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NOTES ON THE LAPLACE-BELTRAMI OPERATOR ON A FOLIATED RIEMANNIAN MANIFOLD WITH A BUNDLE-LIKE METRIC

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1. Introduction

Let $(M, g, \mathcal{F})$ be a $p+q$ dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a Riemannian metric g which is bundle-like with respect to $\mathcal{F}$ ([7], [8]). Let $\Lambda^t(M)$ (resp. $\Lambda^{r,s}(M)$) be the space of all t-forms (resp. (r,s)-forms) on M . Then we have a decomposition (*): $\Lambda^t(M) = \Sigma_{r+s=t} \Lambda^{r,s}(M)$ and a projection $\pi_{r,s} : \Lambda^t(M) \longrightarrow \Lambda^{r,s}(M)$ ($r+s = t$). Let d be the exterior derivative and δ be the formal adjoint operator of d . Then the decomposition (*) implies the following decompositions: $d = d' + d'' + d'''$ and $\delta = \delta' + \delta'' + \delta'''$ ([4], [8], [11], [12]).

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An operator $\square = \delta d + d\delta$ acting on $\Lambda^t(M)$ is called the Laplace-Beltrami operator. Moreover, we can consider two operators: $\square' = \delta' d' + d'\delta'$ and $\square'' = \delta'' d'' + d''\delta''$.

First, we consider three operators $\square$, $\square'$ and $\square''$ acting on $C^\infty(M) = \Lambda^0(M)$, that is, $\square = \delta d$, $\square' = \delta' d'$ and $\square'' = \delta'' d''$. It is trivial that $\square = \square' + \square''$. Next, we consider the operators $\square$ and $\square''$ acting on $\Delta^1(M) = \{ \varphi \in \Lambda^{0,1}(M) \mid d'\varphi = 0 \}$. Then we have that $\square = (\delta''' d'' + d'\delta'') + (\delta'' d'' + d''\delta'') = (\delta''' d'' + d'\delta'') + \square''$.

Let H be the mean curvature vector field of $\mathcal{F}$ that is a vector field on M ([8], [9], [10], [13]).

The purpose of this note is as follows:

(i) To show the decompositions of the operator $\square''$ (Theorems A and B below) and to prove those statements (sections 3 and 5).

(ii) To show concrete forms of the decompositions of the Laplace-Beltrami operators on foliated Lie groups (Examples 2 and 3 in section 4).

(iii) To give an application of Theorems A and B — the non-existence of harmonic basic 1-forms (Theorem C in section 6).

Theorem A. <u>Let</u> $(M,g,\mathcal{F})$ <u>be a</u> p+q <u>dimensional Riemannian manifold with a foliation</u> $\mathcal{F}$ <u>of codimension</u> q <u>and a bundle-like metric</u> g <u>with respect to</u> $\mathcal{F}$. <u>Then the Laplace-Beltrami operator</u> $\square$ <u>acting on</u> $C^\infty(M)$ <u>has a decomposition</u>:

$$\square = \square' + \square''_0 + H \quad .$$

Moreover, if $\mathcal{F}$ is minimal (or, totally geodesic) then $\square = \square' + \square''_0$.

In the above theorem, if $\mathcal{F}$ is regular and $p : M \longrightarrow B = M/\mathcal{F}$ is a Riemannian submersion, then it holds that $\square''_0(u \circ p) = (\square_B u) \circ p$ for any $u \in C^\infty(B)$, where $\circ$ denotes the composition of mappings and $\square_B$ is the Laplace-Beltrami operator acting on $C^\infty(B)$. The definition of $\square''_0$ is precisely given in section 3.

Remark 1. We can consider that the operator $\square'' = \square''_0 + H$ is the normal part ([5]) or the radial part ([1], [2]) of the Laplace-Beltrami operator $\square$ acting on $C^\infty(M)$ (see Example 1 in section 3).

Theorem B. Let $(M, g, \mathcal{F})$ be as Theorem A. Then the Laplace-Beltrami operator $\square$ acting on $\Delta^1(M)$ has a decomposition:

$$\square = (\delta''' d'' + d' \delta'') + \square''_0 + \pi_{0,1} \circ L_H \ ,$$

where L_H denotes the Lie differentiation with respect to H .

The definition of $\square''_0$ is precisely given in section 5.

Remark 2. It holds that $L_H\varphi \in \Lambda^{1,0}(M) + \Lambda^{0,1}(M)$ for any $\varphi \in \Delta^1(M)$. Thus we have that $(\pi_{0,1} \circ L_H)\varphi = \pi_{0,1}(L_H\varphi) \in \Lambda^{0,1}(M)$.

Remark 3. If H is an infinitesimal automorphism of $\mathcal{F}$ ([3]), then $L_H\varphi \in \Delta^s(M)$ and $\delta''\varphi \in \Delta^{s-1}(M)$ for any $\varphi \in \Delta^1(M)$. This fact was pointed out by the referee.

Remark 4. Let $(M,g,\mathcal{F})$ be as Theorem A. If $\mathcal{F}$ is a Clairaut foliation, then H is an infinitesimal automorphism of $\mathcal{F}$ ([13, Propositions 6.1 and 6.2]). Thus the operator $\square$ acting on $\Delta^1(M)$ has a decomposition: $\square = (\delta'''d'') + \square''_O + L_H$.

Remark 5. For the Laplace-Beltrami operator $\square$ acting on $\Delta^s(M)$ ($s \geq 2$), we have

$$\square = (\delta'''d'' + d'\delta'' + d''\delta''') + (d'\delta''') + \square''_O + \pi_{0,s} \circ L_H + d'''\delta''' \quad .$$

This decomposition was given by J. H. Park[6] too.

We shall be in C^∞-category. Manifolds are connected and orientable, and foliations are transversally orientable ([10]). We agree on the following ranges of indices: $1 \leq i, j, k, \cdots \leq p$, $p+1 \leq \alpha, \beta, \gamma, \cdots \leq p+q$ unless otherwise stated. The authors thanks the referee for his suggestions.

2. Foliated manifold

Let $(M,g,\mathcal{F})$ be a $p+q$ dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a Riemannian metric g which is bundle-like with respect to $\mathcal{F}$. Let $\{ U, (x^i,x^\alpha) \}$ be a flat coordinate neighborhood system, that is, in U, the foliation $\mathcal{F}$ is defined by $dx^\alpha = 0$ ([7], [8]). Let $\{ X_i, X_\alpha \}$ be the basic adapted frame to $\mathcal{F}$ and $\{ \theta^i, \theta^\alpha \}$ be the dual frame to $\{ X_i, X_\alpha \}$ ([13]). Here we notice that X_i is tangent to the leaves of $\mathcal{F}$ in U and $g(X_i,X_\alpha) = 0$ ([13]). We set that $g_{ij} = g(X_i,X_j)$ and $g_{\alpha\beta} = g(X_\alpha,X_\beta)$. Then the metric g is locally expressed in the form: $g|_U = \Sigma_{ij}\, g_{ij}(x^k,x^\gamma)\, \theta^i\cdot\theta^j + \Sigma_{\alpha\beta}\, g_{\alpha\beta}(x^\gamma)\, \theta^\alpha\cdot\theta^\beta$ ([7]).

If a form φ on M has a local expression:

$$\varphi|_U = \frac{1}{r!s!} \sum_{\substack{i_1\cdots i_r \\ \alpha_1\cdots\alpha_s}} \varphi_{i_1\cdots i_r\alpha_1\cdots\alpha_s}(x^k,x^\gamma)\, \theta^{i_1}\wedge\cdots\wedge\theta^{i_r}\wedge\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_s} ,$$

then we call φ an (r,s)-form. Hereafter we omit " $|_U$ " for simplicity. Let $\Lambda^t(M)$ (resp. $\Lambda^{r,s}(M)$) be the space of all t-forms (resp. (r,s)-forms) on M ([11], [12]). Then the following decomposition holds:

$$\Lambda^t(M) = \Sigma_{r+s=t}\, \Lambda^{r,s}(M) .$$

Then, for each r and s satisfying $r+s = t$, we have a projection $\pi_{r,s} : \Lambda^t(M) \longrightarrow \Lambda^{r,s}(M)$. The above decomposition induces decompositions of the exterior derivative d and the its formal adjoint operator δ :

$$d = d' + d'' + d''' \quad \text{and} \quad \delta = \delta' + \delta'' + \delta'''$$

([4], [8], [11], [12]). We notice that $d'' : \Lambda^{r,s}(M) \longrightarrow \Lambda^{r,s+1}(M)$ and $\delta'' = \varepsilon * d'' *$, where $\varepsilon = \pm 1$ and $*$ denotes the Hodge star operator ([4], [11]).

An operator $\square = d\delta + \delta d$ acting on $\Lambda^{t}(M)$ is elliptic. But two operators $\square' = d'\delta' + \delta'd'$ and $\square'' = d''\delta'' + \delta''d''$ acting on $\Lambda^{r,s}(M)$ are not elliptic ([11]), nevertheless both $\square'$ and $\square''$ are interesting operators which are studyed by many people. The operator $\square$ acting on $\Lambda^{t}(M)$ is called the Laplace-Beltrami operator. If $\varphi \in \Lambda^{t}(M)$ satisfies $\square\varphi = 0$ then we call φ a harmonic t-form.

Let $C^{\infty}(M)$ be the space of all functions on M , that is, $C^{\infty}(M) = \Lambda^{0,0}(M) = \Lambda^{0}(M)$. Let $\Delta^{s}(M)$ be the space of all basic s-forms on M , that is, $\Delta^{s}(M) = \{ \varphi \in \Lambda^{0,s}(M) \mid d'\varphi = 0 \}$. The three operators $\square$, $\square'$ and $\square''$ acting on $C^{\infty}(M)$ is given by

$$\square = \delta d \quad , \quad \square' = \delta' d' \quad , \quad \square'' = \delta'' d'' \quad ,$$

and we have that $\square = \square' + \square''$ on $C^{\infty}(M)$. Next, the operator $\square$ acting on $\Delta^{1}(M)$ has a decomposition:

$$\square = (\delta''' d'' + d' \delta'') + \square'' \quad .$$

Here, for any $\varphi \in \Delta^1(M)$, $(\delta''' d'' + d'\delta'')\varphi \in \Lambda^{1,0}(M)$ and $\square''\varphi = (\delta'' d'' + d''\delta'')\varphi \in \Lambda^{0,1}(M)$. We notice that $\square'\varphi = 0$ for any $\varphi \in \Delta^s(M)$.

Now we introduce the mean curvature vector field H of $\mathcal{F}$, that is, H is a vector field on M which has a local expression:

$$H = \Sigma_\alpha H^\alpha X_\alpha \quad ; \quad H^\alpha = \Sigma_\beta g^{\alpha\beta} g(\Sigma_{ij} g^{ij}(\nabla_{X_i} X_j), X_\beta) ,$$

where ∇ denotes the Levi-Civita connection with respect to g , and the restriction of H to a leaf $\mathcal{L}$ of $\mathcal{F}$ is the mean curvature vector field on the submanifold $\mathcal{L}$ of M ([8], [9], [10]). If $H = 0$, then $\mathcal{F}$ is called minimal, that is, all leaves of $\mathcal{F}$ are minimal submanifolds of M ([8], [9]).

3. Proof of Theorem A

For any $f \in C^\infty(M)$, we have

$$\begin{aligned}
\square f = & - \Sigma_{ij} g^{ij} (X_i(df(X_j))) + \Sigma_{ij} g^{ij} df(\nabla_{X_i} X_j) \\
& - \Sigma_{\alpha\beta} g^{\alpha\beta} (X_\alpha(df(X_\beta))) + \Sigma_{\alpha\beta} g^{\alpha\beta} df(\nabla_{X_\alpha} X_\beta) \\
= & - \Sigma_{ij} g^{ij} (X_i(d'f(X_j))) + \Sigma_{ij} g^{ij} d'f((\nabla_{X_i} X_j)_T) \\
& + \Sigma_{ij} g^{ij} d''f((\nabla_{X_i} X_j)_N) \\
& - \Sigma_{\alpha\beta} g^{\alpha\beta} (X_\alpha(d''f(X_\beta))) + \Sigma_{\alpha\beta} g^{\alpha\beta} d'f((\nabla_{X_\alpha} X_\beta)_T)
\end{aligned}$$

$$+ \Sigma_{\alpha\beta}\ g^{\alpha\beta}\ d''f(\ (\nabla_{X_\alpha} X_\beta)_N\ ) \quad ,$$

where $(\ )_T$ (resp. $(\ )_N$) denotes the component of $(\ )$ tangent (resp. normal) to the leaves of $\mathcal{F}$. For example, $(\nabla_{X_i} X_j)_T = \Sigma_k \Gamma^k_{ij} X_k$ and $(\nabla_{X_i} X_j)_N = \Sigma_\gamma \Gamma^\gamma_{ij} X_\gamma$. On the other hand, $\square' f$ and $\square'' f$ are given by

$$\begin{aligned}
\square' f &= \delta' d' f \\
&= \delta d' f \\
&= - \Sigma_{ij}\ g^{ij}\ (X_i(\ d'f(X_j)\ )) + \Sigma_{ij}\ g^{ij}\ d'f(\ (\nabla_{X_i} X_j)_T\ ) \\
\square'' f &= \delta'' d'' f \\
&= \delta d'' f \\
&= - \Sigma_{\alpha\beta}\ g^{\alpha\beta}\ (X_\alpha(\ d''f(X_\beta)\ )) + \Sigma_{\alpha\beta}\ g^{\alpha\beta}\ d''f(\ (\nabla_{X_\alpha} X_\beta)_N\ ) \ . \\
&\quad + \Sigma_{ij}\ g^{ij}\ d''f(\ (\nabla_{X_i} X_j)_N\ ) \\
&= \square''_O f + Hf \quad .
\end{aligned}$$

Here $\square''_O$ is an operator given by

$$\square''_O f = - \Sigma_{\alpha\beta}\ g^{\alpha\beta}\ (X_\alpha(\ d''f(X_\beta)\ )) + \Sigma_{\alpha\beta}\ g^{\alpha\beta}\ d''f(\ (\nabla_{X_\alpha} X_\beta)_N\ ) \ .$$

Since the metric g is bundle-like with respect to $\mathcal{F}$, we have that $\Sigma_{\alpha\beta}\ g^{\alpha\beta}\ (\nabla_{X_\alpha} X_\beta)_T = 0$ ([13, Lemma 5.1]). Therefore, we have that $\square f = \square' f + \square''_O f + Hf$.

Example 1. Let $O(n)$ be the orthogonal group acting on the Euclidean space (R^n, g_0) . Then we have a foliated Riemannian manifold $(M, g, \mathcal{F})$, where $M = R^n - \{\text{the origin}\}$, $g = g_0|_M$ and each leaf of $\mathcal{F}$ is an orbit of $O(n)$. It is clear that g is bundle-like with respect to $\mathcal{F}$. By direct calculation (using the polar coordinates on R^n), we have

$$(\#) \qquad \square_0'' + H = -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} \quad .$$

According to S. Helgason[1, 2], the right hand side of (#) is the radial part of the Laplace-Beltrami operator L_{R^n} on R^n ([2, p.266]), that is, $\Delta(L_{R^n}) = -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r}$. Here we notice that our definition of the Laplace-Beltrami operator has the opposite sign from that in [1, 2].

4. Concrete form of the decomposition

Our discussion in this section is due to [9].

Let G be a $p+q$ dimensional Lie group and $\mathfrak{g}$ be the associated Lie algebra consisting of all vector fields on G that are invariant under left translations. We take a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, then we have a foliated manifold $(G, \mathcal{F}(\mathfrak{h}))$ as follows: We denote by L_x the left translation of G by $x \in G$. Let H be a connected subgroup of G whose Lie algebra is $\mathfrak{h}$. Regard a submanifold $L_x(H)$ as a leaf through x , we have a foliation $\mathcal{F}(\mathfrak{h})$ on G . If we take a left invariant metric $< , >$ on G , we have a foliated Riemannian manifold $(G, \langle\ ,\ \rangle, \mathcal{F}(\mathfrak{h}))$. We assume that the foliation $\mathcal{F}(\mathfrak{h})$ is of codimension q .

Let $\{ e_i, e_\alpha \}$ be an orthonormal adapted frame field on $(G, \langle\ ,\ \rangle, \mathcal{F}(\mathfrak{h}))$, that is, $\{ e_i, e_\alpha \}$ is an orthonormal basis for $\mathfrak{g}$ such that $\{e_i\}$ is a basis for $\mathfrak{h}$. Let C_{AB}^D be the structure constants of $\mathfrak{g}$ with respect to $\{e_A\}$, that is, $[e_A, e_B] = \Sigma_D C_{AB}^D e_D$.

Now, let $\mathfrak{g}$ be a non-compact simple Lie algebra and τ be an involutive automorphism of $\mathfrak{g}$. We set $\mathfrak{k} = \{ X \in \mathfrak{g} ; \tau(X) = X \}$, $\mathfrak{p} = \{ X \in \mathfrak{g} ; \tau(X) = - X \}$, then it holds that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. The Killing form B of $\mathfrak{g}$ induces a left invariant metric $(\ ,\ )$ on G , that is, $(X,Y) = - B(X, \tau(Y))$, and, for any $X \in \mathfrak{p}$, $ad(X)$ is a symmetric linear transformation of $\mathfrak{g}$ with respect to the metric. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{a}^*$ be the dual space of $\mathfrak{a}$. For $\lambda \in \mathfrak{a}^*$, we set $\mathfrak{g}_\lambda = \{ X \in \mathfrak{g} ; [A,X] = \lambda(A)X \text{ for } A \in \mathfrak{a} \}$. If $\mathfrak{g}_\lambda \neq 0$ then λ is called a root, and let Δ be the set of all roots. Then we have

$$\mathfrak{g} = \Sigma_{\lambda\in\Delta}\, \mathfrak{g}_\lambda , \quad \mathfrak{a} \subset \mathfrak{g}_0 , \quad [\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu} \quad \text{for } \lambda, \mu \in \Delta .$$

We take an ordering in $\mathfrak{a}^*$. We denote by Δ^+ the set of positive roots.

We take two subspaces Δ_1 and Δ_2 of Δ^+ satisfying (i) $\Delta_1 \supset \Delta_2$, (ii) $\lambda, \mu \in \Delta_r$, $\lambda+\mu \in \Delta^+$ implies $\lambda+\mu \in \Delta_r$ ($r = 1, 2$) , and we also take two subspaces $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of $\mathfrak{a}$ such that $\mathfrak{a}_1 \supsetneqq \mathfrak{a}_2$. We set

$$\mathfrak{n}_r = \mathfrak{a}_r + \Sigma_{\lambda \in \Delta_r} \mathfrak{g}_\lambda \quad (r = 1, 2) ,$$

then $\mathfrak{n}_1$ is an algebra and $\mathfrak{n}_2$ is a subalgebra of $\mathfrak{n}_1$. Let N_1 (resp. N_2) be a connected Lie subgroup of G with the Lie algebra $\mathfrak{n}_1$ (resp. $\mathfrak{n}_2$). Thus we have a foliated manifold $(N_1, \mathcal{F}(\mathfrak{n}_2))$. Here we use the following ranges of indices:

$$1 \le a, b \le \dim \mathfrak{a}_2 , \quad \dim \mathfrak{a}_2 + 1 \le i, j \le \dim N_2$$
$$\dim N_2 + 1 \le \alpha, \beta \le \dim N_2 + \dim \mathfrak{a}_1 - \dim \mathfrak{a}_2$$
$$\dim N_2 + \dim \mathfrak{a}_1 - \dim \mathfrak{a}_2 + 1 \le \xi, \eta \le \dim N_1 .$$

We set

$\{ e_a, e_i \}$: basis for $\mathfrak{n}_2$, $\{ e_a, e_\alpha \}$: basis for $\mathfrak{a}_1$,
$\{ e_i, e_\xi \}$: basis for $\Sigma_{\lambda \in \Delta_1} \mathfrak{g}_\lambda$,

where we may take e_i (resp. e_ξ) in the root space $\mathfrak{g}_{\lambda_i}$ (resp. $\mathfrak{g}_{\lambda_\xi}$) , and it may happen that $\lambda_s = \lambda_t$ for $s \neq t$. When we take a left invariant metric $\langle \, , \rangle$ on N_1 so that $\{ e_a, e_i, e_\alpha, e_\xi \}$ is orthonormal, we have a foliated Riemannian manifold $(N_1, \langle \, , \rangle, \mathcal{F}(\mathfrak{n}_2))$. Then it holds

$$H^\alpha = \Sigma_i \lambda_i(e_\alpha) , \quad H^\xi = 0 .$$

Thus we have

Lemma([9]). <u>Let</u> $(N_1, \langle \, , \rangle, \mathcal{F}(\mathfrak{n}_2))$ <u>be as above</u>. <u>Then the metric</u> $\langle \, , \rangle$ <u>is bundle-like with respect to</u> $\mathcal{F}(\mathfrak{n}_2)$ <u>if and only if</u> $\lambda_\xi(e_a) = 0$ <u>and</u> $\lambda_i + \lambda_\xi \notin \Delta_1 \setminus \Delta_2$ <u>for all</u> a, i, <u>and</u> ξ .

Example 2. We consider the following case: $\Delta_1 = \Delta_2 = \Delta^+$, $\mathfrak{a}_1 = \mathfrak{a}$, $\mathfrak{a}_2 = \{0\}$. Then $(N_1, \langle\ ,\ \rangle, \mathcal{F}(\mathfrak{n}_2))$ is a foliated Riemannian manifold whose metric $\langle\ ,\ \rangle$ is bundle-like with respect to $\mathcal{F}(\mathfrak{n}_2)$. We have

$$H = \Sigma_\alpha \left(\Sigma_{\lambda \in \Delta^+} \lambda(e_\alpha) \right) e_\alpha .$$

Thus the Laplace-Beltrami operator $\square$ acting on $C^\infty(N_1)$ has a following decomposition:

$$(\#\#) \qquad \square = - \Sigma_i\, e_i \cdot e_i - \Sigma_\alpha\, e_\alpha \cdot e_\alpha + \Sigma_\alpha \left(\Sigma_{\lambda \in \Delta^+} \lambda(e_\alpha) \right) e_\alpha .$$

where $(\Sigma_i\, e_i \cdot e_i) f = \Sigma_i\, e_i(\, e_i(f)\,)$.

In [2, Proposition 3.8, p.267], we can find an expression corresponding to the second and third terms of the right hand of (##). Here we have to notice a formula (49) in [2, p.265].

Example 3. We consider the following case: $\mathfrak{g} = \mathfrak{sl}(4,\mathbf{R})$, $\theta(X) = - {}^tX$ ($X \in \mathfrak{g}$) ([9, Example 4.2]). In this case, we have

$$\mathfrak{k} = \mathfrak{so}(4) \quad , \quad \mathfrak{p} = \{ X \in \mathfrak{gl}(4,\mathbf{R}) \; ; \; {}^tX = X \; , \; \mathrm{Tr}(X) = 0 \} \quad ,$$

$$\mathfrak{a} = \left(\begin{bmatrix} H_1 & & 0 \\ & \ddots & \\ 0 & & H_4 \end{bmatrix} ; H_1 + H_2 + H_3 + H_4 = 0 \right) ,$$

$$\lambda_i \in \mathfrak{a}^* : \quad \lambda_i\left(\begin{bmatrix} H_1 & & 0 \\ & \ddots & \\ 0 & & H_4 \end{bmatrix} \right) = H_i ,$$

$$\Delta = \{ \lambda_i - \lambda_j \ ; \ 1 \le i, \ j \le 4 \} \ ; \quad \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 ,$$

$$\Delta^+ = \{ \lambda_i - \lambda_j \ ; \ 1 \le i < j \le 4 \} ,$$

$$\Delta_1 = \{ \lambda_1 - \lambda_2, \lambda_1 - \lambda_3, \lambda_1 - \lambda_4, \lambda_2 - \lambda_3, \lambda_2 - \lambda_4 \} ,$$

$$\Delta_2 = \{ \lambda_1 - \lambda_2, \lambda_1 - \lambda_3, \lambda_1 - \lambda_4, \lambda_2 - \lambda_3 \} ,$$

$$\mathfrak{a}_1 = \left(a \cdot \begin{bmatrix} 2 & & & 0 \\ & -2 & & \\ & & 1 & \\ 0 & & & -1 \end{bmatrix} \;\middle|\; a \in \mathbf{R} \right) , \quad \mathfrak{a}_2 = \{0\} ,$$

$$e_\alpha : \begin{bmatrix} 2 & & & 0 \\ & -2 & & \\ & & 1 & \\ 0 & & & -1 \end{bmatrix} , \quad e_i : E_{12} , E_{13} , E_{14} , E_{23} ,$$

$$e_\xi : E_{24} .$$

Here E_{ab} denotes a square matrix with entry 1 where the a-th row and b-th column meet, all other entries being 0 . Then we have a foliated Riemannian manifold $(N_1, \langle \ , \ \rangle, \mathcal{F}(\mathfrak{n}_2))$ and, by Lemma, $\langle \ , \ \rangle$ is bundle-like with respect to $\mathcal{F}(\mathfrak{n}_2)$. We have that $H = 5 \cdot e_\alpha$. Thus the Laplace-Beltrami operator $\square$ acting on $C^\infty(N_1)$ has a following decomposition:

$$\square = - \Sigma_i \, e_i \cdot e_i - e_\alpha \cdot e_\alpha + 5 \cdot e_\alpha .$$

5. Proof of Theorem B

For any $\varphi \in \Delta^1(M)$, we have

$$\delta'' d'' \varphi(X_\gamma)$$
$$= \delta d'' \varphi(X_\gamma)$$
$$= - \Sigma_{ij}\ g^{ij} (\nabla_{X_i} d''\varphi)(X_j, X_\gamma) - \Sigma_{\alpha\beta}\ g^{\alpha\beta} (\nabla_{X_\alpha} d''\varphi)(X_\beta, X_\gamma)$$
$$= d''\varphi(H, X_\gamma) - \Sigma_{\alpha\beta}\ g^{\alpha\beta} (\nabla_{X_\alpha} d''\varphi)(X_\beta, X_\gamma)$$

and

$$d'' \delta'' \varphi(X_\gamma)$$
$$= d \delta'' \varphi(X_\gamma)$$
$$= X_\gamma(\delta''\varphi)$$
$$= X_\gamma(\delta\varphi)$$
$$= - X_\gamma\{\Sigma_{ij}\ g^{ij} (\nabla_{X_i} \varphi)(X_j) + \Sigma_{\alpha\beta}\ g^{\alpha\beta} (\nabla_{X_\alpha} \varphi)(X_\beta)\}$$
$$= X_\gamma(\varphi(H)) - X_\gamma\{\Sigma_{\alpha\beta}\ g^{\alpha\beta} (\nabla_{X_\alpha} \varphi)(X_\beta)\} \quad .$$

Thus we have

$$\begin{aligned} \square''\varphi(X_\gamma) &= \delta'' d''\varphi(X_\gamma) + d''\delta''\varphi(X_\gamma) \\ &= (L_H\varphi)(X_\gamma) - \Sigma_{\alpha\beta}\ g^{\alpha\beta} (\nabla_{X_\alpha} d''\varphi)(X_\beta, X_\gamma) \\ &\qquad - X_\gamma\{\Sigma_{\alpha\beta}\ g^{\alpha\beta} (\nabla_{X_\alpha} \varphi)(X_\beta)\} \quad . \end{aligned}$$

If we set

$$\square''_O \varphi(X_\gamma)$$
$$= - \Sigma_{\alpha\beta}\ g^{\alpha\beta} (\nabla_{X_\alpha} d''\varphi)(X_\beta, X_\gamma) - X_\gamma\{\Sigma_{\alpha\beta}\ g^{\alpha\beta} (\nabla_{X_\alpha} \varphi)(X_\beta)\} \quad ,$$

then we have that $\square''\varphi(X_\gamma) = (L_H\varphi)(X_\gamma) + \square''_O\varphi(X_\gamma)$, which completes the proof of Theorem B.

We notice that if $\mathcal{F}$ is regular, $p : M \longrightarrow B = M/\mathcal{F}$ is Riemannian submersion, and $\varphi \in \Delta^1(M)$ is given by $\varphi = p^*\psi$ for $\psi \in \Lambda^1(B)$, then $\square''_O\varphi = p^*\square_B\psi$ ([6]).

6. Non-existence of harmonic basic 1-forms

Let $(M,g,\mathcal{F})$ be as section 2. Let Q be the normal bundle of $\mathcal{F}$ and $\pi : \Gamma(TM) \longrightarrow \Gamma(Q)$ be the natural projection, where TM is the tangent bundle over M and $\Gamma(\ )$ denotes the set of all sections of a bundle ([3], [10]). The metric g induces a metric g_Q on Q ([3], [10]). Then we notice that $g_Q(\pi(X_\alpha),\pi(X_\beta)) = g(X_\alpha,X_\beta) = g_{\alpha\beta}$ and $(g_Q)^{\alpha\beta} = g^{\alpha\beta}$ ([14]). We denote by D the transverse Riemannian connection on Q , and let ρ_D be the Ricci operator of $\mathcal{F}$, that is, $\rho_D(\pi(X_\gamma)) = \Sigma_{\alpha\beta}\, g^{\alpha\beta} R_D(\pi(X_\gamma),\pi(X_\alpha))\pi(X_\beta)$, where R_D is the curvature of D ([3], [10]). We notice that $D_{X_\alpha}\pi(X_\beta) = \pi(\nabla_{X_\alpha}X_\beta)$ and $D_{X_i}\pi(X_\beta) = 0$ ([3], [10], [14]). The Ricci operator ρ_D of $\mathcal{F}$ is non-negative (resp. positive) at a point x of M if $g_Q(\rho_D(\nu),\nu)_x \geq 0$ (resp. > 0) for any $\nu \in \Gamma(Q)$ satisfying $\nu(x) \neq 0$.

Theorem C. Let $(M,g,\mathcal{F})$ be as Theorem A. Suppose that M is compact and without boundary. If $\mathcal{F}$ is minimal and the Ricci operator ρ_D of $\mathcal{F}$ is non-negative everywhere and *positive for at least one point of* M *, then every harmonic basic* 1-*form on* M *vanishes identically*.

We set that $\rho^N(X_\gamma) = \Sigma_{\alpha\beta}\, g^{\alpha\beta}\Big[\Big(\nabla_{X_\gamma}\big(\nabla_{X_\alpha}X_\beta\big)_N\Big)_N - \Big(\nabla_{X_\alpha}\big(\nabla_{X_\gamma}X_\beta\big)_N\Big)_N - \Big(\nabla_{(\nabla_{X_\gamma}X_\alpha)_N}X_\beta\Big)_N + \Big(\nabla_{(\nabla_{X_\alpha}X_\gamma)_N}X_\beta\Big)_N\Big]$. Since it holds that $(\nabla_{X_\alpha}X_\beta)_N = (\nabla_{X_\beta}X_\alpha)_N$ ([12]) and $\pi((\nabla_{X_\alpha}X_\beta)_N) = D_{X_\alpha}\pi(X_\beta)$, we have that $\pi(\rho^N(X_\gamma)) = \rho_D(\pi(X_\gamma))$.

Let $<\ ,\ >$ be the local scalar product on $\Lambda^{r,s}(M)$, and let φ be a basic 1-form on M , that is, $\varphi \in \Delta^1(M)$. We have, by Theorems A and B,

$$
\begin{aligned}
< \Box''\varphi,\varphi > &= \Sigma_{\gamma\tau}\, g^{\gamma\tau}\, \Box''\varphi(X_\gamma)\cdot\varphi(X_\tau) \\
&= \Sigma_{\gamma\tau}\, g^{\gamma\tau}\, (L_H\varphi)(X_\gamma)\cdot\varphi(X_\tau) - H(\tfrac{1}{2}< \varphi,\varphi >) \\
&\quad + \Box(\tfrac{1}{2}< \varphi,\varphi >) \\
&\quad + \Sigma_{\alpha\beta\gamma\tau}\, g^{\alpha\beta}\, g^{\gamma\tau}\, (\nabla_{X_\alpha}\varphi)(X_\gamma)\cdot(\nabla_{X_\beta}\varphi)(X_\tau) \\
&\quad + \Sigma_{\gamma\tau}\, g^{\gamma\tau}\, \varphi(\rho^N(X_\gamma))\cdot\varphi(X_\tau) \quad .
\end{aligned}
$$

Here we notice that $\Box'(\frac{1}{2}< \varphi,\varphi >) = 0$. Since $\varphi \in \Delta^1(M)$ and $\Box\varphi = 0$, we have that $\Box''\varphi = 0$. And we have that $H = 0$ because $\mathcal{F}$ is minimal. By the condition for ρ_D , we have that $\Sigma_{\gamma\tau}\, g^{\gamma\tau}\, \varphi(\rho^N(X_\gamma))\cdot\varphi(X_\tau) \geq 0$. Thus, by the standard method, we can complete the proof of Theorem C.

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