Nihonkai Math. J.
Vol.10 (1999), 63–69

Regeneration in Quaternionic Analysis

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In Complex Analysis of Several Variables, Matsugu [6] gave a necessaryand sufficient condition that any pluriharmonic function g on a Rieman domain Ω over a Stein manifold is a real part of a holomorphic function on Ω. In Quaternionic Analysis, Nôno [8] gave a necessary and sufficient condition that any harmonic function f_1 on a domain Ω in C^2 has a hyper-conjugate harmonic function f_2 so that the function $f_1 + f_2 j$ is hyperholomorphic on Ω. Marinov [5] developped systematically a theory of regenerations of regular functions. The main purpose of the present paper is to add a regeneration in Quaternionic Analysis.

The author would like to express his hearty gratitude to the referee for many valuable suggestions.

1. Regeneration

Let Ω be a complex manifold and f be a holomorphic function on Ω. Then its real part f_1 is a pluriharmonic function on Ω. Let (Ω, φ) be a Rieman domain over a Stein manifold S and $(\tilde{\Omega}, \tilde{\varphi})$ be its envelope of holomorphy over S. Then, Matsugu [6] proved that the necessary and sufficient condition that, for any pluriharmonic function f_1 on Ω, there exists a pluriharmonic function f_2 on Ω so that $f_1 + f_2 i$ is holomorphic on Ω is that there holds $\mathrm{H}^1(\tilde{\Omega}, \mathrm{Z}) = 0$, where Z is the ring of integers.

The field $\mathcal{H}$ of quaternions

$$z = x_1 + ix_2 + jx_3 + kx_4, \qquad x_1, x_2, x_3, x_4 \in \mathrm{R} \tag{1}$$

is a four dimensional non-commutative R-field generated by four base elements 1, i, j and k with the following non commutative multiplication rule:

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j. \tag{2}$$

x_1, x_2, x_3 and x_4 are called, respectively, the real, i, j and k part of z. In the papers Nôno [7], [8], [9], [10] and Marinov [5] loco citato, two complex numbers

$$z_1 := x_1 + ix_2, \quad z_2 := x_3 + ix_4 \in \mathrm{C} \tag{3}$$

are associated to (1), regarded as

$$z = z_1 + z_2 j \in \mathcal{H}. \tag{4}$$

The quaternionic conjugate z^* of $z = z_1 + z_2 j \in \mathcal{H}$ is defined by

$$z^* := \overline{z_1} - z_2 j. \tag{5}$$

They identify $\mathcal{H}$ with $C^2 \cong R^4$, denote a quaternion valued function f by $f = f_1 + f_2 j$ and use fully the theory of functions of several complex variables. Concerning further notations, definitions and citations, please refer to a paper [15] of a colleague of the author in a back number of the present Journal.

Using Laufer's results [4], Nôno [8] proved that the necessary and sufficient condition that, for any complex valued harmonic function f_1 on a domain Ω in C^2, there exists a complex valued harmonic function f_2 on Ω so that $f_1 + f_2 j$ is hyperholomorphic on Ω is that Ω is a domain of holomorphy.

Marinov [5] named those constructions of conjugate functions, regenarations and developped the theory of regenerations in Quaternionic Analysis using $\bar{\partial}$-analysis of Hörmander [3]. The main purpose of the present paper is to add a regeneration, using Dolbeault Isomorphism from resolution of sheaves. Because we use the results of Son[13], we adapt the notations $x = x_1 + x_2 i + x_3 j + x_4 k$ for quaternions x.

2. Main Theorems

Let Ω be a domain in $\mathcal{H} \times \mathcal{H} \cong R^4 \times R^4 = R^8$ of two quaternionic variables $x = x_1 + x_2 i + x_3 j + x_4 k \cong (x_1, x_2, x_3, x_4)$ and $y = y_1 + y_2 i + y_3 j + y_4 k \cong (y_1, y_2, y_3, y_4)$, and $f = f_1 + f_2 i + f_3 j + f_4 k$ be a quaternion valued function of class C^∞ in Ω. The differential operators D_x and D_y are represented under the multiplication rule (2) as follows:

$$(6) \qquad D_x f := (\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} i + \frac{\partial}{\partial x_3} j + \frac{\partial}{\partial x_4} k)(f_1 + f_2 i + f_3 j + f_4 k) =$$

$$(\frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} - \frac{\partial f_4}{\partial x_4}) + (\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} - \frac{\partial f_3}{\partial x_4} + \frac{\partial f_4}{\partial x_3})i +$$
$$(\frac{\partial f_1}{\partial x_3} + \frac{\partial f_2}{\partial x_4} + \frac{\partial f_3}{\partial x_1} - \frac{\partial f_4}{\partial x_2})j + (\frac{\partial f_1}{\partial x_4} - \frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2} + \frac{\partial f_4}{\partial x_1})k.$$

and

$$(7) \qquad f D_y := (f_1 + f_2 i + f_3 j + f_4 k)(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} i + \frac{\partial}{\partial y_3} j + \frac{\partial}{\partial y_4} k) =$$

$$(\frac{\partial f_1}{\partial y_1} - \frac{\partial f_2}{\partial y_2} - \frac{\partial f_3}{\partial y_3} - \frac{\partial f_4}{\partial y_4}) + (\frac{\partial f_1}{\partial y_2} + \frac{\partial f_2}{\partial y_1} + \frac{\partial f_3}{\partial y_4} - \frac{\partial f_4}{\partial y_3})i +$$
$$(\frac{\partial f_1}{\partial y_3} - \frac{\partial f_2}{\partial y_4} + \frac{\partial f_3}{\partial y_1} + \frac{\partial f_4}{\partial y_2})j + (\frac{\partial f_1}{\partial y_4} + \frac{\partial f_2}{\partial y_3} - \frac{\partial f_3}{\partial y_2} + \frac{\partial f_4}{\partial y_1})k.$$

The conjugate operators $\overline{D_x}$ and $\overline{D_y}$ of D_x and D_y are defined as follows:

$$(8) \qquad \overline{D_x} := \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} i - \frac{\partial}{\partial x_3} j - \frac{\partial}{\partial x_4} k, \overline{D_y} := \frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2} i - \frac{\partial}{\partial y_3} j - \frac{\partial}{\partial y_4} k.$$

Theorem 1. *Let Ω be a domain in the space R^8 of 8 real variables $x := (x_1, x_2, x_3, x_4)$ and $y := (y_1, y_2, y_3, y_4)$, f_1, f_2, f_3 be functions of class C^∞ on Ω. If there exits a function f_4 of class C^∞ on Ω such that the quaternion valued function $f = f_1 + f_2 i + f_3 j + f_4 k$ is a biregular function on Ω, the real valued functions f_1, f_2, f_3 satisfies the integrability condition*

$$(9) \qquad d\omega = 0$$

on Ω, where the differential form ω of degree 1 is given by

$$(10) \qquad \omega =$$

$$(-\frac{\partial f_1}{\partial x_4}+\frac{\partial f_2}{\partial x_3}-\frac{\partial f_3}{\partial x_2})\mathrm{d}x_1+(\frac{\partial f_1}{\partial x_3}+\frac{\partial f_2}{\partial x_4}+\frac{\partial f_3}{\partial x_1})\mathrm{d}x_2+(-\frac{\partial f_1}{\partial x_2}-\frac{\partial f_2}{\partial x_1}+\frac{\partial f_3}{\partial x_4})\mathrm{d}x_3+(\frac{\partial f_1}{\partial x_1}-\frac{\partial f_2}{\partial x_2}-\frac{\partial f_3}{\partial x_3})\mathrm{d}x_4+$$
$$(-\frac{\partial f_1}{\partial y_4}-\frac{\partial f_2}{\partial y_3}+\frac{\partial f_3}{\partial y_2})\mathrm{d}y_1+(-\frac{\partial f_1}{\partial y_3}+\frac{\partial f_2}{\partial y_4}-\frac{\partial f_3}{\partial y_1})\mathrm{d}y_2+(\frac{\partial f_1}{\partial y_2}+\frac{\partial f_2}{\partial y_1}+\frac{\partial f_3}{\partial y_4})\mathrm{d}y_3+(\frac{\partial f_1}{\partial y_1}-\frac{\partial f_2}{\partial y_2}-\frac{\partial f_3}{\partial y_3})\mathrm{d}y_4.$$

Conversely, if f_1, f_2, f_3 satisfies the integrability condition (9)-(10) on Ω and if the domain Ω satisfies $\mathrm{H}^1(\Omega, \mathrm{Z}) = 0$ for the ring Z of integers, then there exits a function f_4 of class C^∞ on Ω such that the quaternion valued function $f = f_1 + f_2 i + f_3 j + f_4 k$ is a biregular function on Ω.

Proof. If there exists such a real valued function f_4 on Ω, by definition, its differential ω is given by

$$(11) \qquad \omega := \frac{\partial f_4}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f_4}{\partial x_2}\mathrm{d}x_2 + \frac{\partial f_4}{\partial x_3}\mathrm{d}x_3 + \frac{\partial f_4}{\partial x_4}\mathrm{d}x_4 +$$
$$\frac{\partial f_4}{\partial y_1}\mathrm{d}y_1 + \frac{\partial f_4}{\partial y_2}\mathrm{d}y_2 + \frac{\partial f_4}{\partial y_3}\mathrm{d}y_3 + \frac{\partial f_4}{\partial y_4}\mathrm{d}y_4.$$

Solving $D_x f = 0$ from (6) and $f D_y = 0$ from (7) as linear equations with partial derivatives of f_4 unknown and substituting them in (11), we have the representation (10) of ω by f_1, f_2, f_3. Since ω is the differential of f_4, we have the integrability condition (9).

Let p be a non negative integer, R be the constant sheaf of real numbers over $\Omega, \mathcal{E}^p$ be the sheaf of germs of differential forms of degree p with coefficients of class C^∞ over the domain $\Omega \subset \mathrm{R}^8$, d be the usual differential operator $\mathrm{d}^p : \mathcal{E}^p \to \mathcal{E}^{p+1}$ and $\iota : \mathrm{R} \to \mathcal{E}^0$ be the canonical injection. Then, by the lemma of Poincaré, the above operators give a fine resolution

$$(12) \qquad 0 \to \mathrm{R} \to \mathcal{E}^0 \to \mathcal{E}^1 \to \cdots \to \mathcal{E}^p \to \mathcal{E}^{p+1} \cdots.$$

of the constant sheaf R over Ω. By the theorem of Dolbeault [1], we have the following Dolbeault's isomorphism

$$(13) \qquad \mathrm{H}^p(\Omega, \mathrm{R}) \cong \mathrm{H}^0(\Omega, (\mathrm{d}^p)^{-1}(0))/\mathrm{d}^{p-1}(\mathrm{H}^0(\Omega, \mathcal{E}^{p-1}))$$

for any positive integer p. By the universal coefficient theorem [12], we have $\mathrm{H}^p(\Omega, \mathrm{R}) \cong \mathrm{H}^p(\Omega, \mathrm{Z}) \otimes \mathrm{R}$ and, hence, $\mathrm{H}^p(\Omega, \mathrm{R}) = 0$ if and only if $\mathrm{H}^p(\Omega, \mathrm{Z}) = 0$, for any positive integer p. Therefore, from the assumptions $\mathrm{H}^1(\Omega, \mathrm{Z}) = 0$ and (9), we have $\omega \in \mathrm{H}^0(\Omega, (\mathrm{d}^1)^{-1}(0)) = \mathrm{d}^0(\mathrm{H}^0(\Omega, \mathcal{E}^0))$ and there exists $f_4 \in \mathrm{H}^0(\Omega, \mathcal{E}^0)$ such that $\omega = \mathrm{d}^0 f_4$. The quaternion valued function $f := f_1 + f_2 i + f_3 j + f_4 k$ of class C^∞ on Ω satisfies $D_x f = 0$ by (6) and $\omega = \mathrm{d}^0 f_4$, and $f D_y = 0$ by (7) and $\omega = \mathrm{d}^0 f_4$. Hence the function f is the desired biregular function on Ω with f_4 as k part for the real part f_1, i part f_2 and j part f_3 given.

Corollary. *Let Ω be a domain in R^8 with $\mathrm{H}^1(\Omega, \mathrm{Z}) = 0$ for the ring Z of integers, f_1, f_2, f_3 be functions of class C^∞ on Ω satisfying the integrability condition (9)-(10). Then f_1, f_2, f_3 are harmonic functions on Ω.*

Proof. By the theorem, there exists a real valued function f_4 of class C^∞ on Ω such that the quaternion valued function $f = f_1 + f_2 i + f_3 j + f_4 k$ is biregular on Ω. Since we have

$$\Delta_x f := \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} + \frac{\partial^2 f}{\partial x_4^2} = \overline{D_x} D_x f = 0, \tag{14}$$

$$f\Delta_y := \frac{\partial^2 f}{\partial y_1^2} + \frac{\partial^2 f}{\partial y_2^2} + \frac{\partial^2 f}{\partial y_3^2} + \frac{\partial^2 f}{\partial y_4^2} = f D_y \overline{D_y} = 0,$$

f_1, f_2, f_3, f_4 are harmonic on Ω. q.e.d.

An open set Ω in R^8 is said to be *a Son domain* if, for any pair of quaternion valued functions $g = g_1 + g_2 i + g_3 j + g_4 k$ and $h = h_1 + h_2 i + h_3 j + h_4 k$ of class C^∞ on Ω with $gD_y = D_x h$, there exists a quaternion valued function $f = f_1 + f_2 i + f_3 j + f_4 k$ of class C^∞ on Ω with $D_x f = g$ and $fD_y = h$. By Son [13], a product domain Ω of a simply connected domain Ω_x in the space R^4 of variables $x := (x_1, x_2, x_3, x_4)$ and a simply connected domain Ω_y in the space R^4 of variables $y := (y_1, y_2, y_3, y_4)$ is a Son domain.

Lemma. *Let Ω be a Son domain in the space R^8 of 8 real variables $x := (x_1, x_2, x_3, x_4)$ and $y := (y_1, y_2, y_3, y_4)$ and $\mathcal{R}$ be the sheaf of germs of biregular functions $f = f_1 + f_2 i + f_3 j + f_4 k$ over Ω. Then, there holds $\mathrm{H}^1(\Omega, \mathcal{R}) = 0$.*

Proof. Let $\mathcal{Q}$ be the sheaf of germs of quaternion valued functions $q = q_1 + q_2 i + q_3 j + q_4 k$ of class C^∞ over Ω in the space R^8, $\mathcal{U} = \{U_\lambda; \lambda \in \Lambda\}$ be an open covering of the Son domain Ω and $\mathcal{C} = \{f_{(\lambda_1,\lambda_2)}; \lambda_1, \lambda_2 \in \Lambda\}$ be a 1-cocycle of the covering $\mathcal{U}$ with coefficients in the sheaf $\mathcal{R}$. By the definition, the 1-cocycle $\mathcal{C}$ satisfies the condition of compatibility

$$f_{(\lambda_1,\lambda_2)} + f_{(\lambda_2,\lambda_3)} + f_{(\lambda_3,\lambda_1)} = 0 \tag{15}$$

in $U_{\lambda_1} \cap U_{\lambda_2} \cap U_{\lambda_3}$ for any $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ with non empty $U_{\lambda_1} \cap U_{\lambda_2} \cap U_{\lambda_3}$. Since each $f_{(\mu,\nu)}$ is biregular in each $U_\mu \cap U_\nu$ for any $\mu, \nu \in \Lambda$, each $f_{(\mu,\nu)}$ is of class C^∞ in each $U_\mu \cap U_\nu$. Hence the cocycle $\mathcal{C}$ is a cocycle of the covering $\mathcal{U}$ with coefficients in the sheaf $\mathcal{Q}$ of germs of quaternion valued functions of class C^∞. Since we have $\mathrm{H}^1(\mathcal{U}, \mathcal{Q}) = 0$ by the partition of the unity, there exists a 0-cochain $\{f_{(\mu)}; \mu \in \Lambda\}$ of the covering $\mathcal{U}$ with coefficients in the sheaf $\mathcal{Q}$ such that its coboundary is the 1-cocycle $\mathcal{C}$, i. e., each $f_{(\mu)}$ is a function of class C^∞ in each U_μ and there holds $f_{(\mu,\nu)} = f_{(\nu)} - f_{(\mu)}$ in each $U_\mu \cap U_\nu$. Since $f_{(\mu,\nu)}$ is biregular in $U_\mu \cap U_\nu$, we have $0 = D_x f_{(\mu,\nu)} = D_x f_{(\nu)} - D_x f_{(\mu)}$ and $0 = f_{(\mu,\nu)} D_y = f_{(\nu)} D_y - f_{(\mu)} D_y$ in each U_μ. This means that, if we put $g = D_x f_{(\mu)}, h = f_{(\mu)} D_y$ in each U_μ, the pair (g, h) of the functions g and h is a well-defined pair of quaternion valued functions of class C^∞ on Ω satisfying the condition of compatibility $gD_y = D_x h$. Since Ω is a Son domain, there exits a function f of class C^∞ on Ω such that $D_x f = g, fD_y = h$. We put $r_{(\mu)} = f_{(\mu)} - f$ on U_μ. Then, the revised 0-cochain $\{r_{(\mu)} : \mu \in \Lambda\} \in \mathrm{C}^0(\mathcal{U}, \mathcal{R})$ has the 1-cocycle $\mathcal{C}$ as its coboundary. q.e.d.

Theorem 2. *Let Ω be a Son domain in the space R^8 of 8 real variables $x := (x_1, x_2, x_3, x_4)$ and $y := (y_1, y_2, y_3, y_4)$. Then, there holds $\mathrm{H}^1(\Omega, \mathrm{Z}) = 0$, if and only if, for any functions f_1, f_2, f_3 of class C^∞ on Ω satisfying the integrability condition (9)-(10), there exists a function f_4 of class C^∞ on Ω such that the quaternion valued function*

$f = f_1 + f_2 i + f_3 j + f_4 k$ *is a biregular function on* Ω.

Proof. Let $\mathcal{P}$ be the sheaf of germs of triples (f_1, f_2, f_3) of functions f_1, f_2, f_3 of class C^∞ over Ω satisfying the integrability condition (9)-(10). We consider the sheaf homomorphism $\mathcal{R} \ni f = f_1 + f_2 i + f_3 j + f_4 k \rightsquigarrow \pi(f) := (f_1, f_2, f_3) \in \mathcal{P}$. Then, by (6) and (7), the kernel of the homomorphism π is isomorphic to the constant sheaf R of real number field. So, we consider the inclusion ι associating, to each real number $r \in \mathrm{R}$, the germ of the constant functions $rk \in \mathcal{R}$. By Theorem 1, the short exact sequence

$$(16) \qquad 0 \to \mathrm{R} \to \mathcal{R} \to \mathcal{P} \to 0$$

of sheaves over Ω, given by the homomorphisms ι and π, is exact and induces a long exact sequence

$$(17) \qquad \mathrm{H}^0(\Omega, \mathrm{R}) \to \mathrm{H}^0(\Omega, \mathcal{R}) \to \mathrm{H}^0(\Omega, \mathcal{P}) \to \mathrm{H}^1(\Omega, \mathrm{R}) \to \mathrm{H}^1(\Omega, \mathcal{R}) \to \mathrm{H}^1(\Omega, \mathcal{P})$$

of cohomology of Ω. Since we have $\mathrm{H}^1(\Omega, \mathcal{R}) = 0$ by the above lemma, we have the isomorphism

$$(18) \qquad \mathrm{H}^1(\Omega, \mathrm{R}) \cong \mathrm{H}^0(\Omega, \mathcal{P})/\pi(\mathrm{H}^0(\Omega, \mathcal{R})).$$

By the universal coefficient theorem [12], $\mathrm{H}^1(\Omega, \mathrm{R}) = 0$ if and only if $\mathrm{H}^1(\Omega, \mathrm{Z}) = 0$. Hence we have the equivalence

$$(19) \qquad \mathrm{H}^1(\Omega, \mathrm{Z}) = 0 \Longleftrightarrow \mathrm{H}^0(\Omega, \mathcal{P}) = \pi(\mathrm{H}^0(\Omega, \mathcal{R})),$$

what was to be proved.

3. Weak Solutions

Theorem 3. *Let* Ω *be a domain in the space* R^8 *of 8 real variables* $x := (x_1, x_2, x_3, x_4)$ *and* $y := (y_1, y_2, y_3, y_4)$ *with* $\mathrm{H}^1(\Omega, \mathrm{Z}) = 0$, f_1, f_2, f_3 *be distributions on* Ω *satisfying the integrability condition (9)-(10) in the sense of distribution. Then, the functions* f_1, f_2, f_3 *are distributions defined by functions of class* C^∞ *on* Ω *and there exits a function* f_4 *of class* C^∞ *on* Ω *such that the quaternion valued function* $f = f_1 + f_2 i + f_3 j + f_4 k$ *is a biregular function on* Ω.

Proof. In the proof of Theorem 1, we replace the sheaf $\mathcal{E}^p$ of germs of differential forms of degree p with coefficients real valued functions of class C^∞ over the domain Ω by the sheaf $\mathcal{D}^p$ of germs of differential forms of degree p with coefficients distributions over the domain Ω. Then, we have the other generalized Dolbeault's isomorphism

$$(20) \qquad \mathrm{H}^1(\Omega, \mathrm{R}) \cong \mathrm{H}^0(\Omega, (\mathrm{d}^1)^{-1}(0))/\mathrm{d}^0(\mathrm{H}^0(\Omega, \mathcal{E}^p)),$$

where $(\mathrm{d}^1)^{-1}(0)$ is the sheaf of germs of closed 1-forms $\sum_{\nu=1}^4 (g_\nu \mathrm{d}x_\nu + h_\nu \mathrm{d}y_\nu)$ with coefficients g_ν, h_ν, which are distributions. By assumption, we have $\omega \in \mathrm{H}^0(\Omega, (\mathrm{d}^1)^{-1}(0)) = \mathrm{d}^0(\mathrm{H}^0(\Omega, \mathcal{D}^0))$. Hence, there exists a distribution f_4 on Ω such that ω is its differential in the sense of distribution. Then, we have $\Delta_x f = \overline{D_x} D_x f = 0, f\Delta_y = f D_y \overline{D_y} = 0$ and each part f_ν of $f = f_1 + f_2 i + f_3 j + f_4 k$ is a distribution on Ω which is a weak solution of the typical elliptic equation $(\Delta_x + \Delta_y) f_\nu = 0$ of Laplace. Directly by Theorem 7.2 of Yoshida [14] written in Japanese or, more precisely, by combination of Sobolev's Lemma with the theory of Friedrichs [2] as is indicated there [14] in Japanese, each part f_ν of f is of class C^∞ on the domain Ω and we can apply Theorem 1. q.e.d.

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Received May 25, 1998 **Revised March 15, 1999**