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On the Deformation Theory of Pseudo Hermitian CR-structures Which Preserve the Webster's Scalar Curvature (An Approach to the Local Moduli Theory for Strongly Pseudo Convex Domains)

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Introduction

Recently, for compact projective manifolds which satisfy a certain kind of property, as a generalization of the Teichmüller space to the higher dimensional case, the global moduli theory has been developed by several author (Siu,Fujiki and Schumacher (Si), (Fu-Sc)). On the other hand, for open manifolds, nothing has been known. And last thirty years, several similarities between project algebraic spaces and strongly pseudo convex spaces have been shown. Therefore it seems natural to try to construct a global moduli space for strongly pseudo convex space.

Let X be a strongly pseudo convex space and let r be a C^∞ function on X, which is strictly pluri-subharmonic except a compact set. Let

$$\Omega = \{ x : x \text{ in } X,\ r(x)<0 \}$$

and let $b\Omega$ be its boundary. Then over $b\Omega$, a CR-structure from X is induced. Namely, let

$${}^{0}T'' = \mathbb{C}\otimes T(b\Omega) \cap T''X|_{b\Omega} .$$

Then the pair $(M, {}^{0}T'')$ is called a CR-structure. I should explain why we study such an abstract object. Because,our Ω might have singularities. And furthermore Ω is open (not compact), so these cause several troubles in analysis. However very fortunately, the CR-structure over the boundary $b\Omega$, detrermines Ω almostly (for example, see Rossi's theorem). And furthermore $(M, {}^{0}T'')$ and Ω have the similar property in analysis (if Ω has no singularity). And technically $(M, {}^{0}T'')$ can be handled much easier than Ω. Nothing to say, the CR-structure itself is interested. But from the our stand point,we are always considering Ω.

As you know, for a strongly pseudo convex space,as for local theory, for the first time, a versal family <u>in the sense of Kuranishi</u> is constructed by (A4) under several assumptions from the point of view of CR-structures. Later, by (B-K), by a complete different method without any assumption,the existence of the versal family is shown. Therefore nowadays, it is not necessary to use CR-structures in the local theory. However, in the global theory, in the compact complex manifold case, so called real analysis method is essentially used and indispensable. Therefore we might hope that the CR-structure method could revive in the global moduli theory of strongly pseudo convex spaces.

The first difficulty for constructing a global moduli space is that: the parameter space of the local versal family may not be a local moduli space. Of course this phenomenon appears in the compact manifold case. But for a compact complex manifold X, if we assume:

$$H^0(X,\theta_X) = 0 \quad ,$$

where θ_X means the holomorphic tangent bundle, then the local versal family must be the moduli family. Contrast to the complex manifold case, we cannot expect such a theorem in the CR-case. Namely our standpoint is that: we are always considering ambient open complex spaces. In this sense, if we are given a family of CR-structututres $(M, {}^{\phi(t)}T'')$, t in T, and t_1, t_2 in T, satisfying:

$(V_{t_1}, A_{t_1}) \simeq (V_{t_2}, A_{t_2})$ as a germ of singularities A_{t_1} and A_{t_2},

where (V_{t_1}, A_{t_1}) is a normal stein space determined by $(M, {}^{\phi(t)}T'')$, and (V_{t_2}, A_{t_2}) is a normal stein space determined by $(M, {}^{\phi(t_2)}T'')$, we should regard

$$(M, {}^{\phi(t_1)}T'') \quad \text{and} \quad (M, {}^{\phi(t_2)}T'')$$

as the same point in the moduli space. However, this equivalence is hardly handled. Because even if $(V_{t_1}, A_{t_1}) \simeq (V_{t_2}, A_{t_2})$ as a germ of singularities, we cannot say anything about CR-structures. For example, let $M_{a,b}$ be

$$\Sigma_{i=1}^{n} \left((x_i^2/a_i^2) + (y_i^2/b_i^2) \right) = 1 \quad \text{in} \quad C^n.$$

Obviously for any $a = (a_1,\ldots,a_n)$, $b = (b_1,\ldots\ldots,b_n)$, $(M_{a,b}, {}^0T''_{a,b})$ defines the same stein space (non-singular point). However by Webster's computation (see (W)),

$$(M_{a,b}, {}^0T''_{a,b}) \simeq (M_{a,a}, {}^0T''_{a,a}) \quad \text{as a CR-structure}$$

if and only if essentially $a = b$.

We would like to avoid this difficulty. In the above example, we note that the Webster's scalar curvature with respect to the natural pseudo hermitian structure changes if $a \neq b$ (see Sect.4 in this paper). Hence we would like to propose a deformation theory of pseudo hermitian CR-structure which preserves the Webster's scalar curvature. We must explain the Webster's scalar curvature. Let $(M, {}^0T'')$ be an abstract strongly pseudo convex CR-structure. Let θ a real 1-form satisfying:

$${}^0T'' + {}^0\bar{T}'' = \{ X : X \text{ in } C\otimes TM \ , \ \theta(X) = 0 \},$$

and

$$d\theta \quad \text{is non-degenerate.}$$

We call this triple $(M, {}^0T'', \theta)$ a pseudo hermitian structure. By Chern-

Moser and Tanaka, we have a connection and curvature form over the coframe bundle of M with respect to this pseudo hermitian structure. Let $(M,{}^{\phi}T'',\theta(\phi))$ be a deformation of pseudo hermitian structures of $(M,{}^{0}T'',\theta)$, where $\theta(\phi)$ is the canonical form with respect to $(M,{}^{\phi}T'')$ (for the definition, see Sect.2 in this paper). Let $R(\phi)$ be the Webster's scalar curvature defined by $(M,{}^{\phi}T'',\theta(\phi))$. We consider

$$\{ \phi : \phi \text{ in } \Gamma(M,T'\otimes({}^{0}T'')^{*}) , P(\phi)=0 , R(\phi)=0 \}.$$

Immediately we have the following question. Namely this family has enough deformations or not? We study this problem. Namely we see that this problem can be reduced to a non-linear partial differential equation, of which principal part of the linear term is sub-elliptic (Main Theorem). We see this. Let g be a real valued C^{∞} function on M. And let X_g be a ${}^{0}\bar{T}''$ valued vector field associated with g. For any deformation $(\mathcal{N},\pi,S)$ of a neighborhood N of M, $\psi(s)$, we would like to get a real valued function $g(s)$ satisfying: there is a C^{∞} embedding $f_{X_{g(s)}} : M \to \pi^{-1}(s)$ where $f_{X_{g(0)}}$ = identity, satisfying

$$* \quad R(\psi(s)\cdot f_{X_{g(s)}}) = R.$$

Obviously $*$ is a non-linear partial differential equation, of which unknown function is $g(s)$. And the principal part of the linear term of this equation is sub-elliptic. This is shown by a direct computation.

Section 1. Strongly Pseudo Convex Domains and CR-structures

Let X be a complex manifold and let r be a C^∞ exhausion function which is strictly pluri-subharmonic except a compact subset. Let

$$\Omega = \{ x : x \text{ in } X,\ r(x)<0 \}$$

and we assume that: the boundary of Ω, say $b\Omega$, is smooth. Then, naturally we put a CR-structure over $b\Omega$. Namely we set

$${}^0T'' = C\otimes T(b\Omega) \cap T''X|_{b\Omega}.$$

Then we have

1) ${}^0T'' \cap {}^0\bar{T}'' = 0$, $\text{f-dim}_C(C\otimes T(b\Omega)/({}^0T''+{}^0\bar{T}'')) = 1$,

2) $[\ \Gamma(b\Omega,{}^0T''),\Gamma(b\Omega,{}^0T'')\ ] \subset \Gamma(b\Omega,{}^0T'')$.

This notion is generalized as follows. Let M be a C^∞ orientable real odd dimensional manifold. Let E be a subbundle of the complexfied tangent bundle $C\otimes TM$ satisfying:

1)' $E \cap \bar{E} = 0$, $\text{f-dim}_C(C\otimes TM/(\ E + \bar{E}\ )) = 1$,

2)' $[\ \Gamma(M,E),\Gamma(M,E)\ ] \subset \Gamma(M,E)$.

This pair (M,E) is called an abstract CR-structure or simply a CR-structure. For our pair $(b\Omega, {}^0T'')$,we set a C^∞ vector bundle isomorphism

$$(1\text{-}1) \quad C\otimes T(b\Omega) = {}^0T'' + {}^0\bar{T}'' + C\xi,$$

where ξ is a real vector field, dual to $\sqrt{-1}\partial r$ with respect to a certain hermitian metric.

Section 2. Deformation Theory of CR-structures

We recall deformation theory of CR-structutres which is developed in (A1) ~ (A4), (A-M), (M1) and (M2). As is in Section 1, we assume that we are given a complex manifold X, a strongly pseudo convex domain Ω, and the boundary $b\Omega$, a CR-structure ${}^0T''$ induced from X on M, and we set a C^∞ vector bundle decompsotion (1-1).

Definition 2.1. Let E be a subbundle of the complexfied tangent bundle $C\otimes TM$ satisfying:

$$(2\text{-}1\text{-}1) \quad E \cap \bar{E} = 0, \quad f\text{-}\dim_C(C\otimes TM/(E + \bar{E})) = 1.$$

Then, the pair (M,E) is called an almost CR-structure. As E is a subbundle of $C\otimes TM$, we have a projection map from E to ${}^0T''$

according to (1-1). If this projection map is isomorphism, then we call (M,E) an almost CR-structure which is finite distance from $(M, {}^{0}T'')$ or simply an almost CR-structure.

Then, immediately we have the following proposition.

<u>Proposition 2.2(Proposition 1.6.1 in (A2))</u>. An almost CR-structure ${}^{\phi}T''$ corresponds to an element ϕ of $\Gamma(M, T'\otimes({}^{0}T'')^{*})$ bijectively. The correspondence is that: for ϕ in $\Gamma(M, T'\otimes({}^{0}T'')^{*})$,

$$
{}^{\phi}T'' = \{ X' : X' = X + \phi(X),\ X \text{ in } {}^{0}T'' \},
$$

where $T' = {}^{0}\bar{T}'' + C\xi$.

And we have

<u>Proposition 2.3(Proposition 1.6.2 in (A2))</u>. An almost CR-structure ${}^{\phi}T''$ is an actual CR-structure if and only if ϕ satisfies the non- linear partial differential equation $P(\phi) = 0$.

For a CR-structure $(M, {}^{\phi}T'')$,we assume that ϕ is sufficiently close to 0. Then we can define the canonical C^{∞} vector bundle decomposition

$$
(2\text{-}1) \quad C\otimes TM = {}^{\phi}T'' + {}^{\phi}\bar{T}'' + C\xi,
$$

and so w have the projection ω_{ϕ} from $C\otimes TM$ to $C\xi$ with respect to this decomposition. For this, we set a 1-form $\theta(\phi)$ by:

(2-2) $\theta(\phi)|_{{}^{\phi}T''+{}^{\phi}\bar{T}''} = 0,$

$\theta(\phi)(\xi) = 1.$

By using this $\theta(\phi)$, we have the canonical Levi form for $(M, {}^{\phi}T'')$ as follows. Namely, for X', Y' in ${}^{\phi}T''$,

$$L_{\phi}(X',Y') := d\theta(\phi)(X',\bar{Y}'),$$

that is to say,

$$= -\omega_{\phi}([X',\bar{Y}']).$$

We see this more precisely. By the dfinition, for Z in $C\otimes TM$, we have the unique element $X(\phi)$, $Y(\phi)$ in ${}^{0}T''$, and a function $u(\phi)$ satisfying:

$$Z = X(\phi) + \phi(X(\phi)) + \overline{Y(\phi)+\phi(Y(\phi))} + u(\phi)\xi.$$

Comparing the type according to (1-1), we have

(2-1) $(Z)_{{}^{0}T''} = X(\phi) + \overline{\phi(X(\phi))},$

$$(2\text{-}2)\quad (Z)_{0_{\bar{T}''}} = \overline{Y(\phi)} + \phi(X(\phi)),$$

$$(2\text{-}3)\quad (Z)_{\xi} = u(\phi)\xi .$$

Here $(Z)_{0_{T''}}$ means the $0_{T''}$ - part of Z , $(Z)_{0_{\bar{T}''}}$ means the $0_{\bar{T}''}$ - part of Z, and $(Z)_{\xi}$ means the ξ-part of Z respectively. By (2-1) and (2-2), we have

$$(2\text{-}4)\quad (Z)_{0_{T''}} = X(\phi) + \phi(\overline{((Z)_{0_{\bar{T}''}}}) - \overline{\phi(X(\phi))}).$$

Hence

$$(Z)_{0_{\bar{T}''}} - \phi((\overline{(Z)_{0_{\bar{T}''}}})) = X(\phi) - \phi((\overline{\phi(X(\phi))})).$$

And

$$\overline{(Z)_{0_{\bar{T}''}}} - \overline{\phi((Z)_{0_{T''}}}) = Y(\phi) - \overline{\phi((\overline{\phi(Y(\phi)}))}).$$

Hence $X(\phi)$, $Y(\phi)$ and also $u(\phi)$ are solved by the inverse mapping theorem, written in terms of ϕ and

$$X(0) = (Z)_{0_{T''}},\ Y(0) = (Z)_{0_{\bar{T}''}},\ u(0) = (Z)_{\xi}.$$

Section 3. Webster's Scalar Curvature R for $(M, {}^0T'')$

We recall the Webster's scalar curvature. Let p be a point of M. Let $\{Y_1, \ldots, Y_{n-1}\}$ be a base of ${}^0T''$ on a neighborhood of p. Then we have a hermitian matrix $(g_{\alpha\bar{\beta}})_{1\le\alpha,\beta\le n-1}$ by

$$g_{\alpha\bar{\beta}} = -\sqrt{-1}\,[Y_\alpha, \bar{Y}_\beta]_\xi,$$

where $[Y_\alpha, \bar{Y}_\beta]_\xi$ means the ξ-part of $[Y_\alpha, \bar{Y}_\beta]$ according to (1-1). Let $\theta^\alpha(\bar{Y}_\beta) = \delta_{\alpha\beta}$,

$$\theta^\alpha(Y_\beta) = 0,$$

$$\theta^\alpha(\xi) = 0.$$

And let θ be the dual real 1-form to ξ, namely

$$\theta(\xi) = 1,$$

$$\theta|_{{}^0T'' + {}^0\bar{T}''} = 0.$$

Then our ${}^0T''$ integrable, so these are 1-forms, η_α, satisfying

$$d\theta = \sqrt{-1}\, g_{\alpha\bar{\beta}}\, \theta^\alpha \wedge \theta^{\bar{\beta}} + \theta \wedge (\eta_\gamma \theta^\gamma + \overline{\eta_\gamma \theta^\gamma}),$$

and so there are 1-forms $\omega_\beta{}^\alpha$, τ^α, $\omega_{\bar{\beta}}{}^{\bar{\alpha}}$, $\tau^{\bar{\alpha}}$ by

$$d\theta^{\alpha} = \theta^{\beta}\Lambda\omega_{\beta}{}^{\alpha} + \theta\Lambda\tau^{\alpha},$$

$$d\theta^{\alpha} = \overline{\theta^{\beta}}\Lambda\omega_{\bar{\beta}}{}^{\bar{\alpha}} + \theta\Lambda\tau^{\bar{\alpha}}..$$

And if we assume

$$dg_{\alpha\bar{\beta}} - \omega_{\alpha}{}^{\gamma}g_{\gamma\bar{\beta}} - g_{\alpha\bar{\gamma}}\omega_{\bar{\beta}}{}^{\bar{\gamma}} = 0,$$

then $\omega_{\alpha}{}^{\beta}$, $\omega_{\bar{\alpha}}{}^{\bar{\beta}}$ are uniquely determined. And τ^{α}, $\tau^{\bar{\alpha}}$ are uniquely determined in modθ. See Theorem (1.1) in (W). By using these, we set

$$\Omega_{\beta}{}^{\alpha} = d\omega_{\beta}{}^{\alpha} - \omega_{\beta}{}^{\gamma}\Lambda\omega_{\gamma}{}^{\alpha} - \sqrt{-1}\theta_{\beta}\Lambda\tau^{\alpha} + \sqrt{-1}\tau_{\beta}\Lambda\theta^{\alpha}.$$

From these $\Omega_{\beta}{}^{\alpha}$, we have $R_{\beta}{}^{\alpha}{}_{\rho\bar{\alpha}}$ by:

$$\Omega_{\beta}{}^{\alpha} = R_{\beta}{}^{\alpha}{}_{\rho\bar{\sigma}}\ \theta^{\rho}\Lambda\theta^{\bar{\sigma}} + W_{\beta}{}^{\alpha}{}_{\rho}\theta^{\rho}\Lambda\theta - W^{\alpha}{}_{\beta\bar{\sigma}}\theta^{\bar{\sigma}}\Lambda\theta.$$

So we have the Ricci curvature

$$R_{\rho\bar{\sigma}} = \Sigma_{\alpha}\ R_{\alpha}{}^{\alpha}{}_{\rho\bar{\sigma}},$$

and the Webster's scalar curvature

$$R = \Sigma_{\rho,\alpha}\ g^{\rho\bar{\sigma}}R_{\rho\bar{\sigma}}.$$

Section 4. Webster's Scalar Curvature $R(\phi)$ for $(M, {}^{\phi}T'')$

For a deformation of $(M, {}^{0}T'')$, say $(M, {}^{\phi}T'')$, we will write down Webster's scalar curvature $R(\phi)$ in terms of ϕ. Let θ be as in Section 3. Then we have the canonical $\theta(\phi)$ for this deformation $(M, {}^{\phi}T'')$ (see (2-2) in this paper). Let p be a point of M. Let $\{Y_1, \ldots, Y_{n-1}\}$ be the orthonormal base of ${}^{0}T''$ at p. Let

$$ {}^{\phi}Y_i = Y_i + \phi(Y_i), \quad 1 \le i \le n-1. $$

Then we have a hermitian matrix $(g_{\alpha\bar{\beta}}(\phi))_{1 \le \alpha, \beta \le n-1}$ by

$$ g_{\alpha\bar{\beta}}(\phi) = -\sqrt{-1}\,[{}^{\phi}Y_\alpha, \overline{{}^{\phi}Y_\beta}]_\xi, $$

where $[{}^{\phi}Y_\alpha, {}^{\phi}\bar{Y}_\beta]_\xi$ means the ξ-part of $[{}^{\phi}Y_\alpha, {}^{\phi}\bar{Y}_\beta]$ according to (2-1). Let $\theta^\alpha(\phi)$ be the dual 1-form to ${}^{\phi}Y_\alpha$, namely

$$ \theta^\alpha(\phi)(\overline{{}^{\phi}Y_\beta}) = \delta_{\alpha\beta}, $$

$$ \theta^\alpha(\phi)({}^{\phi}Y_\beta) = 0, $$

$$ \theta^\alpha(\phi)(\xi) = 0. $$

And let $\theta(\phi)$ be the dual real 1-form to ξ, namely

$$\theta(\phi)(\xi) = 1,$$

$$\theta(\phi)|_{\phi_{T''}+\phi_{\bar{T}''}} = 0.$$

Then by the same reason as for $(M, {}^0T'')$, there are 1-forms, $\eta_\alpha(\phi)$, satisfying:

$$d\theta(\phi) = \sqrt{-1}g_{\alpha\bar{\beta}}(\phi)\theta^\alpha(\phi)\Lambda\overline{\theta^\beta(\phi)} + \theta(\phi)\Lambda(\eta_\gamma(\phi)\theta^\gamma(\phi) + \overline{\eta_\gamma(\phi)\theta^\gamma(\phi)})$$

and so there are 1-forms $\omega_\alpha{}^\beta(\phi)$, $\omega_{\bar{\alpha}}{}^{\bar{\beta}}(\phi)$, $\tau^\alpha(\phi)$, $\tau^{\bar{\alpha}}(\phi)$ by

$$d\theta^\alpha(\phi) = \theta^\beta(\phi)\Lambda\omega_\beta{}^\alpha(\phi) + \theta(\phi)\Lambda\tau^\alpha(\phi),$$

$$d\theta^{\bar{\alpha}}(\phi) = \theta^{\bar{\beta}}(\phi)\Lambda\omega_{\bar{\beta}}{}^{\bar{\alpha}}(\phi) + \theta(\phi)\Lambda\tau^{\bar{\alpha}}(\phi).$$

So under

$$dg_{\alpha\bar{\beta}}(\phi) - \omega_\alpha{}^\gamma(\phi)g_{\gamma\bar{\beta}}(\phi) - g_{\alpha\bar{\gamma}}(\phi)\omega_{\bar{\beta}}{}^{\bar{\alpha}}(\phi) = 0,$$

$\omega_\alpha{}^\beta(\phi)$, $\omega_{\bar{\alpha}}{}^{\bar{\beta}}(\phi)$ are uniquely determined and $\tau^\alpha(\phi)$, $\tau^{\bar{\alpha}}(\phi)$ are in modθ, uniquely determined. By the same way as in Section 3, we set

$$(4\text{-}1)\quad \Omega_\beta{}^\alpha(\phi) = d\omega_\beta{}^\alpha(\phi)-\omega_\beta{}^\gamma(\phi)\Lambda\omega_\gamma{}^\alpha(\phi)-\sqrt{-1}\theta_\beta(\phi)\Lambda\tau^\alpha+\sqrt{-1}\tau_\beta(\phi)\Lambda\theta^\alpha(\phi),$$

and we have $R_{\beta}{}^{\alpha}{}_{\rho\bar{\sigma}}$ by

(4-2) $\Omega_{\beta}{}^{\alpha}(\phi) = R_{\beta}{}^{\alpha}{}_{\rho\bar{\sigma}}(\phi)\theta^{\rho}(\phi)\wedge\theta^{\bar{\sigma}}(\phi)+W_{\beta}{}^{\alpha}{}_{\rho}(\phi)\theta^{\rho}(\phi)\wedge\theta(\phi)-W^{\alpha}{}_{\beta\bar{\sigma}}\theta^{\bar{\sigma}}(\phi)\wedge\theta(\phi).$

So we have the Ricci curvature

(4-3) $R_{\rho\bar{\sigma}}(\phi) = \Sigma_{\alpha} R_{\alpha}{}^{\alpha}{}_{\rho\bar{\sigma}}(\phi),$

and the Webster's scalar curvature

(4-4) $R(\phi) = \Sigma_{\rho,\sigma} g^{\rho\bar{\sigma}}(\phi)R_{\rho\bar{q}}(\phi),$

where $(g^{\rho\bar{\sigma}}(\phi))_{1\le\rho,\sigma\le n-1}$ is the inverse matirx of $(g_{\alpha\bar{\beta}}(\phi))_{1\le\alpha,\beta\le n-1}$.

Section 5. The Principal Part of the First Order Term

We compute the principal part of the first order term of $R(\phi)$, the Webster's scalar curvature for $(M,{}^{\phi}T'',\theta(\phi))$, ϕ in $\Gamma(M,T'\otimes({}^{0}T'')^{*})$ with respect to ϕ, by using (4-1)∼(4-4). By (4-4), the principal term of the first order term of $R(\phi)$ with respect to ϕ , is that:

the first order term of $g^{\rho\bar{\sigma}}(\phi)R_{\rho\bar{\sigma}}(\phi)$

$$= \text{the first order term of } g^{\rho\bar{\sigma}}(\phi)\Sigma_{\alpha} d\omega_{\alpha}{}^{\alpha}(\phi)({}^{\phi}Y_{\rho}, \overline{{}^{\phi}Y_{\sigma}})$$

$$= \text{the first order term of } \Sigma_{\alpha,\rho}\, d\omega_{\alpha}{}^{\alpha}({}^{\phi}Y_{\rho}, \overline{{}^{\phi}Y_{\rho}}).$$

Because $g^{\rho\bar{\sigma}}(0)=\delta_{\rho\sigma}$ and $g^{\rho\bar{\sigma}}(\phi)$ includes only the 1-st derivatives of ϕ but $d\omega_{\alpha}{}^{\alpha}(\phi)$ includes the 2-nd derivatives of ϕ, and furthermore from $\theta_{\beta}(\phi)\Lambda\tau^{\alpha}(\phi)$, $\tau_{\beta}(\phi)\Lambda\theta^{\alpha}(\phi)$, only 2-nd derivatives appear. So we can ignore these terms(this is shown in our computation). Namely we have:

the principal term of the first order term of $R(\phi)$

$$= \text{the first order term of } \Sigma_{\alpha,\rho}\, d\omega_{\alpha}{}^{\alpha}(\phi)({}^{\phi}Y_{\rho}, \overline{{}^{\phi}Y_{\rho}}) \text{ with respect to } \phi$$

= the first order term of

$$\Sigma_{\alpha,\rho}\{ (Y_{\rho}+\phi(Y_{\rho}))\omega_{\alpha}{}^{\alpha}(\phi)(\overline{Y_{p}+\phi(Y_{\rho})}) - (\overline{Y_{\rho}+\phi(Y_{\rho})})\omega_{\alpha}{}^{\alpha}(\phi)(Y_{\rho}+\phi(Y_{\rho}))$$

$$- \omega_{\alpha}{}^{\alpha}(\phi)([Y_{\rho}+\phi(Y_{\rho}), \overline{Y_{\rho}+\phi(Y_{\rho})}]) \}.$$

From the term; $\omega_{\alpha}{}^{\alpha}(\phi)([Y_{\rho}+\phi(Y_{\rho}), \overline{Y_{\rho}+\phi(Y_{\rho})}])$, the 2-nd derivative of ϕ doesnot appear. So the above becomes

= the principal term of the first order term of

$$\Sigma_{\alpha,\rho}\{(Y_\rho+\phi(Y_\rho))\omega_\alpha{}^\alpha(\phi)\overline{(Y_\rho+\phi(Y_\rho))} - \overline{(Y_\rho+\phi(Y_\rho))}\omega_\alpha{}^\alpha(\phi)(Y_\rho+\phi(Y_\rho))\}.$$

And so

= the principal term of the first order term of

$$\Sigma_{\alpha,\rho}\{ Y_\rho\omega_\alpha{}^\alpha(\phi)(\overline{Y_\rho}) - \overline{Y_\rho}\omega_\alpha{}^\alpha(\phi)(Y_\rho) \}.$$

We compute $\omega_\alpha{}^\alpha(\phi)(\overline{Y_\rho})$, $\omega_\alpha{}^\alpha(\phi)(Y_\rho)$. By the definition, we have

5-i) $d\theta^\alpha(\phi)({}^\phi Y_i, \overline{{}^\phi Y_j}) = (\theta^\beta(\phi)\Lambda\omega_\beta{}^\alpha(\phi))({}^\phi Y_i, \overline{{}^\phi Y_j}) + (\theta\Lambda\tau^\alpha(\phi))({}^\phi Y_i, \overline{{}^\phi Y_j})$

$= -\omega_j{}^\alpha(\phi)({}^\phi Y_i)$,

5-ii) $d\theta^\alpha(\phi)({}^\phi Y_i, {}^\phi Y_j) = 0$,

5-iii) $d\theta^\alpha(\phi)({}^\phi Y_i, \xi) = (\theta^\beta(\phi)\Lambda\omega_\beta{}^\alpha(\phi))(\overline{{}^\phi Y_i}, \xi) + (\theta\Lambda\tau^\alpha(\phi))(\overline{{}^\phi Y_i}, \xi)$

$= \omega_i{}^\alpha(\phi) - \tau^\alpha(\phi)(\overline{{}^\phi Y_i})$.

Following (W), we set

$$\tau^\alpha(\phi)(\overline{{}^\phi Y_i}) = 0.$$

And

$$\text{5-iv)}\quad d\theta^{\alpha}(\phi)({}^{\phi}Y_i,\xi) = (\theta^{\beta}(\phi)\wedge\omega_{\beta}{}^{\alpha}(\phi))({}^{\phi}Y_i,\xi) + (\theta\wedge\tau^{\alpha}(\phi))({}^{\phi}Y_i,\xi)$$

$$= -\tau^{\alpha}(\phi)({}^{\phi}Y_i),$$

$$\text{5-v)}\quad d\theta^{\alpha}(\phi)(\overline{{}^{\phi}Y_i},\overline{{}^{\phi}Y_j}) = (\theta^{\beta}(\phi)\wedge\omega_{\beta}{}^{\alpha}(\phi))(\overline{{}^{\phi}Y_i},\overline{{}^{\phi}Y_j}) + (\theta\wedge\tau^{\alpha}(\phi))(\overline{{}^{\phi}Y_i},\overline{{}^{\phi}Y_j}).$$

Furthermore

$$\text{5-i)'}\quad d\theta^{\bar{\alpha}}(\phi)({}^{\phi}Y_i,\overline{{}^{\phi}Y_j}) = (\theta^{\beta}(\phi)\wedge\omega_{\bar{\beta}}{}^{\bar{\alpha}}(\phi))({}^{\phi}Y_i,\overline{{}^{\phi}Y_j}) + (\theta\wedge\tau^{\alpha}(\phi))({}^{\phi}Y_i,\overline{{}^{\phi}Y_j})$$

$$= \omega_{\bar{i}}{}^{\bar{\alpha}}(\phi)(\overline{{}^{\phi}Y_j}),$$

$$\text{5-ii)'}\quad d\theta^{\bar{\alpha}}(\phi)({}^{\phi}Y_i,{}^{\phi}Y_j) = (\theta^{\bar{\beta}}(\phi)\wedge\omega_{\bar{\beta}}{}^{\bar{\alpha}}(\phi))({}^{\phi}Y_i,{}^{\phi}Y_j) + (\theta\wedge\tau^{\alpha}(\phi))({}^{\phi}Y_i,{}^{\phi}Y_j)$$

$$= \omega_{\bar{i}}{}^{\bar{\alpha}}({}^{\phi}Y_j) - \omega_{\bar{j}}{}^{\bar{\alpha}}({}^{\phi}Y_i),$$

$$\text{5-iii)'}\quad d\theta^{\bar{\alpha}}(\phi)(\overline{{}^{\phi}Y_i},\xi) = (\theta^{\bar{\beta}}(\phi)\wedge\omega_{\bar{\beta}}{}^{\bar{\alpha}}(\phi))(\overline{{}^{\phi}Y_i},\xi) + (\theta\wedge\tau^{\bar{\alpha}}(\phi))(\overline{{}^{\phi}Y_i},\xi)$$

$$= -\tau^{\bar{\alpha}}(\overline{{}^{\phi}Y_i}),$$

5-iv)' $d\theta^{\bar{\alpha}}(\phi)({}^{\phi}Y_i, \xi) = (\theta^{\bar{\beta}}(\phi) \wedge \omega_{\bar{\beta}}{}^{\bar{\alpha}}(\phi))({}^{\phi}Y_i, \xi) + (\theta \wedge \tau^{\bar{\alpha}}(\phi))({}^{\phi}Y_i, \xi)$

$$= \omega_{\bar{i}}{}^{\bar{\alpha}}(\phi)(\xi) - \tau^{\bar{\alpha}}(\phi)({}^{\phi}Y_i),$$

5-v)' $d\theta^{\bar{\alpha}}(\phi)(\overline{{}^{\phi}Y_i}, \overline{{}^{\phi}Y_j}) = 0.$

On the other hand, we require

$$dg_{\alpha\bar{\beta}}(\phi) - \omega_{\alpha}{}^{\gamma}(\phi) g_{\gamma\bar{\beta}}(\phi) - g_{\alpha\bar{\gamma}}(\phi)\omega_{\bar{\beta}}{}^{\bar{\gamma}}(\phi) = 0.$$

And so

the first order term of $\omega_i{}^{\alpha}(\phi)(\overline{{}^{\phi}Y_j})$

= the first order term of $\overline{\omega_{\bar{i}}{}^{\bar{\alpha}}(\phi)({}^{\phi}Y_j)}$

= the first order term of $\overline{-\omega_{\alpha}{}^{i}(\phi)({}^{\phi}Y_j)}$

= the first order term of $\overline{d\theta^{i}(\phi)({}^{\phi}Y_j, \overline{{}^{\phi}Y_{\alpha}})}$.

Therefore

the first order term of $\omega_\alpha^{\alpha}(\phi)(\overline{{}^{\phi}Y_\rho})$

$= $ the first order term of $\overline{d\theta^{\alpha}(\phi)({}^{\phi}Y_\rho, \overline{{}^{\phi}Y_\alpha})}$

$= $ the first order term of $\overline{d\theta^{\alpha}(\phi)(Y_\rho+\phi(Y_\rho), \overline{Y_\alpha+\phi(Y_\alpha)})}$

$= $ the first order term of $\overline{-\theta^{\alpha}(\phi)([Y_\rho+\phi(Y_p), \overline{Y_\alpha+\phi(Y_\alpha)}]}$.

So the principal term of the first order term of $\omega_\alpha^{\alpha}(\phi)(\overline{{}^{\phi}Y_\rho})$ with respect to ϕ, is that:

$$\overline{-\theta^{\alpha}(\phi)([Y_\rho+\phi(Y_\rho), \overline{Y_\alpha+\phi(Y_\alpha)}])}$$

$$= \overline{\bar{Y}_\alpha \phi_\rho{}^{\alpha}} .$$

Similarly, the principal term of the first order term of $\omega_\alpha^{\alpha}(\phi)({}^{\phi}Y_\rho)$ with respect to ϕ, is that:

$$- \bar{Y}_\alpha \phi_\rho{}^{\alpha}.$$

Therefore we have

Theorem 5.1. The principal term of the first order term of $R(\phi)$

$-$ R with respect to ϕ is that:

$$Y_\rho \overline{Y_\alpha \phi_\rho{}^\alpha} + \bar{Y}_\rho \bar{Y}_\alpha \phi_\rho{}^\alpha .$$

Proof. We have already shown that:the principal term of the first order term of $R(\phi) - R$

= the principal term of the first order term of

$$\Sigma_{\alpha,\rho} \{ Y_\rho \omega_\alpha{}^\alpha(\phi)(\bar{Y}_\rho) - \bar{Y}_\rho \omega_\alpha{}^\alpha(\phi)(Y_\rho) \}.$$

And as above, we obtain that:the principal term of the first order term of $\omega_\alpha{}^\alpha(\phi)(\overline{{}^\phi Y_\rho}) = Y_\rho \overline{\phi_\rho{}^\alpha}$, namely

the principal term of the first order term of $\omega_\alpha{}^\alpha(\phi)(\overline{Y_\rho}) = Y_\alpha \overline{\phi_\rho{}^\alpha}$.

Hence it follows that: the principal term of the first order term of

$R(\phi) - R = Y_\rho \overline{Y_\alpha \phi_\rho{}^\alpha} + \bar{Y}_\rho \bar{Y}_\alpha \phi_\rho{}^\alpha$. Q.E.D.

Section 6. Induced CR-structure by a Real Valued C^∞ Function g

Let g be a real valued C^∞ function on M. From this g, we introduce a $\Gamma(M, {}^0T'')$-valued C^∞ vector field X_g by

$$-\sqrt{-1}[X_g, Y]_\xi = Yg \quad \text{for } Y \text{ in } \Gamma(M, {}^0T''),$$

where $[X_g, Y]_\xi$ means the ξ-part of $[X_g, Y]$ according to (1-1). In the construction of a new CR-structure, we use this vector X_g. Namely, for any family of deformations of a neighborhood of M, say N, in X, $(\mathcal{N}, \pi, S)$, where $\mathcal{N}$ is an analytic space and S is also, and π is a smooth map from $\mathcal{N}$ to S satisfying:

$$\begin{array}{ccl} \mathcal{N} & \longrightarrow & N \times S \\ \pi \downarrow & & \text{projection} \\ S & & \end{array}$$

and o in S, $\pi^{-1}(o) = N$, we would like to set a parametrization of C^∞ embeddinngs of M by $\Gamma(M, T')$, where

$$T' = {}^0\bar{T}'' + C\xi.$$

We put a Riemannian metric on N. And we consider the exponential mapping $\exp_p(X)$ and we restrict this map to M. Namely for a vector field X in θ of $\Gamma(M, TN|_M)$ with respect to the C^0-norm (sup-norm), we have

$$\exp_p(X) : M \longrightarrow N,$$

where TN means the real tangent bundle and $TN|_M$ means the restriction of TN to M. So we have

$$M \times \Theta \longrightarrow N$$

$$(p, X) \longrightarrow \exp_p(X).$$

Let $T'N_s$ means the vector bundle consisting of (0,1) vectors with respect to the complex manifold $\pi^{-1}(s) = N_s$. Then the inclusion map i: $T' \longrightarrow C \otimes TN_s$, where $C \otimes TN_s$ means the complexfied tangent bundle of the real tangent bundle TN_s (we note that T' is a subbundle of $C \otimes TM$, and as M is a submanifold of N, so there is an inclusion map i: $TM \longrightarrow TN_s$), induces the isomorphism map : $T' \longrightarrow T'N_s$, where $T'N_s$ means the vector bundle consisting of (1,0) vectors with respect to the complex manifold $\pi^{-1}(s) = N_s$. We denote ρ_s for this isomorphism map. For η in T', we consider

$$\exp_p(\rho_s(\eta) + \overline{\rho_s(\eta)}).$$

That is to say, we have

$$M \times \Theta' \longrightarrow M \times \Theta \longrightarrow N$$

$$(p, \eta) \longrightarrow (p, \rho_s(\eta) + \overline{\rho_s(\eta)}) \longrightarrow \exp_p(\rho_s(\eta) + \overline{\rho_s(\eta)}),$$

where Θ' is an open set of the origin in $\Gamma(M, T')$. We, briefly, write this composition map by f_η. Let $\{V_i\}_{i \in I}$ be an open coordinate covering of $\mathcal{N}$ satisfying:

$$
\begin{array}{ccc}
\mathcal{V}_i & \xrightarrow{\quad\text{projection}\quad} & U_i \times S \\
\pi \downarrow & & \\
S & &
\end{array},
$$

where $\{U_i\}_{i\in I}$ is an open coordinate covering of N. And let $\{z_i^{\alpha}(s)\}_{i\in I,1\le\alpha\le n}$ be a complex coordinate of $\mathcal{N}$. This means that $z_i^{\alpha}(s)$ is a complex valued C^{∞} function on U_i satisfying: $\{z_i^{\alpha}(s)\}_{i\in I,1\le\alpha\le n}$ is a holomorphic local coordinate with respect to the complex structures, which depends on s complex analytically. Then by using this coordinate, $z_i^{\alpha}(s)$, we have

$$f_{\eta}^{\alpha} = z_i^{\alpha}(s) + \eta_i^{\alpha} + \text{the higher order term of } \eta,\ \bar{\eta},$$

where

$$\eta_i^{\alpha} = \eta(z_i^{\alpha}(s))|_{U_i \cap M}.$$

Namely, we have a C^{∞} embedding f_{η}

$$
\begin{array}{ccc}
M \times S \times \Gamma(M,T') & \xrightarrow{\ f_{\eta}\ } & \mathcal{N} \\
\downarrow & \pi & \\
S & &
\end{array}.
$$

Then, via this C^{∞} embedding f_{η}, from $\pi^{-1}(s)$, we have the induced CR-structure $\psi(s)\cdot f_{\eta}$, where $\psi(s)$ is the corresponding holomorphic tangent bundle valued (0,1) form. We already introduced a vector field X_g in $\Gamma(M,{}^{0}\bar{T}'') \subset \Gamma(M,T')$, in the beginning of this section.

We must study $\psi(s)\cdot f_{X_g}$ and also we compute its Webster's scalar curvature $R(\psi(s)\cdot f_{X_g})$ by using the result obtained in Section 4 in this paper.

Now we compute

$$R(\psi(s)\cdot f_{X_g}) - R.$$

We use the notation L_0 for the linear term of $R(\psi(s)\cdot f_{X_g}) - R$ with respect to g. Then our main theorem in this paper is that:

Theorem 6.1. The principal part of L_0 is sub-elliptic.

Proof. By using a parition of unity, it suffices to show this theorem on a local coordinate open covering U. Let $\{Y_1, \ldots, Y_{n-1}\}$ be a moving frame of ${}^0T''$ on U, which are orthogonal with respect to the Levi-metric defined in (1-1) in Section 1 in this paper. We put the L^2-norm on $\Gamma_0(U,C)$ where $\Gamma_0(U,C)$ means the space which consists of C^∞ functions, supported in U. If we prove that:

$$\text{(6-1)} \quad \| L_0 g \|^2 + \| g \|^2 \geq c\, \Sigma_{\alpha,\beta}(\|Y_\alpha Y_\beta g\|^2 + \|Y_\alpha \bar{Y}_\beta g\|^2 + \|\bar{Y}_\alpha Y_\beta\|^2 + \|\bar{Y}_\alpha \bar{Y}_\beta g\|^2),$$

where c is a positive constant independent of g, and $\| \ \|$ means the L^2-norm defined in the above, then our theorem is complete. In order to prove (6-1), we must prepare something.

Let Y_α^* be the formal adjoint operator with respect to the above Levi metric. Then

$$Y_\alpha^* = -\bar{Y}_\alpha + a_\alpha,$$

where a_α is a C^∞ function on U.

Next we prepare several abbreviations. First, we set $\| \ \|''$ norm by

$$\|g\|''^2 = \Sigma_{\alpha,\beta}\{\|Y_\alpha Y_\beta g\|^2+\|Y_\alpha \bar{Y}_\beta g\|^2+\|\bar{Y}_\alpha Y_\beta g\|^2+\|\bar{Y}_\alpha \bar{Y}_\beta g\|^2\}.$$

Second, as in the standard way, we introduce $\lesssim$. For any real A and B,

$$A \lesssim B$$

means that there is a constant c satisfying

$$A \leq cB.$$

Now in order to prove our theorem, we show some lemmas.

Lemma 6.2. For a g in $\Gamma_0(U,C)$, and for any $\varepsilon > 0$, there is a constant K satisfying:

$$\| Y_\alpha g \|^2 \leq \varepsilon \| g \|''^2 + (K/\varepsilon)\| g \|^2.$$

Proof. In integration by parts, we have

$$\| Y_\alpha g \|^2 = (Y_\alpha g, Y_\alpha g)$$

$$= (Y_\alpha^* Y_\alpha g, g)$$

$$= (-\bar{Y}_\alpha Y_\alpha g, g) + (a_\alpha Y_\alpha g, g).$$

While by the Schwarz inequality,

$$|(-\bar{Y}_\alpha Y_\alpha g, g)| \leq \varepsilon' \|\bar{Y}_\alpha Y_\alpha g\|^2 + (2/\varepsilon')\| g \|^2$$

for any $\varepsilon' > 0$. Smilarly, there is a constant K satisfying:

$$|(a_\alpha Y_\alpha g, g)| \leq \varepsilon' \| Y_\alpha g \|^2 + (K/\varepsilon')\| g \|^2$$

for any $\varepsilon' > 0$ (since M is compact, we can assume that a_α is bounded). Hence

$$\|Y_\alpha g\|^2 \leq \varepsilon' \|g\|''^2 + (2/\varepsilon')\|g\|^2 + \varepsilon' \|Y_\alpha g\|^2 + (K/\varepsilon')\|g\|^2$$

for any $\varepsilon' > 0$. So by choosing K sufficiently large, we have

$$\|Y_\alpha g\|^2 \leq \varepsilon \|g\|''^2 + (K/\varepsilon)\|g\|^2$$

for any $\varepsilon > 0$. Hence we have our lemma. **Q.E.D.**

Next we have

Lemma 6.3. For any g in $\Gamma_0'(U,C)$, and for any $\varepsilon > 0$,

$$(L_0 g, g) + \varepsilon \|g\|''^2 + (K/\varepsilon)\|g\|^2 \geq \Sigma_{\alpha,\beta}\{\|Y_\alpha Y_\beta g\|^2 + \|Y_\alpha Y_\beta g\|^2\}.$$

Proof. By Theorem 5.1, in this paper,

the principal part of the linear term $= \Sigma_{\alpha,\beta}(\bar{Y}_\alpha \bar{Y}_\beta Y_\alpha Y_\beta + Y_\alpha Y_\beta \bar{Y}_\alpha \bar{Y}_\beta)$.

So

$$(L_0 g . g) = \Sigma_{\alpha,\beta}(\bar{Y}_\alpha \bar{Y}_\beta Y_\alpha Y_\beta g, g) + \Sigma_{\alpha,\beta}(Y_\alpha Y_\beta \bar{Y}_\alpha \bar{Y}_\beta g, g)$$

$$= \Sigma_{\alpha,\beta}(\bar{Y}_\beta Y_\alpha Y_\beta g, \bar{Y}_\alpha^* g) + \Sigma_{\alpha,\beta}(Y_\beta \bar{Y}_\alpha \bar{Y}_\beta g, Y_\alpha^* g).$$

While

$$(\bar{Y}_\beta Y_\alpha Y_\beta g, \bar{Y}_\alpha^* g) = (\bar{Y}_\beta Y_\alpha Y_\beta g, -Y_\alpha g) + (\bar{Y}_\beta Y_\alpha Y_\beta g, a_\alpha g).$$

We note that $(\bar{Y}_\beta Y_\alpha Y_\beta g, \bar{a}_\alpha g)$ can be ignored. Because

$$(\bar{Y}_\beta Y_\alpha Y_\beta g, \bar{a}_\alpha g) = (Y_\alpha Y_\beta g, -Y_\beta(\bar{a}_\alpha g)) + (Y_\alpha Y_\beta g, \bar{a}_\beta \bar{a}_\alpha g)$$

$$= -(Y_\alpha Y_\beta g, \bar{a}_\alpha Y_\beta g) + (Y_\alpha Y_\beta g, -(Y_\beta \bar{a}_\alpha) g) + (Y_\alpha Y_\beta g, \bar{a}_\alpha \bar{a}_\beta g)$$

For the first term, there is a constant K_1 satisfying:for any $\varepsilon > 0$,

$$|(Y_\alpha Y_\beta g, \bar{a}_\alpha Y_\beta g)| \leq \varepsilon \|g\|''^2 + (K_1/\varepsilon)\|Y_\beta g\|^2.$$

And for $\|Y_\beta g\|^2$, by Lemma 6.2, we have that: for any $\delta > 0$,

$$\|Y_\beta g\|^2 \leq \delta\|g\|''^2 + (K_2/\delta)\|Y_\beta g\|^2.$$

Hence for any $\varepsilon > 0$, $\delta > 0$,

$$|(Y_\alpha Y_\beta g, \bar{a}_\alpha Y_\beta g)| \leq (\varepsilon + (K_1/\varepsilon))\|g\|''^2 + (K_1 K_2/\varepsilon\delta)\|g\|^2.$$

The second term, namely $(Y_\alpha Y_\beta g, (\bar{Y}_\beta \bar{a}_\alpha)g)$, and the third term $(Y_\alpha Y_\beta g, \bar{a}_\alpha \bar{a}_\beta g)$ are also absorbed in $\varepsilon\|g\|''^2 + (K/\varepsilon)\|g\|^2$. So for any $\varepsilon > 0$, there is a constant K' satisfying:

$$\mathrm{Re}\ (\bar{Y}_\alpha \bar{Y}_\beta Y_\alpha Y_\beta g, g) + \varepsilon\|g\|''^2 + (K'/\varepsilon)\|g\|^2 \geq \mathrm{Re}(\bar{Y}_\beta Y_\alpha Y_\beta g, -Y_\alpha g).$$

By the similar way, we have: for any $\varepsilon > 0$, there is a constant K satisfying:

$$\mathrm{Re}\ (\bar{Y}_\alpha \bar{Y}_\beta Y_\alpha Y_\beta g, g) + \varepsilon\|g\|''^2 + (K/\varepsilon)\|g\|^2 \geq \mathrm{Re}(Y_\alpha Y_\beta g, Y_\beta Y_\alpha g)$$

$$= (Y_\alpha Y_\beta g, Y_\alpha' Y_\beta g) + \mathrm{Re}(Y_\alpha Y_\beta g, [Y_\alpha, Y_\beta]g).$$

We can handle $\mathrm{Re}(Y_\alpha Y_\beta g, [Y_\alpha, Y_\beta]g)$. In fact, there is a constant K' satisfying: for any $\varepsilon' > 0$,

$$\|Y_\nu g\|^2 \leq \varepsilon\|g\|''^2 + (K/\varepsilon)\|g\|^2.$$

So

$$| \operatorname{Re}(Y_\alpha Y_\beta g, [Y_\alpha, Y_\beta]) | \leq \varepsilon \|g\|''^2 + (K'/\varepsilon')(\varepsilon \|g\|''^2 + (K/\varepsilon)\|g\|^2)$$

$$\leq (\varepsilon' + (K'/\varepsilon'))\|g\|''^2 + (K'K/\varepsilon'\varepsilon)\|g\|^2$$

for any ε, $\varepsilon' > 0$. So we can ignore $\operatorname{Re}(Y_\alpha Y_\beta g, [Y_\alpha, Y_\beta]g)$.

Now we prove our theorem. We have obtained that there is a constant K satisfying: for any g in $\Gamma_0(U, C)$, for any $\varepsilon > 0$,

$$(L_0 g, g) + \varepsilon \|g\|''^2 + (K/\varepsilon)\|g\|^2 \geq \Sigma_{\alpha,\beta}(\|Y_\alpha Y_\beta g\|^2 + \|\bar{Y}_\alpha \bar{Y}_\beta g\|^2).$$

While from $\|Y_\alpha Y_\beta g\|^2$,

$$(Y_\alpha Y_\beta g, Y_\alpha Y_\beta g) = (Y_\beta g, Y_\alpha^* Y_\alpha Y_\beta g)$$

$$= (Y_\beta g, -\bar{Y}_\alpha Y_\alpha Y_\beta g) + (Y_\beta g, a_\alpha Y_\alpha Y_\beta g)$$

$$= (Y_\beta g, -Y_\alpha \bar{Y}_\alpha Y_\beta g) + (Y_\beta g, [Y_\alpha, \bar{Y}_\alpha] Y_\beta g) + (Y_\beta g, Y_\alpha Y_\beta g).$$

Henceforth we ignore any term which can be estimated by:

$$\varepsilon \|g\|''^2 + (K/\varepsilon)\|g\|^2$$

for any $\varepsilon > 0$, where K is a constant. So from now on, = means modulo any term of this type. Then by this abbreviation, the above becomes

$$= (Y_\beta g, -Y_\alpha \bar{Y}_\alpha Y_\beta g) + (Y_\beta g, \sqrt{-1}\xi Y_\beta g)$$

$$(\text{by } [Y_\alpha, \bar{Y}_\alpha] = \sqrt{-1}\xi \mod Y_\beta , \bar{Y}_\beta)$$

$$= (\bar{Y}_\alpha Y_\beta g, \bar{Y}_\alpha Y_\beta g) + (Y_\beta g, \sqrt{-1}\xi Y_\beta g).$$

On the other hand,

$$(\bar{Y}_\alpha \bar{Y}_\beta g, \bar{Y}_\alpha \bar{Y}_\beta g) = (\bar{Y}_\beta g, -Y_\alpha \bar{Y}_\alpha \bar{Y}_\beta g)$$

$$= (\bar{Y}_\beta g, -\bar{Y}_\alpha Y_\alpha \bar{Y}_\beta g) + (\bar{Y}_\beta g, [\bar{Y}_\alpha, Y_\beta]\bar{Y}_\beta g)$$

$$= (Y_\alpha \bar{Y}_\beta g, Y_\alpha \bar{Y}_\beta g) + (\bar{Y}_\alpha g, -\sqrt{-1}\xi \bar{Y}_\alpha g).$$

Therefore

$$\Sigma_{\alpha,\beta}\{ \|Y_\alpha Y_\beta g\|^2 + \|\bar{Y}_\beta \bar{Y}_\alpha g\|^2 \}$$

$$= \Sigma_{\alpha,\beta}\{ \|\bar{Y}_\alpha Y_\beta g\|^2 + (Y_\beta g, \sqrt{-1}\xi Y_\beta g) + \|Y_\beta \bar{Y}_\alpha g\|^2 + (\bar{Y}_\beta g, -\sqrt{-1}\xi \bar{Y}_\beta g) \}$$

$$= \Sigma_{\alpha,\beta}\{ \|\bar{Y}_\alpha Y_\beta g\|^2 + \|Y_\beta \bar{Y}_\alpha g\|^2 \}$$

(because the term: $(Y_\beta g, \sqrt{-1}\xi Y_\beta g) + (\bar{Y}_\beta g, -\sqrt{-1}\xi \bar{Y}_\beta g)$ can be estimated by $\varepsilon\|g\|''^2 + (K/\varepsilon)\|g\|^2$ by the similar method as above). So we have

$$\|g\|''^2 \lesssim \|L_0 g\|^2 + \|g\|^2 \quad \text{for any } g \text{ in } \Gamma_0(U,C). \qquad \text{Q.E.D.}$$

Henceforth we use the notation L_0 for this principal part of the linear term.

Section 7. On Versality

By using the result in Section 6, we discuss about versality. By the definition of versality in the sense of Kuranishi, if we prove that there is a solution $g(s)$ satisfying:

$$(7\text{-}1) \quad R(\psi(s)\cdot f_{X_{g(s)}}) = R,$$

where $\psi(s)$ means the corresponding form for a given family of deformations of a neighborhood N of M, $(\mathcal{N},\pi,S)$, $\pi^{-1}(o) = N$, then our family

$$\{ \phi : \phi \text{ in } \Gamma(M,T'\otimes({}^0T'')^*) \ , \ P(\phi) = 0 \ , \ R(\phi) = 0 \}$$

is versal in the sense of Kuranishi. As we have already shown, (7-1) is a non-linear partial differential equation and the principal term of the first order term of (7-1) with respect to $g(s)$, is sub-elliptic.

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