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Weak Projections on Unital Commutative C^*-Algebras

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1. Preliminary

Let Ω be a compact Hausdorff space and let $C(\Omega)$ be the space of complex valued continuous functions on Ω. With the supremum norm, $C(\Omega)$ is a unital commutative C^*-algebra. Let S be a unital C^*-subalgebra of $C(\Omega)$. A bounded linear operator P on $C(\Omega)$ is called a *projection* onto S if $Ph = h$ for every $h \in S$ and the range of P equals to S. A bounded linear operator Q on $C(\Omega)$ is called a *weak projection* for S if $Qh = h$ for every $h \in S$. If P is a projection onto S, then P is a weak projection for S. Converse of this assertion is not true. A counterexample is $S = \{f \in C([0,1]); f(1/3) = f(x)$ for $1/3 \leq x \leq 2/3\}$. For a unital C^*-subalgebra S of $C(\Omega)$, there may not exist a weak projection for S. Our problem in this paper is to find which conditions on S there exists a weak projection for S.

A motivation of this study comes from Korovkin type approximation theorems. A subset E of $C(\Omega)$ is called a *Korovkin set* if for every sequence of bounded linear operators $\{T_n\}_n$ on $C(\Omega)$ such that $\|T_n\| \leq 1$ for every n and $T_n h \to h$ for each $h \in E$, it holds $T_n f \to f$ for every $f \in C(\Omega)$. Korovkin [4] (see also [6]) proved that $\{1, x, x^2\}$ is a Korovkin set of $C([0,1])$. There are many researches on Korovkin type approximation theorems, see [1, 3, 5].

By the definitions, if S is a unital C^*-subalgebra of $C(\Omega)$ and S is a Korovkin set, then there are no weak projections Q for S such that $Q \neq I$ and $\|Q\| = 1$.

Let S be a unital C^*-subalgebra of $C(\Omega)$. For $x \in \Omega$, put

$$E(x) = \{y \in \Omega; f(y) = f(x) \text{ for every } f \in S\}.$$

Then $E(x)$ is a closed subset of Ω, and it holds $E(x) = E(y)$ or $E(x) \cap E(y) = \emptyset$. We call the family $\{E(x)\}_{x\in\Omega}$ the Shilov decomposition for S. We have the following proposition.

Proposition. *Let S be a unital C^*-subalgebra of $C(\Omega)$ and let $\{E(x)\}_{x\in\Omega}$ be the Shilov decomposition for S. Assume that there exist a non-empty open subset U of Ω and a continuous map φ from U to Ω such that*

i) $\varphi(x) \in E(x)$ *for* $x \in U$,

ii) $\varphi(x) \neq x$ *for* $x \in U$.

Then there exists a weak projection Q for S such that $Q \neq I$ and $\|Q\| = 1$.

Proof. Let φ be a continuous map satisfying i) and ii). We shall prove the existence of a weak projection Q for S with $Q \neq I$ and $\|Q\| = 1$. Take a point x_0 in U and a continuous function ψ on Ω such that $0 \leq \psi \leq 1$ on Ω,

$$\psi = 0 \text{ on } \Omega \setminus U \text{ and } \psi(x_0) = 1. \tag{1}$$

We define an operator Q on $C(\Omega)$ as

$$(Qg)(x) = \psi(x)g(\varphi(x)) + (1 - \psi(x))g(x) \text{ for } g \in C(\Omega), x \in \Omega. \tag{2}$$

Then it is not difficult to see that Q is a bounded linear operator on $C(\Omega)$ with $\|Q\| = 1$. Let $h \in S$. Then by i), $h(\varphi(x)) = h(x)$ for $x \in U$. Hence by (2), $(Qh)(x) = h(x)$ for $x \in U$. For $x \in \Omega \setminus U$, by (1) we have $\psi(x) = 0$, so that $(Qh)(x) = h(x)$. Thus we get $Qh = h$ for $h \in S$.

Since $x_0 \in U$, by ii) we have $\varphi(x_0) \neq x_0$, so that there exists $g_0 \in C(\Omega)$ such that $g_0(\varphi(x_0)) \neq g_0(x_0)$. Hence by (1) and (2), $(Qg_0)(x_0) \neq g_0(x_0)$. Therefore Q is a weak projection for S with $Q \neq I$ and $\|Q\| = 1$.

We conjecture that the converse of Proposition is affirmative.

Conjecture. Let S be a unital C^*-subalgebra of $C(\Omega)$ and let $\{E(x)\}_{x\in\Omega}$ be the Shilov decomposition for S. If there exists a weak projection Q for S such that $Q \neq I$ and $\|Q\| = 1$, then there exist a non-empty open subset U of Ω and a continuous map φ from U to Ω such that

i) $\varphi(x) \in E(x)$ for $x \in U$,

ii) $\varphi(x) \neq x$ for $x \in U$.

In the next section, we study this conjecture under some additional conditions.

2. Weak projections

In this section, we shall prove the following theorem.

Theorem 1. *Let S be a unital C^*-subalgebra of $C(\Omega)$ and let $\{E(x)\}_{x\in\Omega}$ be the Shilov decomposition for S. Suppose that $E(x)$ is a countable set for every $x \in \Omega$. If there exists a weak projection Q for S such that $Q \neq I$ and $\|Q\| = 1$, then there exist a non-empty open subset U of Ω and a continuous map φ from U to Ω such that*

i) *$\varphi(x) \in E(x)$ for $x \in U$,*

ii) *$\varphi(x) \neq x$ for $x \in U$.*

Let Ω and Γ be compact Hausdorff spaces, and let μ_x be a positive Borel measure on Ω for every $x \in \Gamma$. Further we assume that (*a*) $\sup\{\mu_x(\Omega); x \in \Gamma\} < \infty$,

μ_x has an atom for every $x \in \Gamma$, that is,

(*b*) for every $x \in \Gamma, \mu_x(\{\zeta\}) > 0$ for some $\zeta \in \Omega$,

and

(*c*) $\int_\Omega f\, d\mu_x$ is continuous in $x \in \Gamma$ for every $f \in C(\Omega)$.

Lemma 1. *Let V be an open subset of Ω. Suppose that $0 < r_1 < \mu_{x_1}(V) \leq \mu_{x_1}(\overline{V}) < r_2$ for a point $x_1 \in \Gamma$. Then there exists an open neighborhood U of x_1 such that $r_1 < \mu_x(V) \leq \mu_x(\overline{V}) < r_2$ for every $x \in U$.*

Proof. By the regularity of the measure μ_{x_1}, there exist a compact subset K of V and an open subset V_1 such that $K \subset V \subset \overline{V} \subset V_1$ and $r_1 < \mu_{x_1}(K) \leq \mu_{x_1}(V_1) < r_2$. Then there exist continuous functions $f_i \in C(\Omega), i = 1, 2$, with $0 \leq f_i \leq 1$ such that

$$f_1 = 1 \text{ on } K \quad \text{and} \quad f_1 = 0 \text{ on } \Omega \setminus V,$$

$$f_2 = 1 \text{ on } \overline{V} \quad \text{and} \quad f_2 = 0 \text{ on } \Omega \setminus V_1.$$

By our assumption (*c*), $\int_\Omega f_i\, d\mu_x \to \int_\Omega f_i\, d\mu_{x_1}$ as $x \to x_1$. We note that

$$r_1 < \mu_{x_1}(K) \leq \int_\Omega f_1\, d\mu_{x_1} \leq \int_\Omega f_2\, d\mu_{x_1} \leq \mu_{x_1}(V_1) < r_2.$$

Since

$$\int_\Omega f_1\, d\mu_x \leq \mu_x(V) \leq \mu_x(\overline{V}) \leq \int_\Omega f_2\, d\mu_x,$$

we have our assertion.

For a closed subset E of Ω, put

(1) $$\lambda_E(x) = \sup\{\mu_x(\{\zeta\}); \zeta \in E\}, \quad x \in \Gamma.$$

By condition (a), sup in (1) is attained, and

(2) $$\lambda_E(x) \leq \mu_x(E), \quad x \in \Gamma.$$

For an open subset U of Γ, put

(3) $$\alpha(E, U) = \sup\{\lambda_E(x); x \in U\} \quad \text{and} \quad \beta(E, U) = \inf\{\lambda_E(x); x \in U\}.$$

Then

(4) $$\beta(E, U) \leq \lambda_E(x) \leq \alpha(E, U), \quad x \in U.$$

Lemma 2. *Let E be a closed subset of Ω and let U be an open subset of Γ such that*

(i) $\mu_x(E) < 4\alpha(E,U)/3$ *for every* $x \in U$,

(ii) $0 < 2\alpha(E,U)/3 \le \beta(E,U)$.

Then there exists a continuous map φ from U to E such that $\mu_x(\{\varphi(x)\}) > 0$ for every $x \in U$.

Proof. By (1), for each $x \in U$ there exists $\zeta(x)$ such that

$$\zeta(x) \in E \quad \text{and} \quad \lambda_E(x) = \mu_x(\{\zeta(x)\}). \tag{5}$$

Then by (ii) and (4),

$$0 < 2\alpha(E,U)/3 \le \lambda_E(x) = \mu_x(\{\zeta(x)\}), \quad x \in U. \tag{6}$$

Here we note that for each $x \in U$, $\zeta \in E$ satisfying $2\alpha(E,U)/3 \le \mu_x(\{\zeta\})$ is unique. For, suppose that $x \in U$, $\zeta, \zeta' \in E$, $\zeta \ne \zeta'$, $2\alpha(E,U)/3 \le \mu_x(\{\zeta\})$, and $2\alpha(E,U)/3 \le \mu_x(\{\zeta'\})$. Then

$$4\alpha(E,U)/3 \le \mu_x(\{\zeta\}) + \mu_x(\{\zeta'\}) \le \mu_x(E).$$

This contradicts (i). Hence $\zeta(x)$ satisfying (5) is unique for each $x \in U$.

Now we shall prove that $\zeta(x)$ is continuous in $x \in U$. Then the map $\varphi(x) = \zeta(x), x \in U$, satisfies our assertion. To prove this, suppose that $\zeta(x)$ is not continuous at $x_0 \in U$. Then there exist two nets $\{x_i\}_i$ and $\{y_i\}_i$ in U which converge to x_0,

$$\zeta(x_i) \to c_1, \ \zeta(y_i) \to c_2, \ \text{and} \ c_1 \ne c_2. \tag{7}$$

By (5), c_1 and c_2 are contained in E.

Take $\varepsilon > 0$ arbitrary. Then there exists a function $h \in C(\Omega)$ such that $0 \le h \le 1$ on Ω,

$$h(c_1) = 1, \tag{8}$$

and

$$\int_\Omega h \, d\mu_{x_0} < \mu_{x_0}(\{c_1\}) + \varepsilon. \tag{9}$$

Now we have

$$\begin{aligned}
\int_\Omega h \, d\mu_{x_0} &= \liminf_{i\to\infty} \int_\Omega h \, d\mu_{x_i} \quad \text{by (c)} \\
&\ge \liminf_{i\to\infty} \mu_{x_i}(\{\zeta(x_i)\}) h(\zeta(x_i)) \\
&= \liminf_{i\to\infty} \mu_{x_i}(\{\zeta(x_i)\}) \quad \text{by (7) and(8)} \\
&\ge 2\alpha(E,U)/3 \quad \text{by (6)}.
\end{aligned}$$

Hence by (9),

$$2\alpha(E,U)/3 < \mu_{x_0}(\{c_1\}) + \varepsilon \quad \text{for every } \epsilon > 0.$$

Thus we get

$$0 < 2\alpha(E,U)/3 \le \mu_{x_0}(\{c_1\}).$$

In the same way, we have

$$0 < 2\alpha(E,U)/3 \le \mu_{x_0}(\{c_2\}).$$

By the first paragraph of the proof, we have $c_1 = c_2$. This contradicts (7).

Lemma 3. *For a closed subset E of Ω, $\lambda_E(x)$ is upper semicontinuous in $x \in \Gamma$.*

Proof. Let $\{x_i\}_i$ be a net in Γ such that $x_i \to x_0 \in \Gamma$ as $i \to \infty$. It is sufficient to prove that

$$\limsup_{i\to\infty} \lambda_E(x_i) \le \lambda_E(x_0).$$

To prove this, suppose that

$$\lim_{i\to\infty} \lambda_E(x_i) = a. \tag{10}$$

We shall prove that $a \le \lambda_E(x_0)$. By (1), there exists $\zeta_i \in E$ such that

$$\mu_{x_i}(\{\zeta_i\}) = \lambda_E(x_i). \tag{11}$$

We may assume moreover that $\zeta_i \to \zeta_0 \in E$. Take a function h in $C(\Omega)$ such that

$$0 \le h \le 1 \quad \text{and} \quad h(\zeta_0) = 1. \tag{12}$$

Then

$$h(\zeta_i)\mu_{x_i}(\{\zeta_i\}) \le \int_\Omega h\, d\mu_{x_i} \to \int_\Omega h\, d\mu_{x_0} \quad \text{as } i \to \infty.$$

Since $h(\zeta_i) \to h(\zeta_0) = 1$, by (10) and (11) we have $a \le \int_\Omega h\, d\mu_{x_0}$. Since this holds for every $h \in C(\Omega)$ satisfying (12), we have $a \le \mu_{x_0}(\{\zeta_0\}) \le \lambda_E(x_0)$.

For a subset E of Γ, we denote by $\operatorname{int} E$ the interior of E. To prove Theorem 1, we use the following theorem.

Theorem 2. *Let Ω and Γ be compact Hausdorff spaces. Suppose that $\mu_x, x \in \Gamma$, is a positive Borel measure on Ω such that $\sup\{\mu_x(\Omega); x \in \Gamma\} < \infty$, μ_x has an atom for every $x \in \Gamma$, and $\int_\Omega f\, d\mu_x$ is continuous in $x \in \Gamma$ for every $f \in C(\Omega)$. Then there exists a continuous map φ from some non-empty open subset U of Γ to Ω such that $\mu_x(\{\varphi(x)\}) > 0$ for every $x \in U$.*

Proof. By our assumption and (1), $\lambda_\Omega(x) > 0$ for every $x \in \Gamma$. Then by the Baire category theorem (see [2, pp.196-197]),there exists $c > 0$ such that $\overline{\{x \in \Gamma; 3c/4 \le \lambda_\Omega(x) < c\}}$ has an interior point. Also by Lemma 3, $\{x \in \Gamma; 3c/4 \le \lambda_\Omega(x)\}$ is a closed subset of Γ and $\{x \in \Gamma; \lambda_\Omega(x) < c\}$ is an open subset of Γ. Therefore $\{x \in \Gamma; 3c/4 \le \lambda_\Omega(x) < c\}$ has also an interior point and so contains a non-empty open subset U_1 of Γ. We may assume moreover that

$$\overline{U_1} \subset \{x \in \Omega; 3c/4 \leq \lambda_\Omega(x) < c\}. \tag{13}$$

For an open subset V of Ω, put

$$W_V = \{x \in \overline{U_1}; 3c/4 \leq \lambda_{\overline{V}}(x)\}. \tag{14}$$

Let $\mathcal{U}(\zeta)$ be the family of open neighborhoods of $\zeta \in \Omega$. We shall prove the existence of a point ζ_0 in Ω such that

$$\operatorname{int} W_V \neq \emptyset \quad \text{for every} \quad V \in \mathcal{U}(\zeta_0), \tag{15}$$

where int W_V denotes the interior of W_V. To prove this, suppose not. Then for each ζ in Ω, there exists $V_\zeta \in \mathcal{U}(\zeta)$ such that int $W_{V_\zeta} = \emptyset$. Since Ω is compact, there exist $\zeta_1, \ldots, \zeta_n \in \Omega$ such that $\Omega = V_{\zeta_1} \cup \ldots \cup V_{\zeta_n}$. Then by (1),

$$\lambda_\Omega(x) = \max\{\lambda_{\overline{V}_{\zeta_j}}(x); 1 \leq j \leq n\} \quad \text{for } x \in U_1,$$

so that by (13) and (14) we have

$$\overline{U_1} = \bigcup_{j=1}^{n} W_{V_{\zeta_j}}.$$

By Lemma 3, $W_{V_{\zeta_j}}$ is a closed subset of $\overline{U_1}$. Hence for some j, $\operatorname{int} W_{V_{\zeta_j}} \neq \emptyset$. This is a desired contradiction.

For $V_1, V_2 \in \mathcal{U}(\zeta_0)$ such that $V_1 \subset V_2$, we have $\lambda_{\overline{V_1}}(x) \leq \lambda_{\overline{V_2}}(x)$, so that by (14) $W_{V_1} \subset W_{V_2}$ and int $W_{V_1} \subset$ int W_{V_2}. Hence by (15), there exists a point $x_0 \in \overline{U}_1$ such that

$$x_0 \in \overline{\operatorname{int} W_V} \subset W_V \subset \overline{U_1} \quad \text{for every} \quad V \in \mathcal{U}(\zeta_0). \tag{16}$$

Then by (13),

$$\mu_{x_0}(\{\zeta_0\}) \leq \lambda_{\overline{V}}(x_0) \leq \lambda_\Omega(x_0) < c \tag{17}$$

for $V \in \mathcal{U}(\zeta_0)$. Since $\mu_{x_0}(\{\zeta_0\}) = \inf\{\mu_{x_0}(\overline{V}); V \in \mathcal{U}(\zeta_0)\}$, by (17) there exists $V_0 \in \mathcal{U}(\zeta_0)$ such that $\mu_{x_0}(\overline{V_0}) < c$. Then by Lemma 1, there exists an open subset U_2 such that $x_0 \in U_2 \subset \Gamma$ such that

$$\sup_{x \in U_2} \mu_x(\overline{V_0}) < c. \tag{18}$$

By (16), $x_0 \in \overline{\operatorname{int} W_{V_0}}$, so that there exists an open subset U of Γ such that

$$U \subset U_2 \cap \operatorname{int} W_{V_0}.$$

Then by (2), (3), (4), (14), and (18),

$$3c/4 \leq \beta(\overline{V_0}, U) \leq \alpha(\overline{V_0}, U) \leq \alpha(\overline{V_0}, U_2) \leq \sup_{x \in U_2} \mu_x(\overline{V_0}) < c.$$

Hence we have

$$0 < 2\alpha(\overline{V_0}, U)/3 \leq 3\alpha(\overline{V_0}, U)/4 < 3c/4 \leq \beta(\overline{V_0}, U)$$

and

$$\mu_x(\overline{V_0}) < c \leq 4\alpha(\overline{V_0}, U)/3 \quad \text{for every } x \in U.$$

Now we can apply Lemma 2. Then there is a continuous map φ from U to Ω such that $\mu_x(\{\varphi(x)\}) > 0$ for every $x \in U$.

As an application of Theorem 2, we prove Theorem 1.

Proof of Theorem 1. Assume the existence of a weak projection Q for S such that $Q \neq I$ and $\|Q\| = 1$. For each $x \in \Omega$, by the Riesz representation theorem there exists a bounded Borel measure ν_x on Ω such that

(19) $$(Qg)(x) = \int_\Omega g\, d\nu_x \quad \text{for every } g \in C(\Omega).$$

Since $Q1 = 1$ and $\|Q\| = 1$, ν_x is a probability measure. Since Q is a weak projection for S, $Qh = h$ for every $h \in S$. Since S is a C^*-subalgebra, by (19) we have

(20) $$\operatorname{supp} \nu_x \subset E(x),$$

where $\operatorname{supp} \nu_x$ is a closed support set of ν_x. Hence by our assumption, ν_x is a discrete measure for every x. Since $Q \neq I$, there exists $g \in C(\Omega)$ such that $\int_\Omega g\, d\nu_{x_0} \neq g(x_0)$ for some $x_0 \in \Omega$. Then there exists ζ_0 in Ω such that $x_0 \neq \zeta_0$ and $\nu_{x_0}(\{\zeta_0\}) > 0$. Take $V_1, V_2 \in \mathcal{U}(\zeta_0)$ such that $\overline{V_1} \subset V_2$ and $x_0 \notin \overline{V_2}$. Since $0 < \nu_{x_0}(\{\zeta_0\}) \leq \nu_{x_0}(V_1)$, by Lemma 1 there exists an open subset W of Ω such that $x_0 \in W$, $\overline{V_2} \cap \overline{W} = \emptyset$, and $\nu_x(V_1) > 0$ for every $x \in \overline{W}$. We note that

(21) $$(x, x) \notin W \times V_2 \quad \text{for every } x \in \Omega.$$

Take a function $g_0 \in C(\Omega)$ such that $0 \leq g_0 \leq 1$,

(22) $$g_0 = 1 \text{ on } \overline{V_1} \text{ and } g_0 = 0 \text{ on } \Omega \setminus V_2.$$

Put

(23) $$d\mu_x = g_0\, d\nu_x, \quad x \in \overline{W}.$$

Then μ_x and $\Gamma = \overline{W}$ satisfy assumptions of Theorem 2. Hence there is a continuous map φ from some non-empty open subset U of W to Ω such that $\mu_x(\{\varphi(x)\}) > 0$ for every

$x \in U$. By (22) and (23), $\varphi(x) \in V_2$ and $\nu_x(\{\varphi(x)\}) > 0$ for every $x \in U$. Then by (20), $\varphi(x) \in E(x)$ for every $x \in U$. Since $U \subset W$, $(x, \varphi(x)) \in U \times V_2 \subset W \times V_2$ for every $x \in U$. Hence by (21), we obtain $\varphi(x) \neq x$ for $x \in U$. This completes the proof.

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