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Joint Spectra of *-Hyponormal Operators on Uniformly c-Convex Spaces

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1. Introduction

In the case of operators on Hilbert spaces, in [30] F.-H. Vasilescu characterized the (Taylor) joint spectrum for a commuting pair and in [16] R. Curto did it for a commuting n-tuple.

For those on Banach spaces, we have not yet such a characterzation. In [24] A. McIntosh, A. Pryde and W. Ricker characterized the joint spectrum for a strongly commuting n-tuple of operators. In [7] M. Chō proved that the joint spectrum for such an n-tuple is the joint approximate point spectrum of it. And in [11] he characterized the joint spectrum for a doubly commuting n-tuple of strongly hyponormal operators on a uniformly smooth space.

In this paper, we will characterize the joint spectrum of a doubly commuting n-tuple of strongly *-hyponormal operators on a uniformly c-convex Banach space.

Let B(X) be the algebra of all bounded linear operators on X. Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a commuting n-tuple of (bounded and linear) operators on X.

Let E^n be the complex exterior algebra on n-generators $e_1, \ldots, e_n$, with product $\wedge$; E^n is graded: $E^n = \bigoplus_{k=-\infty}^{\infty} E_k^n$, where $E_k^n \wedge E_\ell^n \subset E_{k+\ell}^n$ and $\{ e_{j_1} \wedge \cdots \wedge e_{j_k} : 1 \le j_1 < \cdots < j_k \le n \}$ is a basis for E_k^n $(k \ge 1)$, while $E_0^n \cong \mathbb{C}$ and $E_k^n = \{0\}$ for $k < 0$ and $k > n$. Let $E_k^n(X) = X \otimes E_k^n$ and define $D_k^{(n)}: E_k^n(X) \longrightarrow E_{k-1}^n(X)$ by

$$D_k^{(n)}(x \otimes e_{j_1} \wedge \cdots \wedge e_{j_k}) = \sum_{i=1}^{k} (-1)^{i+1} T_{j_i} x \otimes e_{j_1} \wedge \cdots \wedge \check{e}_{j_i} \wedge \cdots \wedge e_{j_k}$$

when $k > 0$ (here $\vee$ means deletion), and $D_k^{(n)} = 0$ when $k \le 0$ and $k > n$. A straightforward computation shows that $D_k^{(n)} \circ D_{k+1}^{(n)} = 0$ for all k, so that $\{ E_k^n(X), D_k^{(n)} \}_{k \in \mathbb{Z}}$ is a chain complex, called the Koszul complex for $\mathbf{T} = (T_1, \ldots, T_n)$ and denoted by $E(X, \mathbf{T})$.

We define $\mathbf{T} = (T_1, \ldots, T_n)$ to be non-singular in case its associated Koszul complex is exact, that is, $\mathrm{Ker}(D_k^{(n)}) = R(D_{k+1}^{(n)})$ for all k. The (Taylor) joint spectrum $\sigma(\mathbf{T})$ of $\mathbf{T}$ is the set of $z \in \mathbb{C}^n$ such that $\mathbf{T} - z = (T_1 - z_1, \ldots, T_n - z_n)$ is singular. A point $z \in \mathbb{C}^n$ is in the joint approximate point spectrum $\sigma_\pi(\mathbf{T})$ of $\mathbf{T} = (T_1, \ldots, T_n)$ if there exists a sequence $\{x_k\}$ of unit vectors in X such that

$$\|(T_i - z_i)x_k\| \longrightarrow 0 \quad \text{as } k \longrightarrow \infty \quad \text{for } i = 1, \ldots, n.$$

A point $z \in \mathbb{C}^n$ is in the joint point spectrum $\sigma_p(\mathbf{T})$ of $\mathbf{T} =$

$(T_1, \ldots, T_n)$ if there exists a non-zero vector x in X such that

$$T_i x = z_i x \quad \text{for} \quad i = 1, \ldots, n.$$

For an operator $T \in B(X)$, the usual spectrum, the approximate point spectrum and the point spectrum of T are denoted by $\sigma(T)$, $\sigma_\pi(T)$ and $\sigma_p(T)$, respectively.

We denote by X^* the dual space of X. Let us set

$$\pi = \{(x,f) \in X \times X^* : \|f\| = f(x) = \|x\| = 1 \}.$$

The numerical range $V(B(X),T)$ and the spatial numerical range $V(T)$ of T are defined by

$$V(B(X),T) = \{F(T) : F \in B(X)^* \text{ and } \|F\| = F(I) = 1 \}$$

and

$$V(T) = \{ f(Tx) : (x,f) \in \pi \},$$

respectively. Then the following results are well-known for $T \in B(X)$:

(1) $\mathrm{co}\,\sigma(T) \subset \overline{V(T)}$ and $\overline{\mathrm{co}}\, V(T) = V(B(X),T)$, where $\mathrm{co}\, E$, $\overline{E}$ and $\overline{\mathrm{co}}\, E$ are the convex hull, the closure and the closed convex hull of E, respectively.

(2) $V(T) \subset V(T^*) \subset \overline{V(T)}$, where T^* is the dual operator of T.

If $V(T) \subset \mathbb{R}$, then T is called hermitian. Hence, T is hermitian iff T^* is hermitian. An operator $T \in B(X)$ is called hyponormal if there are hermitian operators H and K such that $T = H + iK$ and the commutator $C = i(HK - KH) \geq 0$, meaning that $V(C) \subset \mathbb{R}^+ = \{ a \in \mathbb{R} : a \geq 0 \}$. For an operator $T = H + iK$, we denote the operator $H - iK$ by $\bar{T}$. An operator $T = H + iK$ is called *-hyponormal if the inequality

$$\|e^{z\bar{T}}e^{-\bar{z}T}\| \leq 1$$

holds for all complex numbers z.

Remark 1. If T is a *-hyponormal operator on X, then $T - \lambda I$ is *-hyponormal for every $\lambda \in \mathbb{C}$ and $\bar{T}^*$ is also a *-hyponormal operator on X^*.

Normal operators are obviously *-hyponormal. By Proposition 1 in [23], *-hyponormal operators are hyponormal. In particular, subnormal operators on a Hilbert space are *-hyponormal. An example of a hyponormal operator which is not *-hyponormal is shown in [23].

A *-hyponormal operator $T = H + iK$ is called strongly *-hyponormal if H^2 and K^2 are both hermitian.

Remark 2. There is an hermitian operator H such that H^2 is not hermitian. However, if H is hermitian then

$$V(H^2) \subset \{ z \in \mathbb{C} : \text{Re } z \geq 0 \}.$$

Therefore, if T is strongly *-hyponormal, then

$$\sigma(\overline{T}T) \subset \overline{V(\overline{T}T)} \subset \mathbb{R}^+.$$

An n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ is called a doubly commuting n-tuple if $\mathbf{T}$ is a commuting n-tupule and there exist hermitians H_j and K_j such that $T_j = H_j + iK_j$ $(j=1,\ldots,n)$ and $\overline{T}_j T_k = T_k \overline{T}_j$ for $j \neq k$. For a commuting n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ with $T_j = H_j + iK_j$ $(j=1,\ldots,n)$, it is easy to see that $\mathbf{T}$ is a doubly commuting n-tuple iff H_j and K_j commute with H_k and K_k for $j \neq k$, respectively.

A Banach space X will be said to be uniformly c-convex if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|y\| < \varepsilon$ whenever $\|x\| = 1$ and $\|x + \lambda y\| \leq 1 + \delta$ for all complex numbers λ with $|\lambda| \leq 1$.

The space $\mathcal{L}^1(S, \Sigma, \mu)$ and the trace class C_1 are uniformly c-convex. All uniformly convex spaces are uniformly c-convex.

See, for example K. Mattila [22], for details.

We now give an example of a doubly commuting pair of strongly *-hyponormal operators on a uniformly c-convex space. Let $\mathcal{H}$ be a complex Hilbert space. Let C_p be the Schatten p-class for $1 \leq p < \infty$. Then it is well-known that C_p is uniformly c-convex. When A and B^* are *-hyponormal operators on $\mathcal{H}$, the operator $\delta_{A,B}(T) = AT - TB$ $(T \in C_p)$ is a

*-hyponormal operator on $\mathcal{C}_p$ ($1 \le p < \infty$) by Theorem 4 in [23]. And by Corollary 1.3 in [26], it holds that $V(B(\mathcal{C}_p), \delta_{A,B}) = \overline{W(A)} - \overline{W(B)}$, where $W(T) = \{(Tx,x): x \in \mathcal{H} \text{ and } \|x\| = 1\}$. Let

$$L_A(T) = AT \quad (T \in \mathcal{C}_p).$$

When $A = H + iK$ is *-hyponormal operator on $\mathcal{H}$, it is clear that $L_A = L_H + iL_K$ is a strongly *-hyponormal operator on $\mathcal{C}_p$ ($1 \le p < \infty$). Hence, if $\mathbf{A} = (A_1,\ldots,A_n)$ is a doubly commuting n-tuple of strongly *-hyponormal operators on $\mathcal{H}$, then $L_{\mathbf{A}} = (L_{A_1},\ldots,L_{A_n})$ is a doubly commuting n-tuple of strongly *-hyponormal operators on $\mathcal{C}_p$ ($1 \le p < \infty$).

For a commuting n-tuple $\mathbf{T} = (T_1,\ldots,T_n)$ such that $T_j = H_j + iK_j$ $(j=1,\ldots,n)$, a point $z = (z_1,\ldots,z_n) \in \mathbb{C}^n$ is in the complete star spectrum $\sigma_{cs}(\mathbf{T})$ of $\mathbf{T}$ if there is some partition $\{j_1,\ldots,j_k\} \cup \{\ell_1,\ldots,\ell_m\} = \{1,\ldots,n\}$ such that

$$\sum_{\mu=1}^{k} \overline{(T_{j_\mu} - z_{j_\mu})}(T_{j_\mu} - z_{j_\mu}) + \sum_{\nu=1}^{m} (T_{\ell_\nu} - z_{\ell_\nu})\overline{(T_{\ell_\nu} - z_{\ell_\nu})}$$

is not invertible. In particular, the set

$$\{(z_1,\ldots,z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} (T_j - z_j)\overline{(T_j - z_j)}' \text{ is not invertible } \}$$

is called the right spectrum of $\mathbf{T}$ and denoted by $\sigma_r(\mathbf{T})$. Then it is clear that $\sigma_\pi(\mathbf{T}) \subset \sigma_{cs}(\mathbf{T}) \cap \sigma(\mathbf{T})$.

We recall from [1] and [2] the construction of a larger space X^o from a given Banach space X. Then the mapping $T \longrightarrow T^o$ is an isometric isomorphism of $B(X)$ onto a closed subalgebra of $B(X^o)$. Let Lim be a fixed Banach limit on the space of all bounded sequences of complex numbers with the norm $\|\{\lambda_n\}\| = \sup\{|\lambda_n| : n \in \mathbb{N}\}$. Let $\tilde{X}$ be the space of all bounded sequences $\{x_n\}$ of X. Let N be the subspace of $\tilde{X}$ consisting of all bounded sequences $\{x_n\}$ with $\text{Lim}\,\|x_n\|^2 = 0$. The space X^o is defined as the completion of the quotient space $\tilde{X} / N$ with respect to the norm $\|\{x_n\} + N\| = (\text{Lim}\,\|x_n\|^2)^{1/2}$. For an operator $T \in B(X)$, the corresponding operator T^o on X^o is defined by $T^o(\{x_n\} + N) = \{Tx_n\} + N$. Then the following results hold for $T \in B(X)$:

$$\sigma(T) = \sigma(T^o),\ \sigma_\pi(T) = \sigma_\pi(T^o) = \sigma_p(T^o) \text{ and } \overline{\text{co}}\, V(T) = V(T^o).$$

Hence, H is non-negative and hermitian iff H^o is non-negative and hermitian, respectively.

See [1] and [2] for details.

2. Characterization

First we will prepare some results.

Theorem A ([29], Theorem 4.8). Let $\mathbf{T} = (T_1,\ldots,T_n)$ be a commuting n-tuple of operators and f an m-tuple of polynomials in n-variables. Then

$$\sigma(f(\mathbf{T})) = f(\sigma(\mathbf{T})).$$

Theorem B ([15], Theorem 1 and [27], Theorem 3.4). Let $\mathbf{T} = (T_1,\dots,T_n)$ be a commuting n-tuple of operators and f an m-tuple of polynomials in n-variables. Then

$$\sigma_\pi(f(\mathbf{T})) = f(\sigma_\pi(\mathbf{T})).$$

Theorem C ([22], Theorem 2.5). Let X be uniformly c-convex and let H be a hermitian, non-negative operator on X. If there are sequences $\{x_n\} \subset X$ and $\{f_n\} \subset X^*$ such that $\|x_n\| = \|f_n\| = 1$ for each n with $f_n(x_n) \longrightarrow 1$ and $f_n(Hx_n) \longrightarrow 0$, then $Hx_n \longrightarrow 0$.

Theorem D ([28], Theorem 3.6). Let $\mathbf{T} = (T_1,\dots,T_n)$ be a commuting n-tuple of operators on a Banach space X. Then $\sigma(\mathbf{T}) = \sigma(\mathbf{T}^*)$, where $\mathbf{T}^* = (T_1^*,\dots,T_n^*)$.

Theorem E ([12], Theorem 6.6). Let $\mathbf{T} = (T_1,\dots,T_n)$ be a doubly commuting n-tuple of operators on a complex Banach space X. Then $\sigma(\mathbf{T}) \subset \sigma_{cs}(\mathbf{T})$.

Lemma 1. Let $T = H + iK$ be a *-hyponormal operator on a Banach space X. If $\{x_n\}$ is a bounded sequence in X such that $Tx_n \longrightarrow 0$, then $Hx_n \longrightarrow 0$ and $Kx_n \longrightarrow 0$.

Proof. Consider the larger space X^o and the corresponding operator $T^o = H^o + iK^o$. Then T^o is a *-hyponormal operator on X^o. And since $T^o(\{x_n\} + N) = 0$, by Theorem 3 in [23] it follows that $H^o(\{x_n\} + N) = K^o(\{x_n\} + N) = 0$. Therefore, it follows that $\operatorname{Lim} \|Hx_n\|^2 = 0$ and $\operatorname{Lim} \|Kx_n\|^2 = 0$. If the sequence $\{\|Hx_n\|\}$ does not converge to zero, there exist a number $\varepsilon > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|Hx_{n_k}\|^2 \geq \varepsilon$ for all $k \in \mathbb{N}$. Then $\operatorname{Lim} \|Hx_{n_k}\|^2 \geq \varepsilon$. Hence we have $H^o(\{x_{n_k}\} + N) \neq 0$. However, since $Tx_{n_k} \longrightarrow 0$, it follows that $T^o(\{x_{n_k}\} + N) = 0$. Also by Theorem 3 in [23] we have $H^o(\{x_{n_k}\} + N) = 0$. It is a contradiction. $Kx_n \longrightarrow 0$ is proved analogously.

Theorem 2. Let X be uniformly c-convex. Let $T = H + iK$ be a *-hyponormal operator on X. Then $\sigma(T) = \{ z : \overline{z} \in \sigma_\pi(\overline{T}) \}$.

Proof. Since $T - z$ is also *-hyponormal for every $z \in \mathbb{C}$, we may only prove that $0 \in \sigma(T)$ iff $0 \in \sigma_\pi(\overline{T})$. Let 0 be in $\sigma_\pi(\overline{T})$. Then we have $0 \in \sigma(T\overline{T}) = \sigma(\overline{T}^*T^*)$. Since $\operatorname{Re} \sigma(\overline{T}^*T^*) \subset \mathbb{R}^+$, there exists a sequence $\{f_n\}$ of unit vectors in X^* such that $\overline{T}^*T^*f_n \longrightarrow 0$. Since $\overline{T}^*$ is *-hyponormal, from Lemma 1 we have that $T^{*2}f_n \longrightarrow 0$ and also $0 \in \sigma(T^*) = \sigma(T)$. Conversely, Let 0 be in $\sigma(T)$. Since either $\overline{T}T$ or $T\overline{T}$ is not invertible, by Theorem C we may assume that $T\overline{T}$ is not invertible. Since T is *-hyponormal, by Lemma 1 we have that there exists a sequence $\{x_n\}$ of unit vectors in X such that $T^2x_n \longrightarrow 0$. Hence, by the spectral mapping theorem for a approximate point spectrum, we have $0 \in \sigma_\pi(\overline{T})$.

Lemma 3. Let X be uniformly c-convex. Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a doubly commuting n-tuple of strongly *-hyponormal operators on X. If $\sum_{j=1}^{k} \overline{T}_j T_j + \sum_{j=k+1}^{n} T_j \overline{T}_j$ is not invertible $(1 \leq k \leq n)$, then $\sum_{j=1}^{n} T_j \overline{T}_j$ is not invertible.

Proof. Put $\mathbf{S} = (\overline{T}_1 T_1, \ldots, \overline{T}_k T_k, T_{k+1} \overline{T}_{k+1}, \ldots, T_n \overline{T}_n)$. Then $\mathbf{S}$ is a commuting n-tuple. Since by Remark 2

$$\sigma(\overline{T}_j T_j) \cup \sigma(T_j \overline{T}_j) \subset \mathbb{R}^+ \quad (j=1, \ldots, n),$$

by Theorem A it follows that

$$\sigma\Big(\sum_{j=1}^{k} \overline{T}_j T_j + \sum_{j=k+1}^{n} T_j \overline{T}_j\Big) \subset \mathbb{R}^+.$$

Hence, 0 is in the approximate point spectrum of $\sum_{j=1}^{k} \overline{T}_j T_j + \sum_{j=k+1}^{n} T_j \overline{T}_j$. By Theorem B, it holds that $0 = (0, \ldots, 0) \in \sigma_\pi(\mathbf{S})$. Therefore, there exists a sequence $\{x_\ell\}$ of unit vectors in X such that $\overline{T}_j T_j x_\ell \longrightarrow 0$ $(j=1, \ldots, k)$ and $T_j \overline{T}_j x_\ell \longrightarrow 0$ $(j=k+1, \ldots, n)$. When $T_j = H_j + iK_j$, we put that $C_j = i(H_j K_j - K_j H_j) \geq 0$ for $j = 1, \ldots, k$. Choose a linear functional $f_\ell \in X^*$ such that $\|f_\ell\| = f_\ell(x_\ell) = 1$ for each ℓ. Since then $f_\ell((H_j^2 + K_j^2)x_\ell) \geq 0$ $f_\ell(C_j x_\ell) \geq 0$ and $f_\ell(\overline{T}_j T_j x_\ell) = f_\ell((H_j^2 + K_j^2)x_\ell) + f_\ell(C_j x_\ell) \longrightarrow 0$ for $j = 1, \ldots, k$, it follows by Theorem C that

$$(H_j^2 + K_j^2)x_\ell \longrightarrow 0 \quad \text{and} \quad C_j x_\ell \longrightarrow 0 \quad \text{for } j = 1, \ldots, k.$$

Therefore, it follows that $T_j\overline{T}_j x_\ell = (H_j^2 + K_j^2 - C_j)x_\ell \longrightarrow 0$ for $j = 1,\ldots,n$.

Theorem 4. Let X be uniformly c-convex. Let $\mathbf{T} = (T_1,\ldots,T_n)$ be a doubly commuting n-tuple of strongly *-hyponormal operators on X. Then

$$\sigma_{cs}(\mathbf{T}) = \sigma_r(\mathbf{T}) = \{(z_1,\ldots,z_n) \in \mathbb{C}^n : (\overline{z}_1,\ldots,\overline{z}_n) \in \sigma_\pi(\overline{\mathbf{T}})\},$$

where $\overline{\mathbf{T}} = (\overline{T}_1,\ldots,\overline{T}_n)$.

Proof. It is clear that

$$\{(z_1,\ldots,z_n) \in \mathbb{C}^n : (\overline{z}_1,\ldots,\overline{z}_n) \in \sigma_\pi(\overline{\mathbf{T}})\} \subset \sigma_r(\mathbf{T}) \subset \sigma_{cs}(\mathbf{T}).$$

Assume that $\alpha = (\alpha_1,\ldots,\alpha_n)$ is in $\sigma_{cs}(\mathbf{T})$. Since $\mathbf{T} - z = (T_1 - z_1,\ldots,T_n - z_n)$ is a doubly commuting n-tuple of strongly *-hyponormal operators for every $z \in \mathbb{C}^n$, so we may assume that $\alpha = 0$. By Lemma 3, it follows that $\sum_{j=1}^n T_j\overline{T}_j$ is not invertible and there exists a sequence $\{x_\ell\}$ of unit vectors in X such that

$$T_j\overline{T}_j x_\ell \longrightarrow 0 \quad \text{for} \quad j = 1,\ldots,n.$$

And by Lemma 1 it follows that $\overline{T}_j^2 x_\ell \longrightarrow 0$ for $j = 1,\ldots,n$. Consider a function $g(z) := (z_1^2,\ldots,z_n^2)$ for $z = (z_1,\ldots,z_n) \in \mathbb{C}^n$ Applying the spectral mapping theorem, Theorem B, for g, we have $g(\sigma_\pi(\overline{\mathbf{T}})) = \sigma_\pi(g(\overline{\mathbf{T}}))$. Hence, we have $0 \in \sigma_\pi(\overline{\mathbf{T}})$.

Theorem 5. Let X be uniformly c-convex. Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a doubly commuting n-tuple of strongly *-hyponormal operators on X. Then

$$\sigma(\mathbf{T}) = \sigma_{cs}(\mathbf{T}) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : (\bar{z}_1, \ldots, \bar{z}_n) \in \sigma_\pi(\bar{\mathbf{T}}) \},$$

where $\bar{\mathbf{T}} = (\bar{T}_1, \ldots, \bar{T}_n)$.

Proof. By Theorems E and 4, we may only prove that if $0 \in \sigma_\pi(\bar{\mathbf{T}})$, then $0 \in \sigma(\mathbf{T})$. So assume that $0 \in \sigma_\pi(\bar{\mathbf{T}})$. Then there exists a sequence $\{x_k\}$ of unit vectors in X such that

$$\bar{T}_j x_k \longrightarrow 0 \quad \text{for} \quad j = 1, \ldots, n.$$

Therefore, from $0 \in \sigma(\sum_{j=1}^{n} T_j \bar{T}_j)$, it follows that

$$0 \in \sigma((\sum_{j=1}^{n} T_j \bar{T}_j)^*) = \sigma(\sum_{j=1}^{n} \bar{T}_j^* T_j^*).$$

Also $(\bar{T}_1^*, \ldots, \bar{T}_n^*)$ is a doubly commuting n-tuple of strongly *-hyponormal operators on X^*. From the proof of Lemma 3 there exists a sequence $\{f_k\}$ of norm one functionals in X^* such that

$$\bar{T}_j^* T_j^* \, f_k \longrightarrow 0 \qquad \text{for } j = 1, \ldots, n.$$

Since $\bar{T}_j^*$ is a *-hyponormal operator, by Lemma 1 it follows that $T_j^{*2} f_k \longrightarrow 0 \quad (j = 1, \ldots, n)$. Hence, by the proof of Theorem 4, we have $0 \in \sigma_\pi(\mathbf{T}^*)$, where $\mathbf{T}^* = (T_1^*, \ldots, T_n^*)$.

Therefore, by Theorem D, it follows that $0 \in \sigma(\mathbf{T})$.

Theorem 6. Let $\mathbf{T} = (T_1,\dots,T_n)$ be a doubly commuting n-tuple of strongly *-hyponormal operators on a Banach space X such that $T_j = H_j + iK_j$ $(j=1,\dots,n)$. If $\lambda - i\mu = (\lambda_1 - i\mu_1,\dots,\lambda_n - i\mu_n) \in \sigma_\pi(\mathbf{T})$, then $\lambda = (\lambda_1,\dots,\lambda_n) \in \sigma(\mathbf{H})$ and $\mu = (\mu_1,\dots,\mu_n) \in \sigma(\mathbf{K})$, where $\mathbf{H} = (H_1,\dots,H_n)$ and $\mathbf{K} = (K_1,\dots,K_n)$.

Proof. We will prove the theorem by the method of induction. For n = 1, let $T = H + iK$ be a strongly *-hyponormal operator on X and let $\lambda - i\mu \in \sigma_\pi(\overline{T})$. Then, from the first part of the proof of Theorem 2, it follows that $\lambda + i\mu \in \sigma(T)$. Hence we can choose a real number μ' such that $\lambda + i\mu'$ is in the boundary of $\sigma(T)$. So there exists a sequence $\{x_k\}$ of unit vectors in X such that $(T - (\lambda + i\mu'))x_k \longrightarrow 0$. By Lemma 1 it holds that $(H - \lambda)x_k \longrightarrow 0$. $\mu \in \sigma(K)$ is proved analogously.

Next we assume that the theorem is true for (n-1)-tuples. Consider the larger space X^o of X and the mapping $T \longrightarrow T^o$. Since $\lambda - i\mu \in \sigma_p(\overline{\mathbf{T}}^o)$ for $\overline{\mathbf{T}}^o = (\overline{T}_1^o,\dots,\overline{T}_n^o)$, there exists a non-zero vector x^o in X^o such that

$$\overline{T}_j^o\, x^o = (\lambda_j - i\mu_j)x^o \quad \text{for } j = 1,\dots,n.$$

Let $Y = \{\, y^o : \overline{T}_n^o\, y^o = (\lambda_n - i\mu_n)y^o \,\}$. Since $\mathbf{T}$ is a doubly commuting n-tuple, Y is invariant for every H_j^o and K_j^o $(j=1,\dots,n-1)$. Hence it follows that $T_{j|Y}^o = H_{j|Y}^o + iK_{j|Y}^o$ $(j=1,\dots,n-1)$ and $\mathbf{T}' = (T_{1|Y}^o,\dots,T_{n-1|Y}^o)$ is a doubly commuting (n-1)-tuple of strongly *-hyponormal operators on Y. Since x^o is in Y, we have $(\lambda_1 - i\mu_1,\dots,\lambda_{n-1} - i\mu_{n-1}) \in \sigma_p(\overline{\mathbf{T}}')$. So by the assumption of the induction, it follows that

$(\lambda_1,\dots,\lambda_{n-1}) \in \sigma(\mathbf{H}')$ and $(\mu_1,\dots,\mu_{n-1}) \in \sigma(\mathbf{K}')$,

where $\mathbf{H}' = (H^o_1|_Y,\dots,H^o_{n-1}|_Y)$ and $\mathbf{K}' = (K^o_1|_Y,\dots,K^o_{n-1}|_Y)$. Hence, by Theorem 2.1 in [7] and Theorem 6.2 in [12], there exists a non-zero vector $z^o \in Y$ such that $H^o_j z^o = \lambda_j z^o$ for $j = 1,\dots,n-1$. Of course, it holds that $T^o_n z^o = (\lambda_n - i\mu_n)z^o$. Next let $Z = \{ w^o : H^o_j w^o = \lambda_j w^o$ for $j=1,\dots,n-1\}$. It then also follows that $T^o_n|_Z$ is a strongly *-hyponormal operator on Z and $\lambda_n - i\mu_n \in \sigma_p(\bar{T}^o_n|_Z)$. Thus there exists a non-zero vector u^o in Z such that $H^o_n u^o = \lambda_n u^o$. Therefore, we have $(\lambda_1,\dots,\lambda_n) \in \sigma(\mathbf{H})$. $(\mu_1,\dots,\mu_n) \in \sigma(\mathbf{K})$ is proved analogously.

By theorems 5 and 6 we have the following

Corollary 7. Let X be uniformly c-convex. Let $\mathbf{T} = (T_1,\dots,T_n)$ be a doubly commuting n-tuple of *-hyponormal operators on X such that $T_j = H_j + iK_j$ $(j=1,\dots,n)$. If $\lambda + i\mu = (\lambda_1 + i\mu_1,\dots,\lambda_n + i\mu_n) \in \sigma(\mathbf{T})$, then $\lambda \in \sigma(\mathbf{H})$ and $\mu \in \sigma(\mathbf{K})$, where $\mathbf{H} = (H_1,\dots,H_n)$ and $\mathbf{K} = (K_1,\dots,K_n)$.

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