On the differentiability of semi-groups of linear operators in locally convex spaces

By

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1. Introduction

Strongly continuous semi-groups and holomorphic ones of bounded linear operators in Banach spaces have been investigated by a number of authors, and many results on them have been, thanks to the restriction of equicontinuity, generalized to the case of equicontinuous semi-groups in locally convex spaces (See for example K. Yosida [4]). It is, however, in 1968 that the notion of the differentiability of semi-groups was introduced systematically by A. Pazy, who gave in his paper [3] among other things a necessary and sufficient conditions for a strongly continuous semi-groups in a Banach space to be differentiable. In this note, we intend to generalize his results and to deal with the differentiability of semi-groups in a locally convex space.

Let X be a locally convex linear topological space which is assumed to be sequentially complete, and T_t, $t\geqq 0$ be a semi-group of continuous linear operators on X of class (C^0) such that

$$\textit{for some } b>0,\ T_t,\ 0\leqq t\leqq b \quad \textit{is equicontinuous.}$$

This semi-group is called *differentiable* at $t=a$ $(\leqq b)$ if T_aX is included in the domain of the *infinitesimal generator* A (Definition). We are concerned particularly with the characterization of this A in terms of its spectral properties. Throughout this paper, we make frequent use of the following formula:

$$(\lambda I-A)^{-1}=\left\{T_a+\lambda\int_0^a e^{\lambda(a-s)}T_s ds\right\}(\lambda e^{\lambda a}I-AT_a)^{-1}.$$

This formula holds as long as $\lambda e^{\lambda a}I-AT_a$ has the everywhere defined continuous inverse and enables us represent the resolvent $R(\lambda; A)=(\lambda I-A)^{-1}$ of A in a different manner from the usual one by the Laplace transform (Theorem 2. 2), playing an essential role in our theory. Actually we make, for the validity of the above formula, an additional assumption:

$$\textit{for some } C>0,\ (CAT_a)^n,\ n=1, 2,\ldots\ldots \quad \textit{is equicontinuous.}$$

Thus, we characterize the infinitesimal generator of an equicontinuous or not necessarily an equicontinuous semi-group which is differentiable in the above sense (Theorems 2. 1 and 5). Also we obtain necessary and sufficient conditions for a class of semi-group to be differentiable or to be C^∞ (Theorems 3. 2 and 4. 3).

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2. A class of differentiable semi-groups.

Throughout the rest of this paper, we assume that X is a locally convex linear topological space which is to be sequentially complete. A semi-group T_t, $t \geqq 0$ of continuous linear operators on X is called of class (C^0) if

for every $x \in X$, $T_t x$ *is continuous in* $t \geqq 0$.

The *infinitesimal generator* A of a semi-group T_t, $t \geqq 0$ is defined as usual by

$$A = \lim_{h \downarrow 0} h^{-1}(T_h - I).$$

Definition. A semi-group T_t, $t \geqq 0$ of class (C^0) is called *differentiable* at $t=a$ if there exists an $a>0$ such that $T_a X$ is included in the domain $D(A)$ of A. It is called C^∞ if it is diffierentiable at every $t>0$.

By this definition, it is easy to see that if T_t, $t \geqq 0$ is differentiable at $t=a$, then it is differentiable at every $t \geqq a$, and moreover $T_t X$ is included in $D(A^n)$ for $t \geqq na$, $n=1, 2, \ldots$.

In this section, we are cencerned mainly with the properties of the resolvent set $\rho(A)$ and the resolvent $R(\lambda; A) = (\lambda I - A)^{-1}$ of the infinitesimal generator A of a differentiable semi-group T_t, $t \geqq 0$. Here we always assume that T_t, $t \geqq 0$ is of class (C^0) and satisfies

for a constant $b>0$, T_t, $0 \leqq t \leqq b$ *is equicontinuous.*

The above condition is satisfied by a locally equicontinuous semi-group T_t, $t \geqq 0$ in the sense (due to T. Kōmura [2]):

for any fixed $b>0$, T_t, $0 \leqq t \leqq b$ *is equicontinus.*

Our main result in this section is the following

Theorem 2. 1. *Let* T_t, $t \geqq 0$ *be a semi-group of class* (C^0) *with the infinitesimal generator* A, *and satisfy the condition that for a constant* $b>0$, T_t, $0 \leqq t \leqq b$ *is equicontinuous. If this is differentiable at* $t=a$ *for some positive* $a \leqq b$ *and* $(CAT_a)^n$, $n=1, 2, \ldots\ldots$ *is equicontinuous for some positive constant* C, *then the domain*

$$\Sigma = \left\{\lambda\,;\, \mathrm{Re}\lambda \geqq \frac{1}{a}\log\frac{2}{C} - \frac{1}{a}\log|\mathrm{Im}\lambda|\right\}$$

is included in $\rho(A)$ *and* $\lambda^{-1}R(\lambda; A)$, $\lambda \in \Sigma$, $\mathrm{Re}\lambda \leqq \gamma$ *is equicontinuous for any fixed* $\gamma \geqq 0$.

REMARK. This problem was discussed first by E. Hille-R. S. Phillips [1] and recently by A. Pazy [3] both in the case where T_t, $t\geqq 0$ is a strongly continuous semi-group in a Banach space. In this case, the equicontinuity of T_t, $0\leqq t\leqq b$ and $(CAT_a)^n$, $n=1, 2, \ldots$ is always satisfied and hence need not be assumed.

We shall now prepare several lemmas.

LEMMA 2. 1. *Let T be a continuous linear operator on X such that $(CT)^n$, $n=1, 2, \ldots\ldots$ is equicontinuous for a constant $C>0$. Then every complex number λ such that $|\lambda|>1/C$ belongs to $\rho(T)$ and the domain of $R(\lambda; T)$ is all of X.*

PROOF. For any continuous semi-norm p there exists a continuous semi-norm q such that for all $x\in X$

$$p(\lambda^{-n}T^n x)\leqq (C|\lambda|)^{-n}\, q(x),\ n=1, 2, \ldots\ldots$$

Hence, because of $C|\lambda|>1$, the Nemann series

$$R(\lambda; T)=\sum_{n=0}^{\infty}\lambda^{-n-1}T^n$$

converges to be a continuous operator on X. Q. E. D.

REMARK. If X is a Banach space, $(CT)^n$, $n=1, 2, \ldots\ldots\ldots$ is equicontinuous with $1/C=\|T\|$.

LEMMA 2. 2. *Let B be a densely defined closed linear operator, and T, S continuous linear operators on X. Assume further that T has the continuous inverse T^{-1} defined on X. If for every $y\in D(B)$, Sy belongs to $D(B)$ and $Ty=BSy=SBy$, then B has the continuous inverse $B^{-1}=ST^{-1}$ defined on all of X.*

PROOF. Clearly B is a one-one operator. From the fact that B is closed and densely defined, it follows that for all $x\in X$, $Tx=BSx$ as well as $Sx\in D(B)$ and hence that for all $x\in X$, $x=BST^{-1}x$. Thus B is a one-one and onto operator and has the continuous inverse $B^{-1}=ST^{-1}$, as was to be proved.

LEMMA 2. 3. *Let A be the infinitesimal generator of a semi-group T_t, $t\geqq 0$ of class (C^0) such that T_t, $0\leqq t\leqq b$ is equicontinuous for a constant $b>0$. Then, A is a densely defined closed linear operator in X.*

PROOF. It is easily verified that

$$x\in X \quad \text{belongs to } D(A) \quad \text{and} \quad Ax=y$$

if and only if

$$T_t x-x=\int_0^t T_s y\, ds \quad \text{for every } t \text{ with } 0\leqq t\leqq b.$$

Making use of this relation, we prove the lemma. For any $x\in X$, $h^{-1}\int_0^h T_s x\, ds$ belongs to $D(A)$ and tends to x as $h\downarrow 0$, which shows that $D(A)$ is dense in X. Next, let

$\{x_\alpha\}$ be a net in $D(A)$ such that $x_\alpha \to x$ and $Ax_\alpha \to y$. Nothing that

$$T_t x_\alpha - x_\alpha = \int_0^t T_s A x_\alpha \, ds, \; 0 \leqq t \leqq b$$

and T_t, $0 \leqq t \leqq b$ is equicontinuous, we obtain

$$T_t x - x = \int_0^t T_s y \, ds, \; 0 \leqq t \leqq b.$$

That is, x belongs to $D(A)$ and $Ax = y$, which implies that A is closed. Q. E. D.

As a consequence of the above lemmas, we have

THEOREM 2. 2. *Under the assumptions of* Theorem 2. 1,

i) *A is a densely defined closed linear operator;*

ii) *every complex number λ such that* $\mathrm{Re}\lambda > \frac{1}{a} \log \frac{1}{C} - \frac{1}{a} \log |\lambda|$ *belongs to* $\rho(A)$, *and* $R(\lambda; A)$ *is an everywhere defined continuous linear operator given by*

$$R(\lambda; A) = \left\{ T_a + \lambda \int_0^a e^{\lambda(a-s)} T_s \, ds \right\} (\lambda e^{\lambda a} I - A T_a)^{-1}.$$

PROOF. We have only to prove ii). By Lemma 2. 1, for every λ saisfying

$$|\lambda e^{\lambda a}| > \frac{1}{C}, \text{ that is, } \mathrm{Re}\lambda > \frac{1}{a} \log \frac{1}{C} - \frac{1}{a} \log |\lambda|,$$

$\lambda e^{\lambda a}$ belongs to $\rho(A T_a)$ and $(\lambda e^{\lambda a} I - A T_a)^{-1}$ is a continuous operator everywhere defined on X. On the other hand, for every λ and $y \in D(A)$

$$\begin{aligned}(\lambda e^{\lambda a} I - A T_a) y &= (\lambda I - A) T_a y + \lambda (e^{\lambda a} I - T_a) y \\ &= (\lambda I - A) \left\{ T_a + \lambda \int_0^a e^{\lambda(a-s)} T_s \, ds \right\} y \\ &= \left\{ T_a + \lambda \int_0^a e^{\lambda(a-s)} T_s \, ds \right\} (\lambda I - A) y,\end{aligned}$$

where $x \to T_a x + \lambda \int_0^a e^{\lambda(a-s)} T_s x \, ds$ is a continuous operator on X. Thus, with the aid of Lemma 2. 2, we conclude this theorem.

PROOF OF THEOREM 2. 1. Because of Theorem 2. 2, a subset Σ of the set $\{\lambda; \mathrm{Re}\lambda > a^{-1} \log C^{-1} - a^{-1} \log |\lambda|\}$ is included in $\rho(A)$. Furthermore, there exists, for any continuous semi-norm p, a continuous semi-norm q such that for all $x \in X$ and $\lambda = \sigma + i\tau \in \Sigma$ with $\sigma \leqq \gamma$

$$\begin{aligned}p(R(\lambda; A)x) &\leqq \left\{ 1 + |\lambda| \int_0^a e^{\sigma(a-s)} \, ds \right\} (|\lambda| e^{\sigma a} - C^{-1})^{-1} q(x) \\ &\leqq (1 + |\lambda| a e^{a\gamma}) C q(x).\end{aligned}$$

Nothing that for any $\lambda \in \Sigma$, $|\lambda| e^{\sigma a} \geqq 2/C$, we obtain that $p(R(\lambda; A)x)$ is dominated by $Ce^{a\gamma}(C/2 + a) |\lambda| q(x)$ for every $\lambda \in \Sigma$ with $\sigma \leqq \gamma$, which completes the proof.

3. Conditions for differentiability

Our main problem in this section is to find sufficient conditions for the infinitesimal generator A of a semi-group T_t, $t\geqq 0$ satisfying the condition stated below to generate a differentiable semi-group:

$$\text{for a constant } \omega\geqq 0,\quad e^{-\omega t}T_t,\ t\geqq 0 \quad \text{is equicontinuous.}$$

A semi-group which satisfies this condition with $\omega=0$ is called generally an *equicontinuous* semi-group. The following theorems will be established analogously to the case of an equicontinuous semi-group or to that of a strongly continuous semi-group in a Banach space. See [1] and [2].

THEOREM A. *A necessary and sufficient condition for a closed linear operator A with dense domain to generate a semi-group T_t, $t\geqq 0$ of class* (C^0) *such that for a constant $\omega\geqq 0$, $e^{-\omega t}T_t$, $t\geqq 0$ is equicontinuous is that every complex number λ with* $\mathrm{Re}\lambda>\omega$ *belongs to $\rho(A)$ and*

$$(\mathrm{Re}\lambda-\omega)^n R(\lambda; A)^n,\ \mathrm{Re}\lambda>\omega,\ n=1, 2,\ldots\ldots \quad \text{is equicontinuous.}$$

THEOREM B. *Let T_t, $t\geqq 0$ be a semi-group of class* (C^0) *such that for a constant $\omega\geqq 0$, $e^{-\omega t}T_t$, $t\geqq 0$ is equicontinuous, and A be the infinitesimal generator. Then for every positive number ε,*

$$T_t y=\lim_{|\tau|\to\infty}\frac{1}{2\pi i}\int_{\omega+\varepsilon-i\tau}^{\omega+\varepsilon+i\tau} e^{\lambda t}R(\lambda; A)y\, d\lambda,\ t>0,\ y\in D(A).$$

Since $e^{-\omega t}T_t$, $t\geqq 0$ is an equicontinuous semi-group with the infinitesimal generator $-\omega I+A$, many other results on equicontinuous semi-groups will be extended to such semi-groups as this T_t, $t\geqq 0$. So that there may be little that is new. But, we intend to prove in Section 5 that the infinitesimal generators of differentiable semi-groups dealt with in Section 1 do generate, even if they are not necessarily assumed to be equicontinuous, semi-groups of the above type.

Our first assertion is

THEOREM 3. 1. *Let A be the infinitesimal generator of a semi-group T_t, $t\geqq 0$ of class* (C^0) *such that for a constant $\omega\geqq 0$, $e^{-\omega t}T_t$, $t\geqq 0$ is equicontinuous. If for some positive numbers α, β, $\rho(A)$ contains the domain*

$$\textstyle\sum=\{\lambda\,;\ \mathrm{Re}\lambda\geqq\alpha-\beta\log|\mathrm{Im}\lambda|\}$$

and if for some constant $p\geqq 0$, $\lambda^{-p}R(\lambda; A)$, $\lambda\in\sum$ is equicontinuous, then T_t, $t\geqq 0$ is differentiable at every $t>(p+2)/\beta$ and

$$\{t-(p+2)/\beta\}\,(CA\,T_t)^n,\ (p+2)/\beta<t\leqq(p+2)/\beta+1,\ n=1, 2,\ldots\ldots\ldots$$

is equicontinuous for some positive constant C independent of t and n.

COROLLARY. *Let A be the infinitesimal generator of a strongly continuous semi-group T_t, $t\geqq 0$ in a Banach space X. If for some positive numbers α and β, $\rho(A)$ contains the domain $\Sigma=\{\lambda;\ \mathrm{Re}\lambda\geqq\alpha-\beta\log|\mathrm{Im}\lambda|\}$ and if for some constant $p\geqq 0$ and $C>0$, $\|R(\lambda;A)\|\leqq C|\lambda|^p$, $\lambda\in\Sigma$, then T_t, $t\geqq 0$ is differentiable at every $t>\frac{p+2}{\beta}$ and*

$$\left(t-\frac{p+2}{\beta}\right)\|AT_t\|\leqq C' \text{ for all } t \text{ with } \frac{p+2}{\beta}<t\leqq\frac{p+2}{\beta}+1,$$

where C′ is a positive constant independent of t (cf. A. Pazy [3]).

Before the proof of this theorem, we must prove several preparatory lemmas.

Now let us consider a domain $\Sigma=\Sigma_1\cup\Sigma_2$ in the complex number plane, where $\Sigma_1=\{\lambda;\ \mathrm{Re}\lambda>\omega\}$ and $\Sigma_2=\{\lambda;\ \omega+1\geqq\mathrm{Re}\lambda\geqq\alpha-\beta\log|\mathrm{Im}\lambda|\}$ for positive numbers α, β and $\omega\geqq 0$. Let $R(\lambda)$ be a complex valued continuous function defined on Σ such that

$$|R(\lambda)|\leqq(Re\lambda-\omega)^{-1} \text{ on } \Sigma_1 \text{ and } \leqq|\lambda|^p \text{ on } \Sigma_2$$

for a constant $p\geqq 0$. Assuming these things, we study the successive derivatives with respect to t of the Riemann integral

$$I(t)=\frac{1}{2\pi i}\int_C e^{\lambda t}R(\lambda)d\lambda,$$

where $C=C_1\cup C_2\cup C_3$ and C_1, C_2 and C_3 are given by

$$\begin{aligned}
C_1&=\{\lambda=\sigma+i\tau;\ \omega+1\geqq\sigma=\alpha-\beta\log(-\tau)\},\\
C_2&=\{\lambda=\omega+1+i\tau;\ -L\leqq\tau\leqq L,\ L=e^{(\alpha-\omega-1)/\beta}\},\\
C_3&=\{\lambda=\sigma+i\tau;\ \omega+1\geqq\sigma=\alpha-\beta\log\tau\}.
\end{aligned}$$

We first establish

LEMMA 3. 1. *$I(t)$ is well defined for $t>(p+1)/\beta$, and n times differentiable for $t>(n+p+1)/\beta$, and $I^{(n)}(t)$ is estimated in absolute value as*

$$|I^{(n)}(t)|\leqq\frac{1}{\pi}e^{(\omega+1)t}H^n\frac{\beta t-n-p}{\beta t-n-p-1},\ n=1,2,\dots\dots\dots,$$

where H is some positive constant independent of t and n.

PROOF. Put $I_i(t)=\frac{1}{2\pi i}\int_{C_i}e^{\lambda t}R(\lambda)d\lambda$, $i=1,2,3$. Clearly $I_2(t)$ is well defined and n times differentiable for every $t>0$, and

$$|I_2^{(n)}(t)|\leqq(1/\pi)(\omega+1+L)^nLe^{(\omega+1)t}.$$

On C_3, $|\lambda|\leqq|\alpha-\beta\log\tau|+\tau\leqq K\tau$, $K=\mathrm{Max}\left(\frac{\omega+1}{L},\ e^{-1-\alpha/\beta}\right)+1$, and therefore

$$|I_3(t)|\leqq\frac{1}{2\pi}\int_{C_3}e^{\sigma t}|\lambda|^p|d\lambda|\leqq\frac{1}{2\pi}e^{\alpha t}K^p(1+\beta/L)\int_L^\infty\tau^{p-\beta t}\,d\tau,$$

which implies that $I_3(t)$ converges uniformly for $t \geqq t' > (p+1)/\beta$. Similarly $\frac{1}{2\pi i}\int_{C_3} \lambda^n e^{\lambda t} R(\lambda)\, d\lambda$ converges uniformly for $t \geqq t' > (n+p+1)/\beta$ and hence $I_3(t)$ is well defined for $t > (p+1)/\beta$ and n times differentiable for $t > (n+p+1)/\beta$. Moreover $I_3^{(n)}(t)$ is estimated as

$$|I_3^{(n)}(t)| \leqq \frac{1}{2\pi} e^{\omega t} K^{n+p}(1+\beta/L)\int_L^\infty \tau^{(n+p-\beta t)} d\tau$$

$$= \frac{1}{2\pi} K^{n+p}(1+\beta/L) L^{n+p+1} e^{(\omega+1)t} / (\beta t - n - p - 1).$$

Similar estimates hold for $I_1(t)$ and $I_1^{(n)}(t)$. Thus we obtain that $I(t)$ is well defined for $t > (p+1)/\beta$ and n times differentiable for $t > (n+p+1)/\beta$, and

$$|I^{(n)}(t)| \leqq (1/\pi) e^{(\omega+1)t} H^n \{1 + (\beta t - n - p - 1)^{-1}\},\ n = 1, 2, \ldots\ldots,$$

where the positive constant $H = \mathrm{Max}\ \{(\omega+1+L)^2,\ K^{p+1}(1+\beta/L)(L+1)^{p+2}\}$ is independent of t and n. Q. E. D.

We then have immediately

LEMMA 3. 2. *For $(p+2)/\beta < t \leqq (p+2)/\beta + 1$*

$$(t - (p+2)/\beta)\, |I^{(n)}(nt)| \leqq (1/\pi) M^n,\ n = 1, 2, \ldots\ldots,$$

where M is a positive constant independent of t and n.

PROOF. $t > (p+2)/\beta$ implies that $nt > n(p+2)/\beta \geqq (n+p+1)/\beta$ and hence, by the previous lemma, $I^{(n)}(nt)$ is estimated as

$$|I^{(n)}(nt)| \leqq (1/\pi) e^{(\omega+1)nt} H^n \{1 + (\beta t - p - 2)^{-1}\}.$$

Thus, for $(p+2)/\beta < t \leqq (p+2)/\beta + 1$

$$(t - (p+2)/\beta)\, |I^{(n)}(nt)| \leqq (1/\pi) M^n,\ n = 1, 2, \ldots\ldots,$$

where $M = H\, e^{(\omega+1)(1+(p+2)/\beta)}(1 + 1/\beta)$. Q. E. D.

PROOF OF THEOREM 3. 1. Put for every $x \in X$ $S_t x = \frac{1}{2\pi i}\int_C e^{\lambda t} R(\lambda; A) x\, d\lambda$, where C is the integral path used in the definition of $I(t)$.

By Theorem A, $(\mathrm{Re}\lambda - \omega) R(\lambda; A)$, $\mathrm{Re}\lambda > \omega$ is equicontinuous and therefore $S_t x$ is, by Lemma 3. 1, well defined for $t > (p+1)/\beta$ and n times differentiable for $t > (n+p+1)/\beta$. With the aid of Lemma 3. 2, we find, for any continuous semi-norm p, a constant semi-norm q such that for all $x \in X$

$$\{t - (p+2)/\beta\}\, p(S_{nt}^{(n)} x) \leqq (1/\pi) M^n\, q(x),$$
$$(p+2)/\beta < t \leqq (p+2)/\beta + 1,\ \ n = 1, 2, \ldots\ldots\ldots,$$

which means that $\{t - (p+2)/\beta\} M^{-n} S_{nt}^{(n)},\ (p+2)/\beta < t \leqq (p+1)/\beta + 1,\ n = 1, 2, \ldots\ldots$ is equicontinuous.

For the proof of the theorem, it remains to show that $T_t x = S_t x$ for all $x \in X$ and $t > (p+1)/\beta$. By Theorem B, it holds that

$$T_t y = \lim_{|t|\to\infty} \frac{1}{2\pi i} \int_{\omega+1-i\tau}^{\omega+1+i\tau} e^{\lambda t} R(\lambda; A) y \, d\lambda, \ y \in D(A), \ t>0.$$

Let $z \in D(A^{[p]+1})$, then

$$R(\lambda; A)z = \frac{z}{\lambda} + \frac{Az}{\lambda^2} + \cdots\cdots + \frac{A^{[p]}z}{\lambda^{[p]+1}} + \frac{R(\lambda; A)A^{[p]+1}z}{\lambda^{[p]+1}}.$$

Hence noting that by Cauchy's intergral theorem one can shift the intergral path from $\omega+1+i\tau$, $-\infty<\tau<\infty$ to C, we obtain that for $t>(p+1)/\beta$ and $z \in D(A^{[p]+1})$, $T_t z = S_t z$. Indeed, this can be verified by nothing for instance that

$$\int_{\alpha-\beta \log N+iN}^{\omega+1+iN} |\lambda|^{p-[p]-1} |d\lambda| \leq |\alpha-\omega-1|/N^{[p]-p+1} + \beta \log N/N^{[p]-p+1} \longrightarrow 0$$

as $N \to \infty$.

Since $D(A^{[p]+1})$ is dense in X, T_t, $x = S_t x$ for all $x \in X$ and $t>(p+1)/\beta$. Thus, $\{t-(p+2)/\beta\} M^{-n} S_{nt}^{(n)} = \{t-(p+2)/\beta\} (M^{-1}AT_t)^n$, $(p+2)/\beta < t \leq (p+2)/\beta+1$, $n=1$, $2, \ldots\ldots$ is equicontinuous. Q. E. D.

We can summarize the results of Theorems 2. 1 and 3. 1 as follows

Theorem 3. 2. *Let A be the infinitesimal generator of a semi-group T_t, $t \geq 0$ of class* (C^0) *such that for a constant $\omega \geq 0$, $e^{-\omega t} T_t$, $t \geq 0$ is equicontinuous. The following conditions are mutually equivalent*:

(I) *T_t, $t \geq 0$ is differentiable at some $t = a$ and for some constant $C>0$, $(CAT_a)^n$, $n=1, 2, \ldots\ldots$ is equicontinuous*;

(II) *For some constant $c>0$, T_t, $t \geq 0$ is differentiable at every $t>c$ and for some constant $E>0$, $(t-c)(EAT_t)^n$, $c<t \leq c+1$, $n=1, 2, \ldots\ldots$ is equicontinuous*;

(III) *For some constants α, $\beta>0$, $\rho(A)$ contains the domain $\sum = \{\lambda;\ \mathrm{Re}\lambda \geq \alpha-\beta \log |\mathrm{Im}\lambda|\}$ and $\lambda^{-1}R(\lambda; A)$, $\lambda \in \sum$ is equicontinuous.*

Proof. The implication (I)→(III) is shown by Theorem 2. 1 and (III)→(II) by Theorem 3. 1. The proof of (II)→(I) is trivial. Q. E. D.

4. C^∞ semi-groups.

In this section we shall deal with C^∞ semi-groups, whose definition was given in the beginning of Section 2. The following theorems are corollaries of Theorems 2. 1 and 3. 1:

Theorem 4. 1. *Let T_t, $t \geq 0$ be a semi-group of class* (C^0) *with the infinitesimal generator A such that for a positive constant b, T_t, $0 \leq t \leq b$ is equicontinuous, If this is C^∞ and for* *every t with $0<t\leqq b$, there exists a positive number $C(t)$ such that*

$$(C(t)AT_t)^n,\ 0<t\leqq b,\ n=1,2,\ldots\ldots \quad \textit{is equicontinuous,}$$

then for every t with $0<t\leqq b$ and $D\leqq C(t)$, the domain

$$\Sigma_t=\left\{\lambda;\ \mathrm{Re}\lambda\geqq\frac{1}{t}\log\frac{2}{D}-\frac{1}{t}\log|\mathrm{Im}\lambda|\right\}$$

is included in $\rho(A)$ and $\lambda^{-1}R(\lambda;A)$, $\lambda\in\Sigma_t$, $\mathrm{Re}\lambda\leqq\gamma$ is equicontinuous for any fixed $\gamma\geqq 0$.

Proof. Recalling the proof of Theorem 2. 1, we obtain that $\rho(A)$ contains the domain Σ_t and that for any continuous semi-norm p, there exists a continuous semi-norm q such that

$$p(\lambda^{-1}R(\lambda;A)x)\leqq D\,e^{t\gamma}(D/2+t)q(x)\leqq D(D/2+b)e^{b\gamma}q(x)$$

for all $x\in X$ and $\lambda\in\Sigma_t$ with $\mathrm{Re}\lambda\leqq\gamma$, which completes the proof.

Theorem 4. 2. *Let A be the infinitesimal generator of a semi-group T_t, $t\geqq 0$ of class (C^0) such that for a constant $\omega\geqq 0$, $e^{-\omega t}T_t$, $t\geqq 0$ is equicontinuous. If for every $\beta>0$, there exists a positive number $\alpha(\beta)$ such that $\rho(A)$ contains the domain*

$$\Sigma_\beta=\{\lambda;\ \mathrm{Re}\lambda>\alpha(\beta)-\beta\log|\mathrm{Im}\lambda|\}$$

and that for some constant $p\geqq 0$, $\lambda^{-p}R(\lambda;A)$, $\lambda\in\Sigma_\beta$ is equicontinuous, then T_t, $t\geqq 0$ is C^∞, and for every $t>0$, there exists a positive number $C(t)$ such that $(C(t)AT_t)^n$, $t>0$, $n=1,2,\ldots$ is equicontinuous,

Proof. By Theorem 3. 1, for every $\beta>0$, T_t, $t\geqq 0$ is differentiable at every $t>(p+2)/\beta$, which implies that T_t, $t\geqq 0$ is C^∞.

Remembering the proof of Lemma 3. 2, we obtain that for every $\beta>0$ and $t>(p+1)/\beta$,

$$|I^{(n)}(nt)|\leqq(1/\pi)e^{(\omega+1)nt}H^n\{1+(\beta t-p-2)^{-1}\},\ n=1,2,\ldots\ldots\ldots,$$

where H is a positive constant independent of t and n. Therefore, for any continuous semi-norm p, there exists a continuous semi-norm q such that for all $x\in X$

$$p((AT_s)^n x)\leqq(1/\pi)M(s)^n\frac{p+3}{p+2}q(x),\ s=2(p+2)/\beta,\ n=1,2,\ldots\ldots,$$

where $M(s)$ is a positive function of s. But $\beta>0$ was arbitrary. Thus, $(M(t)^{-1}AT_t)^n$, $t>0$, $n=1,2,\ldots\ldots$ is equicontinuous. Q. E. D.

Thus we have established

Theorem 4. 3. *Let A be the infinitesimal generator of a semi-group T_t, $t\geqq 0$ of class (C^0) such that for a constant $\omega\geqq 0$, $e^{-\omega t}T_t$, $t\geqq 0$ is equicontinuous.*

T_t, $t\geqq 0$ is C^∞ and for every t with $0<t\leqq b$ $(b>0)$, there exists a positive number $C(t)$ such that $(C(t)AT_t)^n$, $0<t\leqq b$, $n=1,2,\ldots\ldots$ is equicontinuous if and only if

for every $\beta \geqq \delta (>0)$, *there exists a positive number* $\alpha(\beta)$ *such that* $\rho(A)$ *contains the domain* $\sum_{\beta} = \{\lambda;\ \mathrm{Re}\lambda > \alpha(\beta) - \beta \log|\mathrm{Im}\lambda|\}$ *and that* $\lambda^{-1}\mathrm{R}(\lambda; A)$, $\lambda \in \sum_{\beta}$ *is equicontinuous.*

REMARK. It is not of little interest that the properties of the above C^{∞} semi-group T_t, $t \geqq 0$ depend upon the function $C(t)$ defined on $(0, b]$ for which $(C(t)AT_t)^n$, $0 < t \leqq b$, $n = 1, 2, \ldots\ldots$ is equicontinuous. When for some positive constant C, $C(t) \geqq Ct$ for example, this semi-group is, as is well known, nothing but a holomorphic one.

5. A property of differentiable semi-groups.

In this section, we shall add refinement to the theory in Section 2. Let us consider again a semi-group T_t, $t \geqq 0$ of class (C^0) such that for a constant $b > 0$, T_t, $0 \leqq t \leqq b$ is equicontinuous, and suppose that it is differentiable at $t = a$ for some positive $a \leqq b$ and that for a constant $C > 0$, $(CAT_a)^n$, $n = 1, 2, \ldots\ldots$ is equicontinuous.

Our final object in this section is to show that the above assumptions on T_t, $t \geqq 0$ are restrictive enough for its infinitesimal generator A to generate a semi-group such that

for some constant $\omega > 0$, $\mathrm{e}^{-\omega t}T_t$, $t \geqq 0$ *is equicontinuous.*

We shall begin with numerical calculations. It is obvious that the unique solution ω of $\lambda \mathrm{e}^{\lambda a} = C^{-1}$ is positive and that $\lambda > \omega$ if and only if $\lambda \mathrm{e}^{\lambda a} > C^{-1}$.

LEMMA 5. 1. *For every integer* $m \geqq 0$, $(1 - \lambda^{-1}\mathrm{e}^{-\lambda a}C^{-1})^{-1}$ *is* m *times differentiable with respect to* $\lambda > \omega$, *and* $(-d/d\lambda)^m (1 - \lambda^{-1}\mathrm{e}^{-\lambda a}C^{-1})^{-1}$ *is positive and expanded for* $\lambda > \omega$ *as*

$$(-d/d\lambda)^m (1 - \lambda^{-1}\mathrm{e}^{-\lambda a}C^{-1})^{-1} = \sum_{k=0}^{\infty} (-d/d\lambda)^m (\lambda^{-1}\mathrm{e}^{-\lambda a}C^{-1})^k,$$

where the convergence is uniform for $\lambda \geqq \omega + \varepsilon$, $\varepsilon > 0$.

PROOF. We have only to note that for any integer $k \geqq 0$

$$(-d/d\lambda)^m (\lambda^{-1}\mathrm{e}^{-\lambda a}C^{-1})^k$$
$$= \sum_{\nu=0}^{m} {}_mC_\nu\, k(k+1)\cdots\cdots(k+\nu-1)\lambda^{-\nu}(ka)^{m-\nu}/(\lambda \mathrm{e}^{\lambda a}C)^k.$$

LEMMA 5. 2. *For every integer* $m \geqq 0$ *and* $\lambda > \omega$

$$(-d/d\lambda)^m (\lambda - \omega)^{-1}$$
$$\geqq (-d/d\lambda)^m (\lambda \mathrm{e}^{\lambda a} - C^{-1})^{-1} + \sum_{\nu=0}^{m} {}_mC_\nu \int_0^a s^{m-\nu}\mathrm{e}^{-\lambda s}ds\,(-d/d\lambda)^\nu (1 - \lambda^{-1}\mathrm{e}^{-\lambda a}C^{-1})^{-1}.$$

PROOF. Consider the equality:

$$(\lambda - \omega)^{-1} = \left\{\mathrm{e}^{\omega a} + \lambda \int_0^a \mathrm{e}^{\lambda(a-s)}\mathrm{e}^{\omega s}\, ds\right\}(\lambda \mathrm{e}^{\lambda a} - C^{-1})^{-1},\ \lambda > \omega,$$

which can be easily verified by calculating the integrating part. Differentiating this m times with respect to λ, we have for $\lambda>\omega$

$$(-d/d\lambda)^m(\lambda-\omega)^{-1}$$

$$=(-d/d\lambda)^m(\lambda e^{\lambda a}-C^{-1})^{-1}\, e^{\omega a}+\sum_{\nu=0}^{m} {}_mC_\nu\int_0^a s^{m-\nu}e^{-\lambda s}e^{\omega s}\,ds(-d/d\lambda)^\nu(1-\lambda^{-1}e^{-\lambda a}C^{-1})^{-1}.$$

But, by the previous lemma, $(-d/d\lambda)^m(1-\lambda^{-1}e^{-\lambda a}C^{-1})^{-1}$ and hence $(-d/d\lambda)^m(\lambda e^{\lambda a}-C^{-1})^{-1}=-(-d/d\lambda)^m C+C(-d/d\lambda)^m(1-\lambda^{-1}e^{-\lambda a}C^{-1})^{-1}$ is positive for every $\lambda>\omega$. Therefore, we have for $\lambda>\omega$

$$(-d/d\lambda)^m(\lambda-\omega)^{-1}$$

$$\geqq(-d/d\lambda)^m(\lambda e^{\lambda a}-C^{-1})^{-1}+\sum_{\nu=0}^{m} {}_mC_\nu\int_0^a s^{m-\nu}e^{-\lambda s}\,ds(-d/d\lambda)^\nu(1-\lambda^{-1}e^{-\lambda a}C^{-1})^{-1},$$

as was to be proved.

Thus we have

THEOREM 5. *Under the assumptions of* Theorem 2. 1, $\rho(A)$ *contains the domain*

$$\Delta=\{\lambda;\ \mathrm{Re}\lambda e^{a\mathrm{Re}\lambda}>C^{-1}\}=\{\lambda;\ \mathrm{Re}\lambda>\omega\}\ (\omega e^{\omega a}=C^{-1})$$

and $\{(\mathrm{Re}\lambda-\omega)R(\lambda;A)\}^n$, $\mathrm{Re}\lambda>\omega$, $n=1, 2,\ldots\ldots$ *is eqnicontinuous.*

PROOF. By virtue of Theorem 2. 2, a subset Δ of $\{\lambda;\ |\lambda|e^{a\mathrm{Re}\lambda}>C^{-1}\}$ is included in $\rho(A)$ and $R(\lambda; A)$ is expressed by

$$R(\lambda;A)=\left\{T_a+\lambda\int_0^a e^{\lambda(a-s)}T_s\,ds\right\}(\lambda e^{\lambda a}I-AT_a)^{-1},\ \mathrm{Re}\lambda>\omega.$$

By Lemma 5. 1, $(I-\lambda^{-1}e^{-\lambda a}AT_a)^{-1}$ is m times differentiable with respect to $\sigma=\mathrm{Re}\lambda>\omega$ and

$$(-d/d\sigma)^m(I-\lambda^{-1}e^{-\lambda a}AT_a)^{-1}=\sum_{k=0}^{\infty}(-d/d\sigma)^m(\lambda^{-1}e^{-\lambda a}AT_a)^k,$$

$$m=0, 1,\ldots\ldots$$

Therefore

$$\{(-d/d\sigma)^n(1-\sigma^{-1}e^{-\sigma a}C^{-1})^{-1}\}^{-1}(-d/d\sigma)^n(I-\lambda^{-1}e^{-\lambda a}AT_a)^{-1},$$

$$\sigma>\omega,\ n=0, 1,\ldots\ldots$$

is equicontinuous and similarly so is

$$\{(-d/d\sigma)^n(\sigma e^{\sigma a}-C^{-1})^{-1}\}^{-1}(-d/d\sigma)^n(\lambda e^{\lambda a}I-AT_a)^{-1},\ \sigma>\omega,$$

$$n=0, 1,\ldots\ldots$$

Since for every integer $n\geqq0$ and λ with $\sigma>\omega$

$$(-d/d\sigma)^nR(\lambda;A)$$

$$=T_a(-d/d\sigma)^n(\lambda e^{\lambda a}I-AT_q)^{-1}+\sum_{\nu=0}^{n}{}_nC_\nu\int_0^a s^{n-\nu}e^{-\lambda s}T_s\,ds(-d/d\sigma)^\nu(I-\lambda^{-1}e^{-\lambda a}AT_a)^{-1},$$

for any continuous semi-norm p, there exists a continuous semi-norm q such that for all $x\in X$, λ with $\sigma>\omega$ and $n=0, 1,\dots\dots$

$$p((-d/d\sigma)^nR(\lambda;A)x)$$

$$\leqq\left\{(-d/d\sigma)^n(\sigma e^{\sigma a}-C^{-1})^{-1}+\sum_{\nu=0}^{n}{}_nC_\nu\int_0^a s^{n-\nu}e^{-\sigma s}\,ds(-d/d\sigma)^\nu(1-\sigma^{-1}e^{-\sigma a}C^{-1})^{-1}\right\}\cdot q(x).$$

Making use of Lemma 5. 2, we have

$$p((-d/d\sigma)^nR(\lambda;A)x)\leqq(-d/d\sigma)^n(\sigma-\omega)^{-1}q(x).$$

This implies that

$$\{(-d/d\sigma)^n(\sigma-\omega)^{-1}\}^{-1}(-d/d\sigma)^nR(\lambda;A),\ \sigma>\omega,\ n=0, 1,\dots\dots$$

and consequently

$$\{(\sigma-\omega)^{-1}R(\lambda;A)\}^n,\ \sigma>\omega,\ n=1, 2,\dots\dots$$

is equicontinuous. Q. E. D.

Thus, combining this theorem and Theorem 2. 2 and remembering Theorem A, we obtain that the above A generates a semi-group S_t, $t\geqq0$ of class (C^0) such that

$$e^{-\omega t}S_t,\ t\geqq0\ (\omega e^{\omega a}=C^{-1})\quad \textit{is equicontinnons.}$$

But, as is well known, a densely defined closed linear operator is the infinitesimal generator of at most one semi-group of class (C^0). Hence it must hold that $S_t=T_t$.

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