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Note on Kaplansky's Commutative Rings

Tomohiro TANABE and Ryûki MATSUDA *

Let L be a torsion-free abelian (additive) group, and let S be a subsemigroup of L. Assume that $S \ni 0$. Then S is called a grading monoid (or a g-monoid) ([8]). Many technical terms in multiplicative ideal theories for commutative rings R may be defined analogously for g-monoids S. For example, a non-empty subset I of a g-monoid S is called an ideal of S if $S + I \subset I$. An ideal P of S is called a prime ideal of S, if $P \neq S$ and if $x + y \in P$ (for $x, y \in S$) implies $x \in P$ or $y \in P$. An element x of S is called a unit of S, if $x + y = 0$ for some element $y \in S$. An element x of S is called a prime element of S, if $S + x$ is a prime ideal of S. If every non-unit element of S is expressible as a finite sum of prime elements of S, S is called a unique factorization semigroup (or a UFS). Let x, y be elements of S. We say that x divides y, if $y = x + s$ for some $s \in S$. S is called a Noetherian semigroup, if each ideal I of S can be expressible as $I = \bigcup_{i=1}^{n}(S + a_i)$ for a finite number of elements $a_1, \cdots a_n$ of S. $\cdots$. Many propositions in multiplicative ideal theories for commutative rings R are known to hold for g-monoids S (cf. [1], [2] and [6]). Of course, every technical term for commutative rings R can not be necessarily defined for g-monoids S, and every proposition for R can not be necessarily formulated for S. However, the second author conjectures that almost all propositions in multiplicative ideal theories for R hold for S.

The aim of this paper is to prove propositions in Kaplansky's Commutative Rings ([4]) for g-monoids. We will prove for g-monoids S all the propositions in [4, Ch.1 and Ch.2] that can be formulated for S. We will give consecutive numbers for all of our propositions. The case that the proof of some proposition is straightforward, we will omit it's proof.

If an ideal I is properly contained in S, then I is called a proper ideal of S. If, for a proper ideal M, there are no ideals properly between M and S, then M is called a maximal ideal of S.

Let I be an ideal of a g-monoid S, and $x, x_1, \cdots, x_n \in S$. Then we set $(x_1, \cdots, x_n) = \bigcup_{i=1}^{n}(S + x_i)$ and $(I, x) = I \cup (S + x)$. If $I = (a)$ for some

*Department of Mathematics, Ibaraki University

$a \in S$, then I is called a principal ideal of S.

1. Let Y be an additively closed set in a g-monoid S, and I an ideal of S maximal with respect to the exclusion of Y. Then I is a prime ideal of S.

Let Y be an additively closed set in a g-monoid S. Then Y is called saturated, if $s_1 + s_2 \in Y$ (for $s_1, s_2 \in S$) implies $s_1, s_2 \in Y$.

2. Let S be a g-monoid and Y a non-empty subset of S. Then the following conditions are equivalent.

(1) Y is a saturated additively closed set.

(2) $S - Y = \bigcup P_\lambda$, the union ranging over all prime ideals disjoint from T.

Let $a, b \in S$. We say that a and b are associated elements of S, if $a - b$ is a unit of S.

3. Let S be a g-monoid, and $p_1, \cdots, p_n, q_1, \cdots, q_m$ be prime elements of S. If $p_1 + \cdots + p_n = q_1 + \cdots + q_m$, then $n = m$ and p_i and q_i are associated up to a permutation.

Proof. We prove by induction on n. Suppose that $n > 1$ and the result is true for $n - 1$. There exists $k \in \mathbf{N}$ such that $q_k \in (p_n)$. Hence $q_k = s + p_n$ for $s \in S$. Then s is a unit. We have $p_1 + \cdots + p_{n-1} = s + q_1 + \cdots + q_{k-1} + q_{k+1} + \cdots + q_m$. By the hypothesis, $n - 1 = m - 1$, and p_i and q_i are associated up to a permutation.

4. Let S be a g-monoid, and Y the union of units and all elements in S expressible as a finite sum of prime elements. Then Y is a saturated additively closed set.

Proposition 5. Let S be a g-monoid. Then the following conditions are equivalent.

(1) S is a UFS.

(2) Every prime ideal of S contains a prime element.

Proof. (2)$\Longrightarrow$ (1): Let T be the union of units and all elements of S expressible as a sum of prime elements. Then T is saturated by 4. Suppose that $T \neq S$. Take $c \in S - T$. Then (c) is disjoint from T. Expand (c) to a prime ideal P disjoint from T. By the hypothesis, P contains a prime element; a contradiction. Hence $S = T$, and therefore S is a UFS.

Let I be an ideal of S. We say that I is finitely generated, if $I = (a_1, \cdots, a_n)$ for a finite number of elements $a_1, \cdots, a_n \in I$.

If a non-empty set A satisfies the following conditions, then A is called an S-module.

(i) $s \in S, a \in A$ implies $s + a \in A$.

(ii) $0 + a = a$.

(iii) $s_1 + (s_2 + a) = (s_1 + s_2) + a$ (for $s_1, s_2 \in S$).

An S-module A is called finitely generated over S, if we can write $A = \bigcup_{i=1}^{n}(S + x_i)$ for a finite number of elements $x_1, \cdots, x_n \in A$.

Let A be an S-module, $x, a_1 \in A$, and $(x : a_1)_S = \{s \in S \mid s + a_1 \in S + x\}$.

Proposition 6. Let A be an S-module, and $x \in A$. Assume that $I = (x : a_1)_S$ is maximal among all $\{(x : a_1)_S \mid a_1 \in A \text{ with } a_1 \notin S + x\}$. Then I is a prime ideal.

Proof. Assume that $s_1, s_2 \in S$ and $s_1 + s_2 \in I$. If $s_1 \notin I$, then $s_1 + a_1 \notin S + x$. Now $I = (x : a_1)_S \subset (x : s_1 + a_1)_S$. By the hypothesis, $(x : a_1)_S = (x : s_1 + a_1)_S$. Since $s_1 + s_2 \in I$, we have $s_1 + s_2 + a_1 \in S + x$, and hence $s_2 \in I$. Therefore I is a prime ideal.

7. Let I be an ideal of a g-monoid S. Assume that I is not finitely generated, and is maximal among all ideals of S that are not finitely generated. Then I is a prime ideal.

Proof. Suppose that $a + b \in I$ with neither a nor b in I. Then the ideal (I, a) is finitely generated. Write $(I, a) = (i_1, \cdots, i_n, a)$ (for $i_1, \cdots, i_n \in I$) and $J = \{y \in S \mid y + a \in I\}$. Then $J \supset I$ and $b \in J$.

Hence J is finitely generated. Write $J = (j_1, \cdots, j_m)$ (for $j_1, \cdots, j_m \in J$). We prove that $I = (i_1, \cdots, i_n, j_1 + a, \cdots, j_m + a)$. Take $z \in I$. Then we have $z = i_k + s_1$ or $z = a + s_2$ since z lies in (I, a). If $z = i_k + s_1$, then $z \in (i_1, \cdots, i_n, j_1 + a, \cdots, j_m + a)$. If $z = a + s_2$, then we can write $s_2 = j_l + s_3$ since $s_2 \in J$. Then $z = a + j_l + s_3 \in (i_1, \cdots, i_n, j_1 + a, \cdots, j_m + a)$. It follows that I is finitely generated; a contradiction. Therefore I is a prime ideal.

By the above 7, we have the following,

Proposition 8. If every prime ideal of a g-monoid S is finitely generated, then S is a Noetherian semigroup.

9. Let $P_1 \subset P_2 \subset P_3 \subset \cdots$ be a chain of prime ideals of a g-monoid S, then $\bigcup_i P_i$ is a prime ideal of S. Let $P_1 \supset P_2 \supset P_3 \supset \cdots$ be a chain of prime ideals of S such that $\bigcap_i P_i \neq \emptyset$. Then $\bigcap_i P_i$ is a prime ideal of S.

Let $P_1 \supset P_2 \supset \cdots$ be a chain of prime ideal of S. Then it is not necessarily true that $\bigcap_i P_i \neq \emptyset$.

10. Let I be an ideal of a g-monoid S, and P a prime ideal containing I. Then P can be shrunk to a prime ideal minimal among all prime ideals containing I.

Proposition 11. Let $P \subset Q$ be distinct prime ideals of a g-monoid S. Then there exist distinct prime ideals P_1, Q_1 with $P \subset P_1 \subset Q_1 \subset Q$ such that there are no prime ideals properly between P_1 and Q_1.

Proof. Insert a maximal chain $\{P_i\}$ of prime ideals between P and Q. Take any element $x \in Q - P$. Define Q_1 to be the intersection of all P_i containing x, and P_1 the union of all P_i not containing x. By 9, P_1 and Q_1 are prime ideals, and $P \subset P_1 \subset Q_1 \subset Q$. By the maximality of $\{P_i\}$, no prime ideals can lie properly between P_1 and Q_1.

Let $S \subset T$ be g-monoids. An element $\alpha \in T$ is called integral over S, if there exists $n \in \mathbf{N}$ such that $n\alpha \in S$. T is called integral over S if all its elements are integral over S.

Proposition 12. Let $S \subset T$ be g-monoids and $u \in T$. Then the following conditions are equivalent.

(1) u is integral over S.

(2) There exists a finitely generated S-submodule A of T such that $u + A \subset A$.

Proof. (1)$\Longrightarrow$ (2): By the hypothesis, $nu \in S$ for some $n \in \mathbf{N}$. Set $A = S \cup (S+u) \cup \cdots \cup (S+(n-1)u)$. Then $u + A \subset A$.

(2)$\Longrightarrow$ (1): Let $A = \bigcup_{i=1}^{n}(S+a_i)$. We may assume that $u + a_1 = s_1 + a_2, u + a_2 = s_2 + a_3, \cdots, u + a_{l-1} = s_{l-1} + a_l$ and $u + a_l = s_l + a_k$ for the elements s_i of S and for $1 \leq k \leq l \leq n$. Then we have $(l-k+1)u = s_k + s_{k+1} + \cdots + s_l$. Thus u is integral over S.

13. Let $S \subset \Gamma$ be g-monoids. Then the set of all elements of Γ that are integral over S is a subsemigroup of Γ.

We define $\mathbf{Z}_0$ as $\mathbf{Z}_0 = \{n \in \mathbf{Z} \mid n \geq 0\}$. Let $S \subset T$ be g-monoids and $u_1, \cdots, u_n \in T$. Then the subset $S + \mathbf{Z}_0 u_1 + \cdots + \mathbf{Z}_0 u_n$ of T is denoted by $S[u_1, \cdots, u_n]$. $S[u_1, \cdots, u_n]$ is a subsemigroup of T.

14. Let S be a g-monoid, and u an element of a g-monoid containing S. Then $-u$ is integral over S if and only if $-u \in S[u]$.

15. Let S be a g-monoid that is contained in a torsion-free abelian (additive) group G. If G is integral over S, then S is a group.

Let $S \subset T$ be g-monoids. If $T = S[x_1, \cdots, x_n]$ for some $x_1, \cdots, x_n \in T$, then T is called a finitely generated g-monoid over S.

Proposition 16. Let $S \subset T$ be g-monoids. Then the following conditions are equivalent.

(1) T is a finitely generated S-module.

(2) As a g-monoid, T is finitely generated over S and is integral over S.

Proof. (1)$\Longrightarrow$ (2): Let $T = \bigcup_{i=1}^{n}(S + x_i)$ for a finite number of elements $x_1, \cdots, x_n \in T$. Then $T = S[x_1, \cdots, x_n]$. By Proposition 12, T is integral over S.

(2)$\Longrightarrow$ (1): Let $T = S[x_1, \cdots, x_n]$ for a finite number of elements $x_1, \cdots, x_n \in T$. Then we can take $k_i \in \mathrm{N}$ such that $k_i x_i \in S$. Then $T = \bigcup_{0 \le m_i < k_i}(S + m_1 x_1 + \cdots + m_n x_n)$.

Let S be a g-monoid and $\mathrm{q}(S) = \{s_1 - s_2 \mid s_1, s_2 \in S\}$. We call $\mathrm{q}(S)$ the quotient group of S.

Proposition 17. Let S be a g-monoid with quotient group G. The following conditions are equivalent.

(1) G is a finitely generated g-monoid over S.

(2) As a g-monoid, G can be generated over S by one element.

Proof. (1)$\Longrightarrow$ (2): Assume that $G = S[u_1, \cdots, u_n]$ and $u_i = a_i - b_i$ (for $a_i, b_i \in S, 1 \le i \le n$). Put $b_1 + \cdots + b_n = c$. Take any element $f \in G$. Then, for $s \in S$ and $k_1, \cdots, k_n \in \mathrm{Z}_0$, we have

$f = s + k_1 u_1 + \cdots + k_n u_n = s + k_1 a_1 + \cdots + k_n a_n - k_1 b_1 - \cdots - k_n b_n.$

For a sufficiently large $k \in \mathrm{Z}_0$, we have

$f = s + k_1 a_1 + \cdots + k_n a_n + (k - k_1) b_1 + \cdots + (k - k_n) b_n - k(b_1 + \cdots + b_n) = s_1 - kc \in S[-c]$ (for $s_1 \in S$).

Hence $G = S[-c]$.

Let S be a g-monoid. If S satisfies either of the conditions in Proposition 17, then S is called a G-semigroup.

18. Let S be a g-monoid with quotient group G. For an element $u \in S$ the following conditions are equivalent.

(1) Any prime ideal of S contains u.

(2) Any ideal of S contains nu for some $n \in \mathrm{N}$.

(3) $G = S[-u]$.

Proof. (1)$\Longrightarrow$ (2): Let I be an ideal of S. Suppose that I contains no multiples of u. By 1, I can be expanded to a prime ideal P disjoint from $T = \{nu \mid n \in \mathrm{N}\}$; a contradiction.

(2)$\Longrightarrow$ (3): Take any element $b \in S$. We can write $nu = s + b$ (for $s \in S, n \in \mathrm{N}$) since $nu \in (b)$. Then $-b = s - nu \in S[-u]$. Hence $G = S[-u]$.

(3)$\Longrightarrow$ (1): Let P be a prime ideal of S. Take any element $b \in P$. We can write $-b = s - nu$ (for $s \in S, n \in \mathrm{N}$). Then $nu = s + b \in P$. Therefore $u \in P$.

Let S be a g-monoid with quotient group G. If T is a g-monoid lying between S and G, then T is called an oversemigroup of S.

19. Let S be a G-semigroup and T an oversemigroup of S. Then T is a G-semigroup.

Let S be a g-monoid, X an indeterminate and $S[X] = \{s + nX \mid s \in S, n \in \mathrm{Z}_0\}$. We call $S[X]$ the polynomial semigroup of X over S.

20. If a g-monoid S is a group, then $S[X]$ is a G-semigroup.

Let $S \subset T$ be g-monoids and $u \in T$. Then u is called algebraic over S, if there exists $s \in S$ and $n \in \mathrm{N}$ such that $s + nu \in S$. If u is not algebraic over S, u is called transcendental over S. T is called algebraic over S if all its elements are algebraic over S.

Proposition 21. Let $S \subset T$ be g-monoids. Assume that T is algebraic over S and finitely generated as a g-monoid over S. Then S is a G-semigroup if and only if T is a G-semigroup.

Proof. Let G, G_1 be quotient groups of S, T respectively. Assume that S is a G-semigroup, say $G = S[-u]$ (for $u \in S$). Let $f \in T[-u]$. Then we can take $n \in \mathrm{N}, g \in G$ such that $nf = g$. Then $-f = (n-1)f - g \in T[-u]$. Hence $T[-u]$ is a group, and hence T is a G-semigroup. Assume that T is a G-semigroup, $G_1 = T[-v]$ (for $v \in T$) and $T = S[w_1, \cdots, w_k]$ (for

$w_i \in T$). Since T is algebraic over S, we have $a+mv = s$ and $s_i+m_iw_i \in S$ for some $a, s \in S$ and $m, m_i \in \mathbf{N}$. Let $S_1 = S[-s, -s_1, \cdots, -s_k]$. Then $G_1 = S_1[-v, w_1, \cdots, w_k]$. Since $-v, w_1, \cdots, w_k$ are integral over S_1, G_1 is integral over S_1. By 15, S_1 is a group. Hence $G = S_1$, and therefore S is a G-semigroup.

Proposition 22. Let $S \subset T$ be g-monoids and $u \in T$. If $S[u]$ is a G-semigroup, then S is a G-semigroup.

Proof. Let G, G' be quotient groups of $S, S[u]$ respectively. Since $S[u]$ is a G-semigroup, $G' = S[u, -v]$ for $v \in S[u]$. Let $-v = g + ku$ for $g \in G$ and $k \in \mathbf{Z}$. Then $G' = S[u, g, ku]$.

(i) Assume that u is transcendental over S. Take any element $g_1 \in G$. We have $g_1 = s + n_1u + n_2g + n_3ku = s + (n_1 + n_3k)u + n_2g$ (for $n_1, n_2, n_3 \in \mathbf{Z}_0$). By the hypothesis, $n_1 + n_3k = 0$. Therefore $G = S[g]$.

(ii) Assume that u is algebraic over S. Then S is a G-semigroup by Proposition 21.

23. Let $S \subset T$ be g-monoids and $u \in T$. Assume that $S[u]$ is a G-semigroup. Then u is not necessarily algebraic over S.

For example, assume that S is a group and X an indeterminate. Then X is transcendental over S, but $S[X]$ is a G-semigroup.

24. Let S be a g-monoid and N a maximal ideal of $S[X]$. If S is a group, then $N \cap S = \emptyset$. If S is not a group, then $N \cap S \neq \emptyset$.

Proof. If S is a group, then $N = S + \mathbf{N}X$. Hence $N \cap S = \emptyset$. If S is not a group, then we can take a maximal ideal M of S. Then $N = M \cup (S + \mathbf{N}X)$, and therefore $N \cap S \neq \emptyset$.

Let T be an additively closed set in a g-monoid S. We define S_T as $S_T = \{s - t \mid s \in S, t \in T\}$. Let I be an ideal of S. We write I_T for $I + S_T$. Let P be a prime ideal of S. We write S_P for S_{S-P}.

25. Let T be an additively closed set in a g-monoid S. Then there is a one-to-one order-preserving correspondence between prime ideals of S_T and prime ideals of S disjoint from T.

25 implies the following,

26. Let P be a prime ideal of a g-monoid S. Then there is a one-to-one order-preserving correspondence between prime ideals of S_P and prime ideals of S contained in P.

25 implies the following too,

27. Let S be a g-monoid with quotient group G, and X an indeterminate. Then there is a one-to-one correspondence between prime ideals of $S[X]$ disjoint from S and prime ideals of $G[X]$.

Proposition 28. Let S be a g-monoid. Then there cannot exist in $S[X]$ a chain of three distinct prime ideals with the same contracted ideal in S.

Proof. Suppose that there exists in $S[X]$ a chain of three distinct prime ideals $Q_1 \subsetneqq Q_2 \subsetneqq Q_3$ with the same contraction P in S. Take $f \in Q_2 - Q_1$. Then $f = s + nX$ for $s \in S, n \in \mathbf{Z}_0$. If $nX \notin Q_2$, then $f \in Q_1$ for $s \in P$; a contradiction. Hence $X \in Q_2$. Take $g \in Q_3 - Q_2$, say $g = s' + n'X$ for $s' \in S, n' \in \mathbf{Z}_0$. If $n' = 0$, then $g = s' \in P \subset Q_1$; a contradiction. Therefore $n' \geq 1$. Then $g = s' + n'X \in Q_2$; a contradiction.

Let $P = P_1 \supsetneqq \cdots \supsetneqq P_n$ be a chain of prime ideals of a g-monoid S. Then $n - 1$ is called the length of the chain. Let k be the supremum of lengths of all chains of prime ideals of S. Then $k + 1$ is called the dimention of S, and is denoted by $\dim(S)$. Let l be the supremum of lengths of all chains of prime ideals $P = P_1 \supsetneqq \cdots \supsetneqq P_n$. Then $l + 1$ is called the height of P, and is denoted by $\mathrm{ht}(P)$.

Let I be an ideal of S. Then we write I^* for $I + S[X]$.

29. Let S be a g-monoid and Q a prime ideal of $S[X]$. If $Q \cap S = \emptyset$, then $Q = (X)$.

Proof. Take any $f \in Q$, say $f = s + nX$ (for $s \in S, n \in \mathbf{Z}_0$). If $n = 0$, then $f = s \in S$; a contradiction. Hence $n \geq 1$, and therefore $f \in (X)$, that is, $Q \subset (X)$. Since $s \notin Q$, we have $nX \in Q$, that is, $X \in Q$. Therefore $Q = (X)$.

By 29, for every prime ideal P of S of height 1, P^* has height 1.

Assume that, for every prime ideals $P \supsetneqq N$ in S with no prime ideals properly between P and N, there cannot exist a prime ideal Q of $S[X]$ such that $P^* \supsetneqq Q \supsetneqq N^*$. Then S is called a strong S-semigroup.

Proposition 30. Let S be a strong S-semigroup, P a prime ideal of height n in S, and Q a prime ideal of $S[X]$ that contracts to P in S and contains P^* properly. Then $\mathrm{ht}(P^*) = n$ and $\mathrm{ht}(Q) = n + 1$.

Proof. Let $P = P_1 \supsetneqq \cdots \supsetneqq P_n$ be a chain of prime ideals of S. Then we have the chain of prime ideals $Q \supsetneqq P_1^* \supsetneqq \cdots \supsetneqq P_n^*$ in $S[X]$. It follows that $\mathrm{ht}(P^*) \geq n$ and $\mathrm{ht}(Q) \geq n + 1$. We prove that $\mathrm{ht}(P^*) \leq n$ and $\mathrm{ht}(Q) \leq n + 1$ by induction on n.

(i) $n = 1$: We have $\mathrm{ht}(P^*) = 1$. Assume that $\mathrm{ht}(Q) > 2$. Then we can take a chain of prime ideals $Q = Q_1 \supsetneqq Q_2 \supsetneqq Q_3$. By 29, $Q_2 \cap S = P$. By 28, $Q_2 = P^*$, and hence $\mathrm{ht}(P^*) > 1$; a contradiction. Therefore $\mathrm{ht}(Q) = 2$.

(ii) Suppose that $n > 1$ and the result is true for $n - 1$. Assume that $\mathrm{ht}(P^*) > n$. Then there exists a prime ideal Q_n of $S[X]$ such that $P^* \supsetneqq Q_n$ and $\mathrm{ht}(Q_n) = n$. By 29, we have $Q_n \cap S \neq \emptyset$. Let P_n be the contraction of Q_n to S. P_n is properly contained in P. Let $\mathrm{ht}(P_n) = m$. Then $m < n$. If $Q_n \supsetneqq P_n^*$, then $\mathrm{ht}(Q_n) = m + 1$ by the induction hypothesis. Hence there are no prime ideals properly between P and P_n, and then S is not a strong S-semigroup; a contradiction. Therefore $Q_n = P_n^*$. Continuing this work we can make a chain of prime ideals of length $n - 1$ descending from P_n; a contradiction. Therefore $\mathrm{ht}(P^*) = n$. Assume that $\mathrm{ht}(Q) > n + 1$. Then there exists a prime ideal Q_{n+1} such that $Q \supsetneqq Q_{n+1}$

and $\mathrm{ht}(Q_{n+1}) = n+1$. Let $Q_{n+1} \cap S = P_{n+1}$ and $\mathrm{ht}(P_{n+1}) = m$. By the hypothesis, $n > m$. If $Q_{n+1} \supsetneqq P^*_{n+1}$, then $\mathrm{ht}(Q_{n+1}) = m+1$, that is, $n = m$; a contradiction. Hence $Q_{n+1} = P^*_{n+1}$. Then $n+1 = m$; a contradiction. Therefore $\mathrm{ht}(Q) = n+1$.

31. Let $S \subset T \subset \Gamma$ be g-monoids and u an element of Γ. Suppose that u is integral over T and that T is integral over S. Then u is integral over S.

Let $S \subset T$ be g-monoids. We may list four properties that might hold for a pair S, T.

Lying over (LO): For any prime ideal P of S there exists a prime ideal Q of T with $Q \cap S = P$.

Going up (GU): (i) (LO) holds, and (ii) Given prime ideals $P_0 \subset P$ of S and Q_0 of T with $Q_0 \cap S = P_0$, there exists a prime ideals Q of T satisfying $Q_0 \subset Q$ and $Q \cap S = P$.

Going down (GD): Given prime ideals $P \supset P_0$ of S and Q of T with $Q \cap S = P$, there exists a prime ideal Q_0 of T satisfying $Q \supset Q_0$ and $Q_0 \cap S = P_0$.

Incomparable (INC): (i) If Q is a prime ideal of T, then $Q \cap S \neq \emptyset$, and (ii) Two different prime ideals of T with the same contracted ideal of S cannot be comparable.

32. The following two conditions are equivalent for g-monoids $S \subset T$:

(a) (GU) holds.

(b) (LO) holds. And if P is a prime ideal of S, J is the complement of P in S, and Q is an ideal of T maximal with respect to the exclusion of J, then $Q \cap S = P$.

Proof. (a)$\Longrightarrow$ (b): Let Q be maximal with respect to the exclusion of J. By 1, Q is a prime ideal of T. We have to prove $Q \cap S = P$. Q lies over the prime ideal $Q \cap S$ of S, and (GU) permits us to expand Q to a prime ideal Q_1 of T lying over P. By the maximality of Q, we have $Q = Q_1$.

(b)$\Longrightarrow$ (a): Let $P_0 \subset P$ be prime ideals of S. Suppose that a prime ideal Q_0 of T contracts to P_0 in S. Then Q_0 is disjoint from J. Expand it to Q, maximal with respect to the exclusion of J. By the hypothesis, $Q \cap S = P$, proving (GU).

33. The following conditions are equivalent for g-monoids $S \subset T$:

(a) (INC) holds.

(b) For any prime ideal Q of T, we have $Q \cap S \neq \emptyset$. And if P is a prime ideal of S, and Q is a prime ideal of T contracting to P in S, then Q is maximal with respect to the exclusion of J, the complement of P in S.

Proof. (a)$\Longrightarrow$ (b): Let Q_1 be a prime ideal of T disjoint from J. If Q_1 properly contains Q, then $Q_1 \cap S = P$; a contradiction. Therefore Q is maximal with respect to the exclusion of J.

(b)$\Longrightarrow$ (a): Let P be a prime ideal of S, and let Q be a prime ideal of T that contracts to P in S. Suppose that there exists a prime ideal Q' of T such that $Q' \cap S = P$. By the hypothesis, Q and Q' are incomparable.

Proposition 34. Let $S \subset T$ be g-monoids with T integral over S. Then the pair S, T satisfies (INC) and (GU).

Proof. (GU): Let P be a prime ideal of S, J the exclusion of P in S, and Q an ideal of T maximal with respect to the exclusion of J. Then $(P + T) \cap S = P$. Suppose that $Q \cap S \neq P$. Then there exists $u \in P$ such that $u \notin Q \cap S$. The ideal (Q, u) is properly larger than Q. Take $j \in (Q, u) \cap J$. We can write $j = t + u$ for $t \in T$. There exists $m \in \mathrm{N}$ such that $mt \in S$. Then $mj = mt + mu \in P$, and hence $j \in P$; a contradiction. Therefore $Q \cap S = P$. By 32, (GU) holds.

(INC): Let P be a prime ideal of S, Q a prime ideal of T contracting to P in S and $J = S - P$. We show that Q is maximal with respect to the exclusion of J. Suppose on the contrary that Q is properly contained in an ideal I with $I \cap J$ void. Pick $v \in I - Q$. There exists $n \in \mathrm{N}$ such that $nv \in S$. Since $I \cap J = \emptyset$, nv lies in P. Then $v \in Q$; a contradiction. By 33, (INC) holds.

35. Assume that g-monoids $S \subset T$ satisfy (INC). Let P, Q be prime ideals of S, T respectively with $Q \cap S = P$. Then $\mathrm{ht}(Q) \leq \mathrm{ht}(P)$.

Let S be a g-monoid and P a prime ideal of S. Let m be the supremum of lengths of all chains of prime ideals $P = P_1 \subsetneqq \cdots \subsetneqq P_n$. Then m is called the depth of P, and is denoted by depth(P).

36. Assume that g-monoids $S \subset T$ satisfy (GU). Let P be a prime ideal of S of height $n < \infty$. Then there exists in T a prime ideal Q lying over P and having height $\geq n$. If, further, (INC) holds, then $\mathrm{ht}(Q) = n$.

37. Assume that g-monoids $S \subset T$ satisfy (GU) and (INC). Let Q be a prime ideal of T and $P = Q \cap S$. Then depth(P) = depth(Q).

37 implies the following,

38. Assume that g-monoids $S \subset T$ satisfy (GU) and (INC). Then the dimension of T equals to the dimention of S.

Let a, b be elements in a g-monoid S. An element $z \in S$ is called a common diviser of a and b, if z divides a and b. An element $x \in S$ is called a greatest common diviser of a and b, if x is a common diviser of a and b, and $(x) \subset (y)$ for any common diviser y of a and b. The greatest common diviser of a and b is denoted by GCD(a, b). A g-monoid S is called a GCD-semigroup if any two elements in S have a greatest common divisor.

Proposition 39. Let S be a GCD-semigroup. Then,
(1) GCD$(a + b, a + c) = a+$GCD(b, c).
(2) GCD$(a, b) = d$ implies GCD$(a - d, b - d) = 0$.
(3) GCD$(a, b) =$ GCD$(a, c) = 0$ implies GCD$(a, b + c) = 0$.

Proof. (1) Let GCD$(a+b, a+c) = x$. Then a divides x, say $x = a+y$. Then y divides b and c. If z divides b and c, then $a + z$ divides $a + b$ and $a + c$. Thus $a + z$ divides $x = a + y$, and hence z divides y. It follows that GCD$(b, c) = y$, and GCD$(a + b, a + c) = a+$ GCD(b, c).

(3) Suppose that t divides a and $b + c$. Then t divides $a + b$ and $b + c$. Hence t divides GCD$(a + b, b + c)$, which is b by (1). Therefore t divides

a and b, and hence $t = 0$.

Proposition 40. A GCD-semigroup S is integrally closed.

Proof. Suppose that $u \in \mathrm{q}(S)$ and that $nu \in S$ for some $n \in \mathrm{N}$. We can write $u = s_1 - s_2$ for $s_1, s_2 \in S$. Let $\mathrm{GCD}(s_1, s_2) = r$. Then we have $\mathrm{GCD}(s_1 - r, s_2 - r) = 0$ by (2) of Proposition 39. Therefore we may assume that $\mathrm{GCD}(s_1, s_2) = 0$. Now $ns_1 = s + (n-1)s_2 + s_2$. It follows that s_2 is a unit because $\mathrm{GCD}(ns_1, s_2) = 0$ by (3) of Proposition 39. Hence $u \in S$, and therefore S is integrally closed.

41. If S is integrally closed and if T an additively closed set in S, then S_T is integrally closed.

42. Let S_i be a family of g-monoids all contained in one large g-monoid. Suppose that each S_i is integrally closed and $\bigcap S_i \neq \emptyset$. Then $\bigcap S_i$ is integrally closed.

Let S be a g-monoid, A an S-module and I an ideal in S with $I + A \neq A$. Set $\mathrm{Z}(A/(I + A)) = \{s \in S \mid s + a \in I + A$ for some $a \in A - (I + A)\}$.

43. Let S be a g-monoid and I a proper ideal of S. Then $\mathrm{Z}(S/I)$ is a prime ideal.

Proof. Assume that $s_1 + s_2 \in \mathrm{Z}(S/I)$ for $s_1, s_2 \in S$. Then we can take $y \notin I$ satisfying $s_1 + s_2 + y \in I$. If $s_1 \notin \mathrm{Z}(S/I)$, then $s_2 + y \in I$. Hence $s_2 \in \mathrm{Z}(S/I)$, and therefore $\mathrm{Z}(S/I)$ is a prime ideal.

Theorem 44. Let S be a g-monoid. Then $S = \bigcap\{S_P \mid P$ ranges over all $\mathrm{Z}(S/I)$ for all proper principal ideals I of $S\}$.

Proof. Take $u \in \bigcap S_P$, say $u = s - t$ for $s, t \in S$. Let $I = (t : s)_S$. If $I = S$, then $s \in (t)$. Then $u \in S$. If $I \neq S$, then $s \notin (t)$. Let $P = \mathrm{Z}(S/(t))$. We can write $u = s - t = s_1 - t_1$ for $s_1 \in S, t_1 \in S - P$. Then

$s+t_1 = s_1 + t \in (t)$. Hence $t_1 \in P$ for $s \notin (t)$; a contradiction. Therefore $S = \bigcap S_P$.

Theorem 45. The following conditions are equivalent for S.

(1) S is integrally closed.

(2) Let I be any proper principal ideal of S and $P = \mathrm{Z}(S/I)$. Then S_P is integrally closed.

Proof. (2)$\Longrightarrow$ (1): By 42, $\bigcap\{S_P \mid P$ ranges over all $\mathrm{Z}(S/I)$ for all proper principal ideals I of $S\}$ is integrally closed. By Theorem 44, $S = \bigcap S_P$. Therefore S is integrally closed.

Let $S \subset T$ be g-monoids and let I be an ideal of S. Then I is called to survive in T if $I + T \neq T$.

Proposition 46. Let $S \subset T$ be g-monoids, u a unit in T and I a proper ideal of S. Then I survives either in $S[u]$ or in $S[-u]$.

Proof. Suppose the contrary. Then we have $I + S[u] = S[u]$ and $I + S[-u] = S[-u]$, and hence $i_1 + n_1 u = 0$ and $i_2 - n_2 u = 0$ (for $i_1, i_2 \in I, n_1, n_2 \in \mathrm{Z}_0$). Then we have $n_2 i_1 + n_1 n_2 u = 0$ and $n_1 i_2 - n_1 n_2 u = 0$. It follows that $n_2 i_1 + n_1 i_2 = 0$. Hence $I = S$; a contradiction.

Let G be a torsion-free abelian group, and Γ a totally ordered abelian group. A homomorphism v of G to Γ is called a valuation on G. The subsemigroup $\{x \in G \mid v(x) \geq 0\}$ of G is called the valuation semigroup of G associated with v. Let T be an oversemigroup of S. If T is a valuation semigroup of $\mathrm{q}(S)$, then T is called a valuation oversemigroup of S.

47 ([5, Lemma 10]). S is a valuation semigroup if and only if $\alpha \in S$ or $-\alpha \in S$ for each $\alpha \in \mathrm{q}(S)$.

Proposition 48. Let G be a group, S a subsemigroup of G and I a proper ideal of S. Then there exists a valuation semigroup V of G such that I survives in V.

Proof. Consider all pairs (S_α, I_α), where S_α is a semigroup between S and G, and I_α is a proper ideal of S_α with $I \subset I_\alpha$. If $S_\alpha \supset S_\beta$ and $I_\alpha \supset I_\beta$, we set $(S_\alpha, I_\alpha) \geq (S_\beta, I_\beta)$. Zorn's lemma is applicable to yield a maximal pair (V, J). We prove that if $u \in G$ then either u or $-u$ lies in V. Suppose the contrary. By Proposition 46, J survives in $V[u]$ or in $V[-u]$; a contradiction to the maximality of the pair (V, J). Therefore V is a valuation semigroup of G by 47.

Proposition 49. Let S be an integrally closed semigroup with quotient group G. Then $S = \bigcap V_\alpha$ where the V_α's are valuation oversemigroups of S.

Proof. Take $y \in \bigcap V_\alpha$. Suppose that $y \notin S$, write $y = -u$. By 14, $-u \notin S[u]$, that is, $u + S[u] \neq S[u]$. Then we can enlarge $S[u]$ to a valuation oversemigroup V of S in such a way that $u + S[u]$ survives in V by Proposition 48. Then $y \notin V$; a contradiction.

Let S be a g-monoid with quotient group G, and I a non-empty subset of G. We say that I is a fractional ideal of S if

(i) $S + I \subset I$.

(ii) There exists $s \in S$ such that $s + I \subset S$.

For a fractional ideal I of S, let I^{-1} be the set of all $x \in G$ with $x + I \subset S$. Then I^{-1} is a fractional ideal of S. We say that I is invertible if $I + I^{-1} = S$.

Proposition 50. Any invertible fractional ideal I of a g-monoid S is principal.

Proof. By the hypothesis, we have $I + I^{-1} = S$. Then we can take $a \in I, b \in I^{-1}$ such that $a + b = 0$. If $x \in I$, then $x = x + a + b \in (a)$. Hence $I = (a)$.

51. Let I be an invertible ideal of a g-monoid S and T an additively closed set in S. Then I_T is an invertible ideal of S_T.

Proposition 52. Let S be a g-monoid. Then the following conditions are equivalent.

(1) S is a valuation semigroup.

(2) Every finitely generated ideal of S is principal.

Proof. (2)$\Longrightarrow$ (1): Take any elements $a_1, a_2 \in S$. Let $I = (a_1, a_2)$. By the hypothesis, we can write $I = (a)$ for $a \in S$. Then $a_1 = s_1 + a$ and $a_2 = s_2 + a$ for $s_1, s_2 \in S$. We may assume that $a \in (a_1)$. Write $a = s_1' + a_1$ for $s_1' \in S$. Then $a_1 = s_1 + s_1' + a_1$, and we have $s_1 + s_1' = 0$. Therefore a_1 divides a_2. By 47, S is a valuation semigroup.

By Proposition 52, we have the following,

53. If S is a valuation semigroup, then for every prime ideal P of S, S_P is a valuation semigroup.

Proposition 54. Let S be a valuation semigroup, and V a valuation oversemigroup of S. Then $V = S_P$ for some prime ideal P of S.

Proof. Let N be a maximal ideal of V and set $P = N \cap S$. We have $S_P \subset V$. By 53, S_P is a valuation semigroup. Suppose that $V \neq S_P$. Then we can take $v \in V - S_P$. We have $-v \in S_P$, say $-v = a - s'$ (for $a \in S, s' \in S - P$). If $a \notin P$, then $s' - a = v \in S_P$; a contradiction. If $a \in P$, then $a \in N$. Hence $a + v = s' \in P$; a contradiction. Therefore $V = S_P$.

Proposition 55. Let G be a group and X an indeterminate. Let V be a valuation semigroup of $\mathrm{q}(G[X])$ with $V \neq \mathrm{q}(G[X])$. If V contains G properly, then $V = G[X]$ or $V = G[-X]$.

Proof. Either X or $-X$ lies in V. If $X \in V$, then $V = G[X]$. If $X \notin V$, then $V = G[-X]$.

56. Let S be an integrally closed semigroup with quotient group G, and let u be an element of G. Assume that $u_1 + nu = 0$ for a unit u_1 of S and $n \in \mathrm{N}$. Then $u \in S$.

57. Any g-monoid S is a strong S-semigroup.

Proof. Let $P \supsetneqq Q$ be prime ideals of S. Suppose that there are no prime ideals properly between P and Q. Let $P^* \supsetneqq N \supsetneqq Q^*$ be prime ideals of $S[X]$. Take $f \in N - Q^*$, say $f = a + nX$. Since $X \notin P^*$, we have $a \in N$. Then $a \in N \cap S = Q$, and hence $f = a + nX \in Q^*$; a contradiction.

58. Let S be a g-monoid and I, J be ideals of $S[X]$. Set $I_n = \{s \in S \mid s + nX \in I\}$ and $J_n = \{s' \in S \mid s' + nX \in J\}$ (for $n \in \mathrm{Z}_0$). Then,

(1) If $I \subset J$, then $I_n \subset J_n$ (for $n = 0, 1, 2, \cdots$).

(2) If $I \subset J$ and $I_n = J_n$ (for $n = 0, 1, 2, \cdots$), then $I = J$.

Theorem 59. If S is a Noetherian semigroup, then so is $S[X]$.

Proof. Let $I_0 \subset I_1 \subset \cdots$ be ideals of $S[X]$ and $I_{ij} = \{a \in S \mid a + jX \in I_i\} (i, j \in \mathrm{Z}_0)$. Then each I_{ij} is an ideal of S. By the hypothesis, there exists $m \in \mathrm{Z}_0$ such that $I_{mj} = I_{(m+1)j} = \cdots$ for any j. By 58, we have $I_0 \subset I_1 \subset \cdots \subset I_m = I_{m+1} = \cdots$, and hence $S[X]$ is a Noetherian semigroup.

60. Let A be an S-module, and A_1, A_2 be submodules of A satisfying $A = A_1 \cup A_2$. If A_1 and A_2 satisfy the ascending chain condition on S-submodules, then so does A.

Proof. Let $D_1 \subset D_2 \subset \cdots$ be an ascending chain of submodules in A. If each D_i is contained in A_1 or in A_2, then the chain must stop. If there exists i such that $D_i \not\subset M_1$ and $D_i \not\subset A_2$, then we may assume that $D_1 \cap A_1 \neq \emptyset$ and $D_1 \cap A_2 \neq \emptyset$. Then $D_1 \cap A_1 \subset D_2 \cap A_1 \subset \cdots$ forms an ascending chain of submodules in A_1. Since A_1 satisfies the ascending chain condition, there exists $m \in \mathrm{N}$ such that $D_m \cap A_1 = D_{m+1} \cap A_1 = \cdots$. Similarly we can take $n \in \mathrm{N}$ such that $D_n \cap A_2 = D_{n+1} \cap A_2 = \cdots$. Let $l = \max(m, n)$. Then $D_1 \subset \cdots \subset D_l = D_{l+1} = \cdots$. Therefore A satisfies the ascending chain condition on submodules.

61. Let S be a Noetherian semigroup, and A a finitely generated S-module. Then A satisfies the ascending chain condition on S-submodules.

Proof. By 60, it suffices to prove in the case of $A = S + a$ for $a \in A$. Let $A_1 \subset A_2 \subset \cdots$ be submodules of A and $M_i = \{s \in S \mid s + a \in A_i\}$. Then $A_i = M_i + a$ for each i. By the hypothesis, we can take $m \in \mathrm{N}$ such that $M_1 \subset M_2 \subset \cdots \subset M_m = M_{m+1} = \cdots$. Hence $A_1 \subset A_2 \subset \cdots \subset A_m = A_{m+1} = \cdots$, and therefore A satisfies the ascending chain condition.

Let I be an ideal of a g-monoid S. We define nI as $nI = \{x_1 + \cdots + x_n \mid x_i \in I\}$.

62. Let S be a Noetherian semigroup, I an ideal of S, A a finitely generated S-module, and B a submodule of A. Let C be a submodule of A which contains $I + B$ and is maximal with respect to the property $C \cap B = I + B$. Then $nI + A \subset C$ for some n.

Proof. Since I is finitely generated, it suffices to prove that, for any x in I, there exists $m \in \mathrm{N}$ with $mx + A \subset C$. Define D_r to be the submodule of A consisting of all $a \in A$ with $rx + a \in C$. The submodules D_r form an ascending chain of submodules. By 60, it must become stable, say at $r = m$. We prove that $((mx + A) \cup C) \cap B = I + B$. Let $t \in ((mx + A) \cup C) \cap B$. Then we have $t \in mx + A$ or $t \in C$. If $t \in C$, then $t \in I + B$. If $t \in mx + A$, then we can write $t = mx + a$ for $a \in A$. Then $(m + 1)x + a \in C$, for $x + t \in x + B \subset I + B \subset C$. We have $mx + a \in C$, that is, $t \in C$ since $D_m = D_{m+1}$. Thus $t \in C \cap B = I + B$. Hence $((mx + A) \cup C) \cap B = I + B$. By the maximality of C, we have $mx + A \subset C$.

Proposition 63. Let S be a Noetherian semigroup, I an ideal of S and A a finitely generated S-module. Suppose that $B = \bigcap_{n=1}^{\infty}(nI + A) \neq \emptyset$. Then $I + B = B$.

Proof. Among all submodules of A containing $I+B$, pick C maximal with respect to the property $C \cap B = I + B$. By 62, we have $nI + A \subset C$ for some n. Since $B \subset nI + A$, B is contained in C. Therefore $B = I + B$.

Let S be a g-monoid and A an S-module. If $s_1 + x = s_2 + x$ implies $s_1 = s_2$ for $s_1, s_2 \in S$ and $x \in A$, then A is called a cancellative S-module.

64. Let S be a g-monoid, I an ideal of S, A a finitely generated cancellative S-module, and x an element of S satisfying $x + A \subset I + A$. Then $mx \in I$ for some $m \in \mathbf{N}$.

Proof. Write $A = \bigcup_{i=1}^{n}(S + a_i)$ for $a_i \in A$. We may assume that $x + a_1 = i_1 + a_2, x + a_2 = i_2 + a_3, \cdots, x + a_m = i_m + a_1$ (for $i_1, i_2, \cdots, i_m \in I$ and $m \leq n$). Then we have $mx = i_1 + i_2 + \cdots + i_m \in I$.

64 implies the following,

65. Let S be a g-monoid, I an ideal of S, and A a finitely generated cancellative S-module satisfying $I + A = A$. Then $I = S$.

Proposition 66. Let S be a Noetherian semigroup, I a proper ideal of S, and A a finitely generated cancellative S-module. Then $\bigcap_{n=1}^{\infty}(nI + A) = \emptyset$.

Proof. Suppose the contrary. Write $B = \bigcap_{n=1}^{\infty}(nI + A)$. Then $B = I + B$ by Proposition 63. By 65, $I = S$; a contradiction.

By 65, we have the following,

Theorem 67. Let S be a g-monoid with maximal ideal M, and let A be a finitely generated cancellative S-module. Then $M + A \subsetneqq A$.

68. Let S be a g-monoid with maximal ideal M, A a finitely generated cancellative S-module, and B an S-submodule of A satisfying $A \subset B \cup (M + A)$. Then $A = B$.

Proof. Let $A = \bigcup_{i=1}^{n}(S + a_i)$. We may assume that $a_j \notin S + a_i$ for $i \neq j$. Suppose that $A \neq B$. We can take $a_j \notin B$. Then we have $a_j = x + a_j$ for $x \in M$. It follows that $0 \in M$; a contradiction.

69. Let S be a Noetherian semigroup and x a non-unit of S. Then $\mathrm{Z}(S/(x))$ is not necessarily of the form $(x : s)_S$ for $s \in S$.

For example, let $S = \mathbf{Z}_0 \oplus \mathbf{Z}_0$ and $x = (1,1)$. Then $\mathrm{Z}(S/(x))$ is a maximal ideal of S, and we cannot take $s \in S$ satisfying $\mathrm{Z}(S/(x)) = (x : s)_S$.

70. Let $I, P_1, \cdots, P_r$ be ideals of a g-monoid S satisfying $I \subset P_1 \cup \cdots \cup P_r$. Assume that $P_1, \cdots, P_r$ are prime ideals. Then I is not necessarily contained in some P_i.

For example, let $S = \mathbf{Z}_o \oplus \mathbf{Z}_0, M = ((1,0),(0,1)), P_1 = ((1,0))$ and $P_2 = ((0,1))$. Then $M \subset P_1 \cup P_2$ and $M \not\subset P_1, M \not\subset P_2$.

71. Let S be a g-monoid, I an ideal of S, and T an additively closed set in S. If I' is an ideal of S_T, then $(I' \cap S)_T = I'$.

By 71, we have the following,

72. Let S be a Noetherian semigroup and T an additively closed set in S. Then S_T is a Noetherian semigroup.

Let I be an ideal of S. Set $\sqrt{I} = \{s \in S \mid ns \in I \text{ for some } n \in \mathbf{N}\}$. We call $\sqrt{I}$ the radical of I. Let J be an ideal of S such that $J = \sqrt{J}$. Then J is called a radical ideal of S.

Proposition 73. Let $S \subsetneq \mathrm{q}(S)$ be a g-monoid satisfying the ascending chain condition on radical ideals. Then any radical ideal of S is the intersection of a finite number of prime ideals.

Proof. Suppose the contrary. Let $\{J_\lambda \mid \lambda \in \Lambda\}$ be the set of all radical ideals that cannot be expressed as the intersection of a finite number of prime ideals. Then we can take a radical ideal I maximal among J_λ's. Since I is not a prime ideal, we can pick $a, b \in S$ satisfying $a \notin I, b \notin I$ and $a+b \in I$. Set $J = \sqrt{(I,a)}$ and $K = \sqrt{(I,b)}$. By the maximality of I, J and K are intersections of a finite number of prime ideals. We prove that $I = J \cap K$. Take $x \in J \cap K$. Assume that $x \notin I$. Then we can take $m, n \in \mathrm{N}$ such that $mx \in (a)$ and $nx \in (b)$. By the hypothesis, $(m+n)x \in I$. It follows that $x \in I$; a contradiction. Hence $I = J \cap K$ and therefore I is expressible as the intersection of a finite number of prime ideals; a contradiction.

73 implies the following,

74. Let S be a g-monoid satisfying the ascending chain condition on radical ideals, and let I be an ideal of S. Then there are only a finite number of prime ideals minimal over I.

Let S be a g-monoid, and the A_i be S-modules such that $A = A_1 \supsetneqq A_2 \supsetneqq \cdots \supsetneqq A_n$. Then $n-1$ is called the length of the chain. If the supremum of lengths of all chains of S-submodules of A is finite, then A is called to have finite length.

Proposition 75. Let S be a g-monoid. Then the following three conditions are equivalent.

(1) S is a group.

(2) Any finitely generated S-module has finite length.

(3) S as an S-module has finite length.

Proof. (1)$\Longrightarrow$ (2): Let $A = \bigcup_{j=1}^{n}(S+x_j)$ be a finitely generated S-module. Let A_1 be any S-submodule of A. We may assume that $x_1, \cdots, x_i \in A_1$ and $x_{i+1}, \cdots, x_n \notin A_1$. It suffices to prove that $A_1 = \bigcup_{j=1}^{i}(S+x_j)$. Take $a_1 \in A_1$, say $a_1 = s + x_j$. Then $x_j = a_1 - s \in A_1$. Hence $A_1 = \bigcup_{j=1}^{i}(S+x_j)$.

(3)$\Longrightarrow$ (1): Assume that S is not a group, and let M be a maximal ideal of S. Take $x \in M$. Then we can make the chain $S+x \supsetneqq S+2x \supsetneqq \cdots$;

a contradiction.

Let a be an element of S which is not a unit. Assume that $a = b + c$ (for $b, c \in S$) implies that either b or c is a unit of S. Then a is called an irreducible element of S.

Proposition 76. The following conditions are equivalent for a g-monoid S with maximal ideal M.

(1) S is a Noetherian semigroup of dimention $= 1$.

(2) Let I be any ideal of S. Then there exists $n \in \mathbf{N}$ such that the length of any chain of ideals between S and I is less than n.

Proof. (1)$\Longrightarrow$ (2): There exist irreducible elements $x_1, \cdots, x_k$ such that $M = (x_1, \cdots, x_k)$. We may assume that $I \subset M$. Let $M = I_m \supsetneqq \cdots \supsetneqq I_1 \supsetneqq I_0 = I$ be a chain of ideals of length m. There exists a natural number h such that $hx_i \in I$ for every i. Set $l = h^k$. Each ideal I_i is generated by a finite number of elements $a_1, \cdots, a_n$, and each element a_j is of the form $n_1x_1 + \cdots + n_kx_k$ up to a unit of S for $n_i \geq 0$. We note that $hx_i \in I$ for every i. It follows that $m \leq l$.

(2)$\Longrightarrow$ (1): Suppose that $\dim(S) \geq 2$. Then we can take a chain $S \supsetneqq P_1 \supsetneqq P_2$ of prime ideals. Take $x \in P_1 - P_2$. Then we can make a chain $S \supsetneqq (P_2, x) \supsetneqq (P_2, 2x) \supsetneqq \cdots \supsetneqq P_2$; a contradiction. Hence $\dim(S) = 1$. Let M be a maximal ideal in $S, y \in M$ and $I = (y)$. We show that M is finitely generated. If $M \supsetneqq I$, we can take $y_1 \in M - I$ and make $I_1 = (y, y_1)$. If $M \supsetneqq I_1$, we can take $y_2 \in M - I_1$ and make $I_2 = (y, y_1, y_2)$. Continuing this work, we have our result. By Proposition 8, S is a Noetherian semigroup.

77. Let S be a 1-dimentional g-monoid, and let a and c be elements of S. Let J be the set of s in S satisfying $s + na \in (c)$ for some n. Then $(J, a) = S$.

Proof. If a or c is a unit, the assertion holds. Assume that a and c are non-units. Let M be a maximal ideal of S and $I = (c)$. We have $\sqrt{I} = M$ since $\dim(S) = 1$. Then there exists $n \in \mathbf{N}$ such that $na \in I$.

Hence $(J, a) = S$.

78 ([7]). Let S be a 1-dimentional Noetherian semigroup with quotient group G, and T any oversemigroup of S. Then T is again Noetherian and $\dim(T) \leq 1$.

79. Let S be a Noetherian semigroup and I a proper ideal of S. Suppose that there exists $x \in I$ such that $I \subset Z(S/(x))$. Then it is not necessarily true that $I^{-1} \supsetneqq S$.

For example, let $S = \mathbf{Z}_0 \oplus \mathbf{Z}_0$ and $x = (1,1)$. The prime ideals of S are $(y), (z)$ and $M = (y,z)$ (for $y = (1,0), z = (0,1)$). We have $x \in M$ and $M \subset Z(S/(x))$, but $M^{-1} = S$.

80. Let S be an integrally closed Noetherian semigroup, and M a maximal ideal of S. Suppose that $M \subset Z(S/(x))$ for some $x \in M$. Then M is not necessarily principal.

For example, let $S = \mathbf{Z}_0 \oplus \mathbf{Z}_0$ and M the maximal ideal of S. S is an integrally closed Noetherian semigroup. Assume that M is a principal ideal, say $M = (x)$. Put $y = (1,0)$. Then $P = (y)$ is a prime ideal and $P \subsetneqq M$. We can write $y = s + x$ (for $s \in S$). Since y is a prime element, $s \in P$. Write $s = s' + y$ (for $s' \in S$). Then $y = s' + y + x$, that is, $s' + x = 0$. Hence $0 \in M$; a contradiction. Therefore M is not a principal ideal. Let $z = (1,1)$. Then $M = Z(S/(z))$.

Let G be a torsion-free abelian group. A homomorphism of G onto $\mathbf{Z}$ is called a discrete valuation (of rank 1) on G. The valuation semigroup of a discrete valuation (of rank 1) is called a discrete valuation semigroup (of rank 1) (or DVS).

Proposition 81. Let S be a g-monoid which is not a group. Then the following conditions are equivalent.

(1) Every ideal of S is principal.

(2) S is Noetherian, integrally closed and of dimension 1.

(3) S is a DVS.

Proof. (1)$\Longrightarrow$ (2): By Proposition 52, S is a valuation semigroup. Hence S is integrally closed.

(2)$\Longrightarrow$ (3): By [2].

82. Let S be a DVS and M a maximal ideal of S. Then any ideal of S is of the form nM uniquely (for $n \in \mathbf{N}$).

Theorem 83. Let S be a DVS with quotient group G, and $L \supset G$ a torsion-free abelian group with $(L : G) < \infty$. Then the integral closure T of S in L is a DVS.

Proof. By the structure theorem of abelian groups, we can take subgroups $L_0, L_1, \cdots, L_m$ of G with $G = L_0 \subset L_1 \subset \cdots \subset L_m = L$ such that each L_{i+1}/L_i is a cyclic group of prime order. Let T_1 be the integral closure of S in L_1 ,$(L_1 : G) = p$ and v the valuation on G with the valuation semigroup S. Then pl lies in G for any $l \in L_1$. Let $w : L_1 \longrightarrow \mathbf{Z}\frac{1}{p}$ be the map defined by $w(l) = \frac{1}{p}v(pl)$. Then w is a valuation on L_1. Let T_1' be the valuation semigroup of w. It is enough to show that $T_1 = T_1'$. Take $t \in T_1'$. Then $v(pt) \geq 0$ since $w(t) \geq 0$. Hence $pt \in S$ and therefore $t \in T_1$. Take $l \in T_1$, then $nl \in S$ for some $n \in \mathbf{N}$. It follows that $w(nl) \geq 0$, and hence $w(l) \geq 0$. We have proved Theorem 83.

84. Let T be a valuation semigroup with quotient group G_1, let G be any non-zero subgroup of G_1, and set $S = T \cap G$. Then S is a valuation semigroup with quotient group G. The value group of S is in a natural way a subgroup of that of T. If T is a DVS, so is S.

Proposition 85. Let S be a valuation semigroup with quotient group G. Let $L \supset G$ be a torsion-free abelian group which is algebraic over G, and T the integral closure of S in L. Then T is a valuation semigroup.

Proof. Take $u \in L$. There exists $n \in \mathbf{N}$ such that $nu = s_1 - s_2$ for $s_1, s_2 \in S$. Then s_1 divides s_2 or s_2 divides s_1. If s_1 divides s_2, then

$s_2 = s + s_1$ for $s \in S$. It follows that $nu + s + s_1 = s_1$, and hence $n(-u) = s \in S$. Therefore $-u \in T$. If s_2 divides s_1, then $s_1 = s' + s_2$ for $s' \in S$. It follows that $nu = s' \in S$. Hence $u \in T$. By 47, T is a valuation semigroup.

86. Let S be a Noetherian semigroup and P a prime ideal of S. Assume that $x \in P \subset Z(S/(x))$. Then it is not necessarily true that either $\mathrm{ht}(P) = 1$ or S_P is a DVS.

For example, set $S = \mathbf{Z}_0 \oplus \mathbf{Z}_0$. Let P be the maximal ideal of S and let $x = (1, 1)$. Then $x \in P \subset Z(S/(x))$. But $\mathrm{ht}(P) \neq 1$ and S_P is not a DVS.

87. If S is an integrally closed Noetherian semigroup, then $S = \bigcap S_P$, where P ranges over the prime ideals of height 1.

Proof. By [2, Proposition 2].

88. Let a g-monoid S be the intersection $V_1 \cap V_2$ of valuation oversemigroups V_1, V_2 of S. If V_1 and V_2 are not comparable, then S is not a valuation semigroup.

Proof. Suppose that S is a valuation semigroup. By Proposition 54, we have $V_1 = S_{P_1}$ and $V_2 = S_{P_2}$ for some prime ideals P_1, P_2. Then $P_1 \supset P_2$ or $P_1 \subset P_2$. If $P_1 \subset P_2$, then $S_{P_1} \supset S_{P_2}$; a contradiction. If $P_1 \supset P_2$, then $S_{P_1} \subset S_{P_2}$; a contradiction. Therefore S is not a valuation semigroup.

89 (A counter example for ([3, (22.8)])). Let $V_1, \cdots, V_n$ be valuation semigroups on a group G such that $V_i \not\subset V_j$ for $i \neq j$, and let $S = \bigcap V_i$. Then it is not necessarily true that the center of each valuation semigroup V_i on S is a maximal ideal of S.

For example, let H be a torsion-free abelian group, and $G = H \oplus \mathbf{Z}$. Let $<_1$ be the usual order on $\mathbf{Z}$. Define a mapping $v : G \longrightarrow \mathbf{Z}$ by

$v((h,n)) = n$, and let V be the valuation semigroup of v. Put $\Gamma = \mathbf{Z}$ and let $<_2$ be the reverse order on $\mathbf{Z}$. Define a mapping $w : G \longrightarrow \Gamma$ by $w((h,n)) = n$, and let W be the valuation semigroup of w. Then $S = V \cap W = H \oplus \{0\}$ and $S \cap Q = \emptyset$ for the maximal ideal Q of V.

90. Let a g-monoid S be the intersection $V_1 \cap \cdots \cap V_n$, where the V_i's are valuation oversemigroups of S. Then it is not necessarily true that each V_i is expressible as the form S_{P_i} for some prime ideal P_i of S.

For example, let S be a 2-dimensional integrally closed Noetherian semigroup. Let M be the maximal ideal of S. Suppose that $P_1, \cdots, P_n$ be all the prime ideals of height 1 in S. Then $V_i = S_{P_i}$ is a discrete valuation oversemigroup of S, and $S = \bigcap_i V_i$ by 87. On the other hand, there exists a valuation oversemigroup W of S such that $Q \cap S = M$ for the maximal ideals Q of W ([5, Lemma 9]). Then $S = W \cap V_1 \cap \cdots \cap V_n$. If $W = S$, then W is a DVS. Hence $\dim(W) = 1$; a contradiction.

91. Let a, b be non-units in a 1-dimentional g-monoid S. Then na is divisible by b for some $n \in \mathbf{N}$.

Proof. Let M be the maximal ideal of S, and $I = (b)$. Then $\sqrt{I} = M$, for $\dim(S) = 1$. There exists $n \in \mathbf{N}$ such that $na \in I$. Hence $na = s + b$ (for $s \in S$).

Proposition 92. Let S be a g-monoid satisfying $S = T_1 \cap T_2$, where the T's are oversemigroups of S. Let Q_1, Q_2 be maximal ideals of T_1, T_2 respectively, and set $P_i = Q_i \cap S$. Assume further that P_1 and P_2 are incomparable, and each T_i is 1-dimensional. Then $T_i = S_{P_i}$ for $i = 1, 2$.

Proof. We take an element t that lies in P_2 but not in P_1. Let $x \in T_1$, and write $x = y - z$ (for $y, z \in T_2$). If z is a unit in T_2, then $x \in S$, that is, $x \in S_{P_1}$. If z is non-unit in T_2, then there exists $n \in \mathbf{N}$ such that z divides nt by 91. Write $nt = z + z_1$ (for $z_1 \in T_2$), then $x + nt = y + z_1$. Since $nt + x \in T_1$, we have $nt + x \in S$, that is, $x \in S_{P_1}$. Take $a \in S_{P_1}$, say $a = s - p$ (for $s \in S, p \in S - P_1$). Then $p \notin Q_1$ for $p \notin P_1$. Hence

$-p \in T_1$, that is, $a \in T_1$ and therefore $T_1 = S_{P_1}$. Similarly $T_2 = S_{P_2}$.

Let $S = \bigcap T_i$, where each T_i is an oversemigroup of S. Let N_i be the maximal ideal of T_i. We say that this representation is locally finite if any element of S lies in only a finite number of the N_i's.

Proposition 93. Let a g-monoid S be a locally finite intersection $\bigcap T_i$ of 1-dimensional oversemigroups of S. Let Q_i be the maximal ideal of T_i, and $P_i = Q_i \cap S$. Let N be a prime ideal in S. Then $N \supset P_i$ for some i.

Proof. Assume the contrary. Let x be an element of N and $P_1, \cdots, P_r$ be the finite number of P_i's containing x. Pick u_j in P_j but not in N (for $j = 1, \cdots, r$). Since T_j is 1-dimensional, we have $n_j u_j = t_j + x$ (for $t_j \in T_j$, and for $n_j \in \mathrm{N}$). Let $u = n_1 u_1 + \cdots + n_r u_r$ and $a = t_1 + \cdots + t_r + (r-1)x$. Then $u = a + x$. By the construction, $a \in T_1 \cap \cdots \cap T_r$. Let $T_k \notin \{T_1, \cdots, T_r\}$. Then $a = u - x \in T_k$. It follows that $a \in S$, and hence $u \in N$. Therefore $u_i \in N$ for some i; a contradiction.

Proposition 94. Suppose in addition to the hypothesis of Proposition 93, that an additively closed set Y of S with $S_Y \subsetneqq \mathrm{q}(S) = G$ is given. Then S_Y is a locally finite intersection of the T_i's that contain S_Y.

Proof. Suppose that $S_Y \not\subset T_i$ for each i. Let M_i be the maximal ideal of W_i. Take $x \in G$. Let $W_1, \cdots, W_k$ be the finite number of T_i's not containing x. Since $S_Y \not\subset T_i$, we can take $y_i \in Y$ which is a non-unit in T_i. Let $I_i = (W_i - x) \cap W_i$. Then I_i is an ideal of W_i. Since W_i is 1-dimensional, $\sqrt{I_i} = W_i$ or $= M_i$. Then there exists $n_i \in \mathrm{N}$ such that $n_i y_i \in I_i \subset W_i - x$. Hence $n_i y_i + x \in W_i$. Then $\Sigma n_j y_j + x$ lies in each W_i and in other T_j's. Hence $\Sigma n_j y_j + x \in S$. Then $x \in S_Y$, that is, $G = S_Y$, a contradiction. Therefore $S_Y \subset T_i$ for some i. Let us use the subscript j for a typical T_j containing S_Y. To prove $S_Y = \bigcap T_j$ we take $x \in \bigcap T_j$ and have to prove $x \in S_Y$. Let $W_1, \cdots, W_r$ be the finite number of T_i's not containing x. Then there exists $y_k \in Y$ with $-y_k \notin W_k$. By 91, $n_k y_k + x \in W_k$ for some n_k. Then $\Sigma n_k y_k + x \in S$ and so $x \in S_Y$. The representation $S_Y = \bigcap T_j$ is again locally finite.

Let S be a g-monoid and V a valuation oversemigroup of S. If $V = S_P$ for some prime ideal P, then V is called essential.

95. Let a g-monoid S be a locally finite intersection of 1-dimensional essential valuation oversemigroups of S, and assume that $\dim(S) = 1$. Then S is one of the $V_i's$.

Let P be a prime ideal of a g-monoid S. If P contains no prime ideal without P, then P is called a minimal prime ideal.

Proposition 96. Let a g-monoid S be a locally finite intersection of 1-dimensional essential valuation oversemigroups of S. Let N be a minimal prime ideal of S. Then S_N is one of the V_i's.

Proof. By Proposition 94, S_N is a locally finite intersection of the V_i's that contain S_N. By 95, S_N is one of the V_i's.

Let V be a valuation semigroup. If the value group of V is isomorphic to a subgroup of the additive group of rational numbers, then V is called rational.

Proposition 97. Suppose, in addition to the hypothesis of Proposition 93, that each V_i is a rational valuation oversemigroup of S. Then $S = \bigcap V_j$, where the intersection is taken over those V_i's that have the form S_N, N a minimal prime ideal of S.

Proof. If V_i has the form S_N, N a maximal ideal of S, let us call the V_i e-type. If V_j is not of e-type, let us call the V_j i-type. We show that one i-type component can be deleted. Let W be i-type, Q a maximal ideal of W and $P = Q \cap S$. If P is a minimal prime ideal, then S_P is one of the V_i's by Proposition 96. Let $S_P = V_i$. Then $V_i \subset W$. Hence we can delete W. So we may assume that $\mathrm{ht}(P) \geq 2$. Then there exists a prime ideal P' such that $P \supsetneqq P'$. By Proposition 94, $P' \supset P_k$ (for

$P_k = Q_k \cap S, Q_k$ is a maximal ideal of V_k). Take any $y \in P_k$. Suppose that W can not be deleted. Then we can take an element x that lies in every V_i but not in W. Let U_W be a group of units of W and $G = \mathrm{q}(S)$. Since W is rational, we can take $m, n \in \mathbf{Z}$ such that $m\bar{x} + n\bar{y} = \bar{0}$ for $\bar{x}, \bar{y} \in G/U_W$. Then $z = mx + ny$ is a unit of W. Since $x \in V_k$ and $y \in Q_k$, we have $z \in Q_k$. On the other hand, z lies in S. Thus z is a unit of W and non-unit of V_k. This contradicts the inclusion $P_k \subset P$. Hence we can delete W if it is i-type. Suppose that u lies in every e-type V_i. We show that $u \in S$. By the locally finiteness, u lies in all but a finite number of the V_i's. The components which do not contain u are i-type. Hence $u \in S$.

An ideal in a g-monoid S is called primary, if $I \neq S$ and if $x + y \in I$ implies either $x \in I$ or $ny \in I$ for some $n \in \mathbf{N}$. Let I be a primary ideal of S. Then $\sqrt{I}$ is the smallest prime ideal containing I. If $P = \sqrt{I}$, then I is called a P-primary ideal.

98. Let S be a Noetherian semigroup with maximal ideal M, and let I be an M-primary ideal. Then there exists $n \in \mathbf{N}$ such that the length of any chain of ideals between I and M is less than n.

Proof. There exist irreducible elements $x_1, \cdots, x_k$ such that $M = (x_1, \cdots, x_k)$. We may assume that $I \subset M$. Let $M = I_m \supsetneqq \cdots \supsetneqq I_1 \supsetneqq I_0 = I$ be a chain of ideals of length m. There exists a natural number h such that $hx_i \in I$ for every i. Set $l = h^k$. Each ideal I_i is generated by a finite number of elements $a_1, \cdots, a_n$, and each element a_j is of the form $n_1x_1 + \cdots + n_kx_k$ up to a unit of S for $n_i \geq 0$. We note that $hx_i \in I$ for every i. It follows that $m \leq l$.

Theorem 99. Let S be a Notherian semigroup, a a non-unit in S, and P a minimal prime ideal over (a). Then $\mathrm{ht}(P) = 1$.

Proof. We may assume that P is a maximal ideal in S. Suppose that there exists a prime ideal P_1 which is properly contained in P. Since P is the only prime ideal which contains (a), (a) is a P-primary ideal. Evidently $P \supset (a, P_1) \supset (a, 2P_1) \supset \cdots$ and each (a, iP_1) contains (a). By

99, there exists $n \in \mathrm{N}$ such that $(a, nP_1) = (a, (n+1)P_1) = \cdots$. Hence $mP_1 \subset (a, (m+1)P_1) \cap mP_1 \subset ((a) \cap mP_1, (m+1)P_1)$ for any $m \geq n$. Since mP_1 is a P_1-primary ideal and $a \notin P_1$, we have $(a) \cap mP_1 = a + mP_1$. Then $mP_1 \subset (a + mP_1, (m+1)P_1) \subset (P + mP_1, (m+1)P_1)$. By 68, $mP_1 = (m+1)P_1$. On the other hand, $\bigcap iP_1 = \emptyset$ by Proposition 66; a contradiction. Therefore $\mathrm{ht}(P) = 1$.

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