), g) + q(\overline{\xi}g, g)| \leq q\|g\|^2 + \|\mathcal{P}g\|^2,\ g \in L^2(\gamma),$$

then we say $\xi \in \mathcal{A}_0(\gamma)$. In this case, by δ_ξ we denote the minimum of $q(1+q)^{-1}$, when q varies over all possible nonnegative numbers satisfying above inequality. For a self-adjoint operator $T \in B(\mathcal{H})$, let us denote

$$m(T) = \inf_{\|x\|=1}(Tx, x) \text{ and } |T|^{2p}_{(\pm)} = \mathcal{S}^{\pm}_{U}(|T|^{2p}),$$

where $\mathcal{S}^{\pm}_{U}(T)$ denote the polar symbols of T (cf. Chapter II of [7]).

Proof of Theorem. Let $S = U|T|^{2p}$. Since S ∈SHU, from condition (3) and Theorem VI. 3. 4 ([7]), we have

$$Re((|S| - w|S|\xi_t(U)^*)f, f) \geq 0,$$

for $f \in \mathcal{H}$ and $|w| = 1$. Hence,

$$Re((|T|^{2p} - w|T|^{2p}\xi_t(U)^*)f, f) \geq 0 \tag{4}$$

for $f \in \mathcal{H}$ and $|w| = 1$. Consider a fixed $t \in [0,1]$. Put $B_t = \xi_t(U)|T|^{2p}$ and $b_t = \xi_t(e^{i\theta})\rho^{2p}$. Then we have $\|(B_t - b_tI)f\|^2 = \|(\xi_t(U)|T|^{2p} - \xi_t(e^{i\theta})\rho^{2p}I)f\|^2$

$$= \|(|T|^{2p} - \rho^{2p}I)f\|^2 + 2\rho^{2p}Re((|T|^{2p} - \xi_t(e^{i\theta})|T|^{2p}\xi_t(U)^*)f, f)). \tag{5}$$

It follows, from (4) and (5), that

$$\|(B_t - b_tI)f\|^2 \geq \|(|T|^{2p} - \rho^{2p}I)f\|^2. \tag{6}$$

Let spectral decomposition of $|T|$ be $|T| = \int_{\sigma(|T|)} s dQ(s)$. Let $m = m(|T|)$ and $M = \|T\|$. Then obviously $\sigma(|T|) \subset [m, M]$. Put $\varphi_t(T) = \xi_t(U) \cdot \psi(|T|^{2p})^{\frac{1}{2p}} (\equiv \varphi_{\{\xi_t, \psi\}}(T))$. By (6), if $\rho \neq 0$ then

$$\|(\varphi_t(T) - \varphi_t(\rho e^{i\theta})I)f\| = \|(\xi_t(U) \cdot \psi(|T|^{2p})^{\frac{1}{2p}} - \xi_t(e^{i\theta}) \cdot \psi(\rho^{2p})^{\frac{1}{2p}}I)f\|$$

$$= \left\| \xi_t(U) \left\{ \psi(|T|^{2p})^{\frac{1}{2p}} - \frac{\psi(\rho^{2p})^{\frac{1}{2p}}}{\rho^{2p}} |T|^{2p} \right\} f + \frac{\psi(\rho^{2p})^{\frac{1}{2p}}}{\rho^{2p}} (B_t - b_tI)f \right\|.$$

Let

$$\Delta = \|(|T|^{2p} - \rho^{2p} I)f\| + \left\| \left(\frac{\rho^{2p}}{\psi(\rho^{2p})^{\frac{1}{2p}}} \cdot \psi(|T|^{2p})^{\frac{1}{2p}} - |T|^{2p} \right) f \right\|$$

and

$$u_t(s,\rho) = \left\{ 1 - \left(\frac{\psi(s^{2p})}{\psi(\rho^{2p})} \right)^{\frac{1}{2p}} \right\} \left\{ 1 - \left(\frac{s}{\rho} \right)^{2p} + \left(\frac{\psi(s^{2p})^{\frac{1}{2p}}}{s^{2p}} - \frac{\psi(\rho^{2p})^{\frac{1}{2p}}}{\rho^{2p}} \right) \cdot \frac{s^{2p}}{\psi(\rho^{2p})^{\frac{1}{2p}}} \right\}.$$

By (6) we have

$$\|(\varphi_t(T) - \varphi_t(\rho e^{i\theta})I)f\| \geq \frac{\psi(\rho^{2p})^{\frac{1}{2p}}}{\rho^{2p}} \|(|T|^{2p} - \rho^{2p} I)f\| - \left\| \xi_t(U) \left\{ \psi(|T|^{2p})^{\frac{1}{2p}} - \frac{\psi(\rho^{2p})^{\frac{1}{2p}}}{\rho^{2p}} |T|^{2p} \right\} f \right\|$$

$$= \frac{\psi(\rho^{2p})^{\frac{1}{2p}}}{\rho^{2p}\Delta} \left\{ \|(|T|^{2p} - \rho^{2p} I)f\|^2 - \left\| \left(\frac{\rho^{2p}}{\psi(\rho^{2p})^{\frac{1}{2p}}} \cdot \psi(|T|^{2p})^{\frac{1}{2p}} - |T|^{2p} \right) f \right\|^2 \right\}$$

$$= \frac{\psi(\rho^{2p})^{\frac{1}{2p}}}{\rho^{2p}\Delta} \int_{\sigma(|T|)} \left\{ |s^{2p} - \rho^{2p}|^2 - \left| \frac{\rho^{2p}}{\psi(\rho^{2p})^{\frac{1}{2p}}} \cdot \psi(s^{2p})^{\frac{1}{2p}} - s^{2p} \right|^2 \right\} d(Q(s)f, f)$$

$$= \frac{\psi(\rho^{2p})^{\frac{1}{2p}} \cdot \rho^{2p}}{\Delta} \int_{\sigma(|T|)} u_t(s,\rho) d(Q(s)f, f). \tag{7}$$

Since $\psi(\rho)$ is a monotone increasing function of ρ and $\frac{\psi(\rho^{2p})^{\frac{1}{2p}}}{\rho^{2p}}$ is a monotone decreasing function of ρ, for $0 < s^{2p} \leq \rho^{2p} - \delta$,

$$u_t(s,\rho) \geq \left\{ 1 - \left(\frac{\psi(\rho^{2p} - \delta)}{\psi(\rho^{2p})} \right)^{\frac{1}{2p}} \right\} \frac{\delta}{\rho^{2p}} > 0,$$

and for $s^{2p} \geq \rho^{2p} + \delta$,

$$u_t(s,\rho) \geq \left\{ \left(\frac{\psi(\rho^{2p} + \delta)}{\psi(\rho^{2p})} \right)^{\frac{1}{2p}} - 1 \right\} \frac{\delta}{\rho^{2p}} > 0.$$

Hence for any positive number $\delta < \rho^{2p}$, there exists a positive number ε such that

$$\inf_{|s^{2p}-\rho^{2p}|\geq\delta} u_t(s,\rho) \geq \varepsilon. \tag{8}$$

On the other hand, let $c = \||T|^{2p} - \rho^{2p}I\| + \left\|\frac{\rho^{2p}}{\psi(\rho^{2p})^{\frac{1}{2p}}} \cdot \psi(|T|^{2p})^{\frac{1}{2p}} - |T|^{2p}\right\|$. Then

$$\Delta \leq c\|f\|. \tag{9}$$

Hence, from (8),

$$\|(|T|^{2p} - \rho^{2p}I)f\|^2 = \int_{\sigma(|T|)} (s^{2p} - \rho^{2p})^2 d(Q(s)f, f)$$

$$= \int_{|s^{2p}-\rho^{2p}|\leq\delta} (s^{2p} - \rho^{2p})^2 d(Q(s)f, f) + \int_{|s^{2p}-\rho^{2p}|\geq\delta} (s^{2p} - \rho^{2p})^2 d(Q(s)f, f)$$

$$\leq \delta^2 + \int_{|s^{2p}-\rho^{2p}|\geq\delta} (s^{2p} - \rho^{2p})^2 d(Q(s)f, f).$$

$$\leq \delta^2 + \frac{1}{\varepsilon}\int_{|s^{2p}-\rho^{2p}|\geq\delta} u_t(s,\rho)(s^{2p} - \rho^{2p})^2 d(Q(s)f, f)$$

$$\leq \delta^2 + \frac{L^2}{\varepsilon}\int_{|s^{2p}-\rho^{2p}|\geq\delta} u_t(s,\rho) d(Q(s)f, f), \tag{10}$$

for $\|f\| = 1$, where $L = \sup_{s\in\sigma(|T|)} |s^{2p} - \rho^{2p}|$. From (7),

$$\frac{L^2}{\varepsilon}\int_{|s^{2p}-\rho^{2p}|\geq\delta} u_t(s,\rho) d(Q(s)f, f) \leq \frac{\Delta \cdot L^2}{\varepsilon\rho^{2p} \cdot \psi(\rho^{2p})^{\frac{1}{2p}}}\|(\varphi_t(T) - \varphi_t(\rho e^{i\theta})I)f\|.$$

Hence, from (9) and (10),

$$\|(|T|^{2p} - \rho^{2p}I)f\|^2 \leq \delta^2 + \frac{c \cdot L^2}{\varepsilon\rho^{2p}\psi(\rho^{2p})^{\frac{1}{2p}}}\|(\varphi_t(T) - \varphi_t(\rho e^{i\theta})I)f\|. \tag{11}$$

If $\varphi_t(\rho e^{i\theta}) \in \sigma_a(\varphi(T))$ and $\rho \neq 0$, then there exists a sequence of unit vectors $\{f_n\}$ in $\mathcal{H}$ such that

$$\lim_{n\to\infty} \|(\varphi_t(T) - \varphi_t(\rho e^{i\theta})I)f_n\| = 0. \tag{12}$$

By (11), we have $\limsup_{n\to\infty} \|(|T|^{2p} - \rho^{2p}I)f_n\| \leq \delta$. Letting $\delta \to 0$, we have $(|T|^{2p} - \rho^{2p}I)f_n \to 0$ as $n \to \infty$. It follows that

$$(|T| - \rho I)f_n \to 0 \text{ as } n \to \infty, \tag{13}$$

and

$$(\psi(|T|^{2p})^{\frac{1}{2p}} - \psi(\rho^{2p})^{\frac{1}{2p}} I)f_n \to 0 \text{ as } n \to \infty. \tag{14}$$

Since $\rho \neq 0$, by (12) and (13), we have

$$(\xi_t(U) - \xi_t(e^{i\theta})I)f_n \to 0 \text{ as } n \to \infty. \tag{15}$$

Since ξ_t is a homeomorphism, it follows that

$$(U - e^{i\theta} I)f_n \to 0 \text{ as } n \to \infty. \tag{16}$$

Hence, by (13) and (16), we have

$$\rho e^{i\theta} \in \sigma_{na}(T) \tag{17}$$

and also, by Theorem 8 of [2], $\rho e^{i\theta} \in \sigma_a(T)$. Furthermore, by (14) and (15), we have

$$\varphi_t(\rho e^{i\theta}) \in \sigma_{na}(\varphi_t(T)). \tag{18}$$

Hence $\sigma_a(\varphi_t(T)) \subset \sigma_{na}(\varphi_t(T))$. In general, $\sigma_a(\varphi_t(T)) \supset \sigma_{na}(\varphi_t(T))$. So we have

$$\sigma_a(\varphi_t(T)) = \sigma_{na}(\varphi_t(T)). \tag{19}$$

Hence, (1) holds for $t \in [0, 1]$. By (17) and (18), we have

$$\sigma_{na}(\varphi_t(T)) = \varphi_t(\sigma_{na}(T)). \tag{20}$$

Hence, from Theorem 8 of [2], (19) and (20), it follows that

$$\sigma_a(\varphi_t(T)) = \varphi_t(\sigma_a(T)). \tag{21}$$

Hence, the case of $\sigma_* = \sigma_a$ of (2) holds for $t \in [0, 1]$. From (21) and Lemma I.3.1 of [7], (1) and the remainder of (2) hold. This completes the proof.

Corollary. *Let $T = U|T| \in p$–HU, $\xi \in \mathcal{S}_0(\sigma(U)) \cap \mathcal{T}_0(\sigma(U))$, $\psi \in \mathcal{T}_0(\sigma(|T|))$ and $\frac{\psi(s^{2p})^{\frac{1}{2p}}}{s^{2p}}$ be monotone decreasing function. Then* (1) *and* (2) *hold.*

Proof. Since $\xi \in \mathcal{S}_0(\sigma(U))$, from Theorem 3 of [5], we have $\xi(U)|T| \in p$–HU. And since $\xi \in \mathcal{T}_0(\sigma(U))$ we have $\delta_\xi = 0$. Hence we can take $\xi_t(z) = \xi(z)$ in the above Theorem. Thus, this corollary is the case of $\delta_{\xi_t} \equiv 0$ for any $t \in [0, 1]$, in the above

Theorem. Therefore, it is clear that (1) and (2) hold.

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