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Applications of norm inequalities equivalent to Löwner-Heinz theorem

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Abstract. A capital letter means a bounded linear operator on a Hilbert space. We define a new class of operators as follows. An operator T is said to be perinormal if $(T^*T)^n \leq T^{*n}T^n$ holds for every natural number n. Our new class of perinormal operators occupies the following place

$$\text{Normal} \subsetneqq \text{Quasinormal} \subsetneqq \text{Heminormal} \subsetneqq \text{Perinormal} \subsetneqq \text{Normaloid}.$$

In this note we shall show the assertion made in its title for perinormal operators A and B* for every natural number n.

§1. Basic Properties

First of all, we show the following result.

THEOREM 1. If A and B are arbitrary bounded linear operators on a Hilbert space, then the following properties hold and follow from each other.

(1) $A \geq B \geq 0$ ensures $A^s \geq B^s$ for any $1 \geq s \geq 0$.

(2) $\|AB\|^q \leq \| |A|^q |B^*|^q \|$ for any $q \geq 1$, namely $\| |A|^q |B^*|^q \|^{1/q} \leq \| |A|^p |B^*|^p \|^{1/p}$ for any $p \geq q > 0$, that is, $f(p) = \| |A|^p |B^*|^p \|^{1/p}$ is an increasing function on p.

(3) $\| |A|^s |B^*|^s \| \leq \|AB\|^s$ for any $1 \geq s \geq 0$, namely $\| |A|^{1/s} |B^*|^{1/s} \|^s \leq \| |A|^{1/t} |B^*|^{1/t} \|^t$ for any $s \geq t > 0$, that is, $g(s) = \| |A|^{1/s} |B^*|^{1/s} \|^s$ is a decreasing function on s.

(4) $\|AB\|^{(p+q)/2} \leq \| |A|^p |B^*|^p \|^{1/2} \| |A|^q |B^*|^q \|^{1/2}$ for any $p \geq 0$, $q \geq 0$ with $(p+q)/2 \geq 1$.

(5) $\|AB\|^{(p+q)/2} \leq \| |A|^p |B^*|^q \|^{1/2} \| |A|^q |B^*|^p \|^{1/2}$ for any $p \geq 0$, $q \geq 0$ with $(p+q)/2 \geq 1$.

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Recently we have given the following several equivalent norm inequalities to the famous Löwner-Heinz inequality inspired by [1].

THEOREM A [4]. If A and B are positive bounded linear operators on a Hilbert space, then the following properties hold and follow from each other.

(1) $A\geq B\geq 0$ ensures $A^s\geq B^s$ for any $1\geq s\geq 0$.

(2) $\|AB\|^q\leq\|A^qB^q\|$ for any $q\geq 1$, namely $\|A^qB^q\|^{1/q}\leq\|A^pB^p\|^{1/p}$ for any $p\geq q>0$, that is, $f(p)=\|A^pB^p\|^{1/p}$ ia an increasing function on p.

(3) $\|A^sB^s\|\leq\|AB\|^s$ for any $1\geq s\geq 0$, namely $\|A^{1/s}B^{1/s}\|^s\leq\|A^{1/t}B^{1/t}\|^t$ for any $s\geq t>0$, that is, $g(s)=\|A^{1/s}B^{1/s}\|^s$ is a decreasing function on s.

(4) $\|AB\|^{(p+q)/2}\leq\|A^pB^p\|^{1/2}\|A^qB^q\|^{1/2}$ for any $p\geq 0$, $q\geq 0$ with $(p+q)/2\geq 1$.

(5) $\|A^{st}B^{st}\|^2\leq\|A^sB^s\|^{2st/(s+t)}\|A^tB^t\|^{2st/(s+t)}$ for any $s>0$, $t>0$ with $2st/(s+t)\leq 1$.

(6) $\|AB\|^{(p+q)/2}\leq\|A^pB^q\|^{1/2}\|A^qB^p\|^{1/2}$ for any $p\geq 0$, $q\geq 0$ with $(p+q)/2\geq 1$.

(7) $\|A^{st}B^{st}\|^2\leq\|A^sB^t\|^{2st/(s+t)}\|A^tB^s\|^{2st/(s+t)}$ for any $s>0$, $t>0$ with $2st/(s+t)\leq 1$.

In order to give the proof of Theorem 1, we state the following lemma.

Lemma 1. $\|AB\| = \||A||B^*|\|$ for arbitrary operators A and B.

Proof.

$$\|AB\|^2 = \|B^*A^*AB\| = \|B^*|A|^2B\| = \||A|B\|^2 = \|B^*|A|\|^2 = \||B^*||A|\|^2 = \||A||B^*|\|^2.$$

Proof of Theorem 1. Combining Theorem A with Lemma 1, so the proof of Theorem 1 is complete.

Lemma 2. If A is quasinormal (i.e., $A(A^*A) = (A^*A)A$), then $|A^n| = |A|^n$ holds for every natural number n.

Proof. In case $n=1$, the result is obvious. Assume $|A^n| = |A|^n$, that is, $(A^*)^n A^n = (A^*A)^n$. Then $A^{*n+1}A^{n+1} = A^*A^{*n}A^nA = A^*(A^*A)^nA = (A^*A)^{n+1}$, namely $|A^{n+1}| = |A|^{n+1}$, so the proof is complete by Induction.

Theorem A, Lemma 1 and Lemma 2 yield the following Theorem 2.

Theorem 2. If A and B* are quasinormal, then (*) holds;

(*) the following (1) and (2) hold and follow from each other.

(1) $\|AB\|^n \le \|A^nB^n\|$ for every natural number n.

(2) $\|AB\|^{n+m} \le \|A^nB^m\|\|A^mB^n\|$ for every natural number n and m.

Proof. By Lemma 1, Lemma 2 and Theorem A, we have

$$\|AB\|^n = \||A||B^*|\|^n \le \||A|^n|B^*|^n\| = \||A^n||B^{*n}|\| = \|A^nB^n\|,$$

so that we have the desired result by Theorem A.

(1) in Theorem 2 easily yields the following result.

Theorem B ([9]). If A and B are normal, then $\|AB\|^n \le \|A^nB^n\|$ holds for every natural number n.

§ 2. Further extension of Theorem 2.

In this section we shall give an extension of Theorem 2.

Lemma 3. If $0 \le B \le A$ and $0 \le D \le C$, then $\|B^{1/2}D^{1/2}\| \le \|A^{1/2}C^{1/2}\|$ holds.

Proof. $\|B^{1/2}D^{1/2}\|^2 = \|D^{1/2}BD^{1/2}\| \le \|D^{1/2}AD^{1/2}\| = \|D^{1/2}A^{1/2}\|^2$

$$= \|A^{1/2}DA^{1/2}\| \le \|A^{1/2}CA^{1/2}\| = \|A^{1/2}C^{1/2}\|^2.$$

Lemma 4. If A and B satisfy $\||A|^n|B^*|^n\| \le \||A^n||B^{*n}|\|$ for every natural number n, then (*) holds.

Proof. By Lemma 1, Theorem A and the hypothesis in Lemma 4, we have

$$\|AB\|^n = \| |A| |B^*| \|^n \le \| |A|^n |B^*|^n \| \le \| |A^n| |B^{*n}| \| \le \|A^n B^n\|.$$

Following after [7], we cite the following definition.

Definition 1. An operator T is said to be perinormal if

$$(T^*T)^n \le T^{*n}T^n$$

holds for every natural number n. For each k, an operator T is k-hyponormal if

$$(TT^*)^k \le (T^*T)^k.$$

An operator T is heminormal ([2]) if T is hyponormal and T^*T commutes with TT^*.

Lemma 5. If T is k-hyponormal, then $(T^*T)^n \le T^{*n}T^n$ holds for $n = 1, 2, \cdots, k, k+1$.

Proof. In case $n = 1$, the result is obvious. By Löwner-Heinz theorem ([8][6]), the hypothesis implies $(TT^*)^{k-m} \le (T^*T)^{k-m}$ for $m = 1, 2, \cdots, k-1$. Assume $(T^*T)^n \le T^{*n}T^n$ for $n = 1, 2, \cdots, k$. Then

$$(T^*T)^{n+1} = T^*(TT^*)^n T \le T^*(T^*T)^n T \le T^*(T^{*n}T^n)T = T^{*n+1}T^{n+1},$$

so the proof is complete by induction.

Remark. It is known that heminormal is k-hyponormal for every k and every k-hyponormal is hyponormal ([2]). There exists an example of hyponormal operator T whose square T^2 is not hyponormal [5, Problem 164], but it can be verified that this square T^2 is perinormal. We remark that perinormal is normaloid, i.e., $\|T\| = r(T)$ where $r(T)$ means the spectral radius of T.

Combining Lemma 5 with the remark stated above, our new class of perinormal operators occupies the place shown in the following schema and the inclusions are all proper.

$$\text{Normal} \subsetneqq \text{Quasinormal} \subsetneqq \text{Heminormal} \subsetneqq \text{Perinormal} \subsetneqq \text{Normaloid}$$

Theorem 3. If A and B* are perinormal, then (*) holds.

Proof. By the hypothesis, $|A|^{2n} \leq |A^n|^2$ and $|B^*|^{2n} \leq |B^{*n}|^2$ for every natural number n, so we have

(C) $$\| |A|^n |B^*|^n \| \leq \| |A^n| |B^{*n}| \|$$

by Lemma 3. This condition (C) satisfies the hypothesis of Lemma 4, so we have the desired conclusion (*) by Lemma 4.

Theorem 3 and Lemma 5 imply the following result.

Corollary 1.
Property (*) holds under any one of the following conditions

(1) A and B* are heminormal

(2) A and B* are quasinormal

(3) A and B are normal.

Proof. If A and B* are heminormal, then they are also k-hyponormal for every natural number k, therefore they are perinormal by Lemma 5, so we have (1) by Theorem 3. (2) is obtained by (1) and (3) is also obtained by (2).

§ 3. Counterexample

We attempt to extend Theorem 3 for normaloid operators which belong to more wider class than that of perinormal operators, but we have a counterexample to this conjecture.

Counterexample. Put A and B as follows;

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then A is self adjoint and B is normaloid. But $1 = \|AB\|^2 \nleq \|A^2B^2\| = 0$.

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