Remarks on dimensions of compact transformation groups

By

Kojun ABE

(Received October 1, 1971)

1. Introduction

In this paper, we consider the dimension of a compact connected group G which acts on a space X effectively. In [1], L. N. Mann treated of the case where G acts on X transitively. We shall try to generalize this result when G acts on X not necessary transitively. The proof of our result is based on the property of a slice on the G-space.

Throughout this paper, all transformation groups are metrizable and all spaces are cohomology manifolds over a characteristic 0–field K.

The author wishes to thank Prof. K. Aoki and Prof. T. Watabe for their kind encouragements and advices.

2. Some propositions

For a G-space X, H and $X_{(H)}$ denote a principal isotropy subgroup and the set of all principal orbits of this action respectively (see [2]).

If X is a finite dimensional G-cohomology manifold over K and $x \in X$, there exists a slice at x which is also a cohomology manifold. Following propositions have been proved when X is a differentiable G-manifold. Combining the methods in [3, p. 6] and [2, p. 121, lemm 3. 2], and those results can be gengralized when X is a G-space similarly.

In this section we assume that G is a compact connected Lie group which acts on an connected n-cm X over K effectively.

Proposition 1 (cf. [4] p. 9). If G_x ($x \in X$) is connected, $X_{(H)}$ *is open dence in* X.

Let X be an n-dimensional G-space. Using Proposition 1, we have following proposition.

Proposition 2 (cf. [4] p. 12). *Let* X *be an effective* G*-space, then we have dim* $G \leqq n(n+1)/2$.

Proposition 3 (cf. [5]). *If the dimension on* G *falls into one of the ranges* $(n-k)(n-k+1)/2+k(k+1)/2<$ *dim* $G<(n-k+1)(n-k+2)/2$ $(k=1, 2, \ldots\ldots\ldots)$, *then there exist only three possibilities*:

(1) $n=4$, $G \sim SU(3)$ *and* $X \approx P^2(C)$.

(2) *$n=6$, $G\simeq G_2$ and $X\approx S^6$ or $P^6(R)$.*

(3) *$n=10$, $G\sim SU(6)$ and $X\approx P^5(C)$.*

where all the above actions are transitive. The notations $\sim$, $\simeq$ and $\approx$ imply locally isomorphic, isomorphic and homeomorphic respectively.

3. Main theorems

In this section G denotes a compact connected group (not necessary a Lie group). We try to generalize some results for differentiable G-action in the case of continuous G-action.

THEOREM 1. *For a compact connected finite dimensional group G, Proposition 3 also holds.*

PROOF. There exists a O-dimensional central subgroup F of G such that G/F is a Lie group (see [6, Chapt. 9]). By [3, Theorem 1], X/F is an n-cm over K. Then it is easy to check the action:

$$G/F\times X/F\longrightarrow X/F$$

is an almost effective action. By Proposion 3, G/F is a simple Lie group.

Let $G^*=A\times G_1$ be a finite covering of G such that A is a compact connected abelian group and G_1 is a compact connected simply connected semi simple Lie group. Since G/F is a simple Lie group, A is a trivial factor. Considering the proof of [6, Example 74], it is easy to see that G is a Lie group. The proof of Theorem 1 is reduced to that of Proposition 3.

L. N. Mann considered dimensions of compact groups which act on spaces transitively and effectively [1]. By the above results, we can remove the assumption "transitively" with somewhat difference from his conclusion.

THEOREM 2. *Let G be a finite dimensional compact connected group which acts on an n-cm X over K effectively. If dim $G>(n-2)(n-1)/2+2$, then one of the following holds.*

(1) *G is a Lie group and X is a manifold,*

(2) *dim $G=n(n-1)/2+1$ and $G\sim A^1\times Spin(n)$,*

(3) *$n=5$ and $G\sim A^1\times SU(3)$,*

(4) *dim $G=n(n-1)/2$ and $G\sim Spin(n)$,*

where A^1 is a 1-dimensional compact connected group.

PROOF. Similarly as the case of the proof of Theorem 1, suppose that $G^*=A\times G_1$ is the finite covering of G and F is a o-dimensional subgroup of G. Further we can choose $T^q\times G_1$ as a finite covering of G/F where T^q is a q-dimensional toral group. It is easy to see that $(T^q\times G_1)\times X/F\longrightarrow X/F$ is an almost effective action. Let H be a principal isotropy subgroup of this action. Set $X_1=(T\times G_1)/H$.

Let G_x be an isotropy subgroup of x such that $G/G_x \cdot F = X_1$. Then $G_x \cdot F/F$ is a principal isotropy subgroup of the action: $G/F \times X/F \longrightarrow X/F$. By a result in [6, Chapt. 7], we have $\dim G/G_x \cdot F = \dim G/G_x = \dim X_1$. From the definition of H, it follows that H/K is also a principal isotropy subgroup of this action, where K is a finite subgroup of $T^q \times G_x$ such that $T^q \times G_1/K = G/F$. Therefore we have

$$\dim X_1 = \dim T^q \times G_1 - \dim H = \dim G/F - \dim H/K = \dim G/G_x \cdot F$$

Since the action: $G_1 \times X_1/T^q \longrightarrow X_1/T^q$ is an almost effective action, Proposition 2 concludes that

$$\dim G_1 \leqq (\dim X_1 - q)(\dim X_1 - q + 1)/2.$$

Therefore we have

$$(*) \quad \dim G \leqq (\dim X_1 - q)(\dim X_1 - q + 1)/2 + q.$$

Combining the assumption of dim G, Proposition 3 and (*), we have following five possibilities:

(I) $\dim X_1 = n$

(a) $\dim G = n(n+1)/2$,

(b) $\dim G = n(n-1)/2 + 1$,

(c) $\dim G = n(n-1)/2$,

(d) $\dim G = (n-1)(n-2)/2 + 3$,

(II) $\dim X_1 = n - 1$

(e) $\dim G = n(n-1)/2$,

Next, we investgate the above cases.

Case (I):

Using the slice theorem one can deduce from $\dim X_1 = \dim X/F$ that $X_1 = X/F$, and hence the action is transitive.

(a) (*) implies that

$$n(n+1)/2 \leqq (n-q)(n-q+1)/2 + q,$$

and hence we have $q^2 + q - 2nq \geqq 0$.

Since T^q acts on X almost effectively, we have $n \geqq q$. Then we have following two possibilities:

(a. 1) $q=0$.

G is a Lie group and hence we have possibility (1).

(a. 2) $q=1$.

We can deduce from $n=1$ that $G \sim A^1$, and this is contrary to our hypothesis.

(b) Similarly (*) implies $q \leqq 2$.

(b. 1) $q=0$.

We have possibility (1).

(b. 2) $q=1$.

The action: $G_1\times X_1/T^1\longrightarrow X_1/T^1$ is an almost effective action. Since dim $G_1=n(n-1)/2$ and dim $X/T^1=n-1$, we shall prove in Theorem 3 that $G_1\sim Spin(n)$. So we have possibility (2).

(b. 3) $q=2$.

It is easy to see that this is contrary to our hypothesis.

(c) (*) implies $q\leqq 3$.

(c. 1) $q=0$.

(c. 2) $q=1$.

Since the action: $G_1\times X_1/T^1\longrightarrow X_1/T^1$ is an almost effective action, it follows from Proposition 3 that $n\leqq 2$. Therefore $n=2$, but this is contradiction to $X/F=X_1$.

(c. 3) $q=2$.

Similar to (c. 2), it can be showed.

(c. 4) $q=3$.

Since $n=3$, this is contrary to our hypothesis.

(d) (*) implies $q\leqq 1$.

(d. 1) $q=0$.

(d. 2) $q=1$.

Combining Proposition 3 and (*), we have $n=5$ and $G_1\sim SU(3)$. Therefore we have possibility (3).

Case (II):

(e) Since $T^q\times X_1\longrightarrow X_1$ is an almost effective action, we have $n\geqq q+1$. Then (*) $q\leqq 1$.

(e. 1) $q=0$.

We shall prove in Theorem 3 that $G_1\sim Spin(n)$, and hence we have possibilities (4).

(e. 2) $q=1$.

Since $n\leqq 2$, this is contradiction.

This completes the proof of Theorem 2.

If we use the above discussion, we have an extension of L. P. Eisenhalt [7] in the case of compact topological group.

THEOREM 3 *Let G be an $n(n+1)/2$ dinensional compact connected group which acts on an n-cm X over K effectively. Then we have $G\sim Spin(n+1)$ or A^1, and if n is even $X\approx S^n$ or $P^n(R)$.*

PROOF. (I) In the case where G is a Lie group.

With the same notation and similar computation as Theorem 2 we have

$$\dim G\leqq (n-q)(n-q+1)/2+q.$$

This implies that $q\leqq 1$, and hence we have following two possibilities.

(I. 1) $q=1$.

Since $n=1$, we have $G=T^1$ and $X=S^1$.

(I. 2) $q=0$.

We require here following lemma [5].

LEMMA. There are integers $t_1, t_2, \ldots\ldots, t_s$ such that

(1) $G=G_1\times G_2\times \ldots\ldots \times G_s$ where G_1 is a simple Lie group or $Spin(4)$,

(2) dim $G_i \leqq (t_i+1)t_i/2$ and $t_1+t_2+\ldots\ldots+t_s \leqq t$, where t=dimension of the principal orbit.

Since dim $G=n(n+1)/2$, there exists some $t_i=n$ and the other $t_j=0(j\neq i)$. Then we have $G=G_i$. By L. N. Mann's list [5], such G_i is locally isomorphic to $Spin(n+1)$.

Let H be a principal isotropy subgroup of the action, and H_1 be identity component of H. Set $X_1=Spin(n+1)/H_1$. Since the dimension of the maximal subgroup of $Spin(n+1)$ is $n(n-1)/2$, we have dim $H_1=n(n-1)/2$. Hence $H_1=Spin(n)$ and $X_1=S^n$. By a result in [8], X_1 is a covering space of X and the order of this covering is one or two when n is even. So we have $X=S^n$ or $P^n(R)$.

(II) In the case where G is a compact group.

Let F be the O-dimensional subgroup of G such that G/F is a Lie group.

$$G/F\times X/F \longrightarrow X/F$$

is an action of a Lie group, and hence it is easy to show that this is reducible to the case (I).

NIIGATA UNIVERSITY

References

[1] L. N. MANN, *Dimensions of compact transformation groups.* Mich. Math. Jour. 14 (1967), 433–444.

[2] A. BOREL, Seminar on Tranceformation Groups, Ann. of Math. Studies 46, Princeton Univerfity Press (1960), 101–115.

[3] F. RAYMOND *The orbit spaces of totally disconnected groups of transformations manifolds.* Proc. of Amer, Math. Soc. 12 (1961), 1–7.

[4] K. JÄNICH, Differenzierbare G-Mannigfaltigkeiten, Springer-Verlag: Heidelberg New York (1968).

[5] L. N. MANN, *Gaps in the dimension of transformation groups,* Illinois Jour. Math. 10(1966), 532–546.

[6] L. S. PONTRJAGIN, Topological Groups, Princeton University Press (1958).

[7] L. P. EISENHALT, Riemannian Geometry, Princeton University Press (1940).

[8] H. C. WANG, *Homogeneous space with non-vanishing Euler characteristic,* Ann. of Math. 50 (1949), 925–953.